R.H.

The Riemann Zeta function is defined as the Analytic Continuation of the Dirichlet series

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} , \quad Re(s) > 1$$

Riemann's Xi Function is defined as

$$\xi(s) = \frac{s(s-1)}{2} \pi^{-s/2} \Gamma\left(\frac{s}{2}\right) \zeta(s)$$

The zero of $s-1$ cancels the pole of $\zeta(s)$, and the real zeros of $s\,\zeta(s)$ are cancelled by the simple poles of $\Gamma\left(\frac{s}{2}\right)$ which never vanishes
Thus, $\xi(s)$ is an entire function whose zeros are the non trivial zeros of $\zeta(s)$.
The non trivial zeros of $\zeta(s)$ lie in the critical strip $0 \leq Re(s) \leq 1$.

<u>Statement of R.H.</u> – The R.H. states that all the non trivial zeros of the Riemann Zeta function lie on the critical line with real part equal to $\frac{1}{2}$.

Riemann's Xi function is defined as

$$\xi(s) = \xi(0) \prod_{\rho} \left(1 - \frac{s}{\rho}\right)$$

The above infinite product is Absolutely Convergent if we combine the factors $\left(1-\frac{s}{\rho}\right)$ and $\left(1 - \frac{s}{1-\rho}\right)$

$$\xi(s) = \xi(0) \prod_{\operatorname{Im}\rho > 0} \left(1-\frac{s}{\rho}\right)\left(1-\frac{s}{1-\rho}\right) \quad \text{—} ①$$

Claim :- $\xi(\rho) = 0 \Rightarrow \operatorname{Re}(\rho) = \frac{1}{2}$

Enough to Prove :- $\operatorname{Re}\rho \neq \frac{1}{2} \Rightarrow \xi(\rho) \neq 0$

We prove that $\xi(\rho) \neq 0$ by contradiction

Let, $\xi(\rho_0) = 0$ for some $\rho_0 \in \mathbb{C}$

$$\Rightarrow \xi(\overline{\rho_0}) = 0$$

From ①, $\xi(0) \prod_{\operatorname{Im}\rho > 0} \left(1 - \frac{\overline{\rho_0}}{\rho}\right)\left(1 - \frac{\rho_0}{1-\rho}\right) = 0$

$\because \operatorname{Re}\rho \neq \frac{1}{2}$ So we split the above product as

$$\prod_{\substack{\operatorname{Im}\rho > 0 \\ \operatorname{Re}\rho < \frac{1}{2}}} \left(1 - \frac{\rho_0}{\rho}\right)\left(1 - \frac{\overline{\rho_0}}{1-\rho}\right) \prod_{\substack{\operatorname{Im}\rho > 0 \\ \operatorname{Re}\rho > 1/2}} \left(1 - \frac{\rho_0}{\rho}\right)\left(1 - \frac{\overline{\rho_0}}{1-\rho}\right) = 0 \quad \text{—} ②$$

Let,

$$I_\rho = \prod_{\substack{Im\,\rho>0 \\ Re\,\rho<\frac{1}{2}}} \left(1-\frac{\rho_0}{\rho}\right)\left(1-\frac{\rho_0}{1-\rho}\right)$$

&

$$J_\rho = \prod_{\substack{Im\,\rho>0 \\ Re\,\rho>1/2}} \left(1-\frac{\rho_0}{\rho}\right)\left(1-\frac{\rho_0}{1-\rho}\right)$$

(2) $\Rightarrow$ $I_\rho \cdot J_\rho = 0$

$\Rightarrow$ $I_\rho = 0$ or $J_\rho = 0$

<u>Case 1:-</u> $I_\rho = 0$

$$\prod_{\substack{Im\,\rho>0 \\ Re\,\rho<\frac{1}{2}}} \left(1-\frac{\rho_0}{\rho}\right)\left(1-\frac{\rho_0}{1-\rho}\right) = 0$$

Let, ρ_0 be a zero of multiplicity k.

$$\left[\left(1-\frac{\bar{\rho}_0}{\rho_0}\right)\left(1-\frac{\bar{\rho}_0}{1-\rho_0}\right)\right]^k \prod_{\substack{Im\,\rho>0 \\ Re\,\rho<\frac{1}{2} \\ \rho\neq\rho_0}} \left(1-\frac{\bar{\rho}_0}{\rho}\right)\left(1-\frac{\bar{\rho}_0}{1-\rho}\right) = 0$$

$$\left[\frac{2i\, Im\,\rho_0\,(1-2\,Re\,\rho_0)}{\rho_0(1-\rho_0)}\right]^k \prod_{\substack{Im\,\rho>0,\ \rho\neq\rho_0 \\ Re\,\rho<1/2}} \left(1-\frac{\bar{\rho}_0}{\rho}\right)\left(1-\frac{\bar{\rho}_0}{1-\rho}\right) = 0$$

$\because$ $\operatorname{Im} \rho_0 > 0$ & $\operatorname{Re} \rho_0 < \frac{1}{2}$

$\Rightarrow$ $\operatorname{Im} \rho_0 > 0$ & $1 - 2\operatorname{Re} \rho_0 > 0$

$$\therefore \prod_{\substack{\operatorname{Im} \rho > 0 \\ \operatorname{Re} \rho < \frac{1}{2} \\ \rho \neq \rho_0}} \left(1 - \frac{\overline{\rho_0}}{\rho}\right)\left(1 - \frac{\overline{\rho_0}}{1-\rho}\right) = 0$$

Value of a convergent infinite product is 0 iff atleast one of the factors is 0.

$\left(1 - \frac{\overline{\rho_0}}{\rho_1}\right)\left(1 - \frac{\overline{\rho_0}}{1-\rho_1}\right) = 0$ for some $\rho_1 \in \mathbb{C}$ $\operatorname{Im} \rho_1 > 0$ & $\operatorname{Re} \rho_1 < \frac{1}{2}$, $\rho_1 \neq \rho_0$

$\Rightarrow$ $1 - \frac{\overline{\rho_0}}{\rho_1} = 0$ or $1 - \frac{\overline{\rho_0}}{1-\rho_1} = 0$

$\Rightarrow$ $\rho_1 = \overline{\rho_0}$ or $\overline{\rho_0} = 1 - \rho_1$

$\operatorname{Im} \rho_1 > 0$

$\Rightarrow$ $\operatorname{Im} \overline{\rho_0} > 0$

$\Rightarrow$ $-\operatorname{Im} \rho_0 > 0$

$\Rightarrow$ $\operatorname{Im} \rho_0 < 0$

Contradicts $\operatorname{Im} \rho_0 > 0$

or Let, $\rho_0 = \sigma_0 + it_0$

$\rho_1 = \sigma_1 + it_1$

$\sigma_0 - it_0 = 1 - \sigma_1 - it_1$

$\sigma_0 = 1 - \sigma_1$

$\operatorname{Re}(\rho_0) < \frac{1}{2}$

$\Rightarrow$ $\sigma_0 < 1/2$

$$1-\sigma_1 < \frac{1}{2}$$

$$\Rightarrow \quad \sigma_1 > 1/2$$

$$\operatorname{Re} \rho_1 > 1/2$$

Contradicts $\operatorname{Re} \rho < \frac{1}{2}$ in the above product.

<u>Case 2:-</u> $J_\rho = 0$

$$\prod_{\substack{\operatorname{Im}\rho > 0 \\ \operatorname{Re}\rho > 1/2}} \left(1 - \frac{\overline{\rho_0}}{\rho}\right)\left(1 - \frac{\overline{\rho_0}}{1-\rho}\right) = 0$$

~~Let ρ_0 be a zero with multiplicity~~

Let ρ_0 be a zero with multiplicity K

$$\left[\frac{2i \operatorname{Im}\rho_0 (1 - 2\operatorname{Re}\rho_0)}{\rho_0(1-\rho_0)}\right]^K \prod_{\substack{\operatorname{Im}\rho > 0 \\ \operatorname{Re}\rho > 1/2 \\ \rho \neq \rho_0}} \left(1 - \frac{\overline{\rho_0}}{\rho}\right)\left(1 - \frac{\overline{\rho_0}}{1-\rho}\right) = 0$$

$\because \operatorname{Im}\rho_0 > 0$ & $\operatorname{Re}\rho_0 > 1/2 \Rightarrow 1 - 2\operatorname{Re}(\rho_0) < 0$

$$\therefore \prod_{\substack{\operatorname{Im}\rho > 0 \\ \operatorname{Re}\rho > 1/2 \\ \rho \neq \rho_0}} \left(1 - \frac{\overline{\rho_0}}{\rho}\right)\left(1 - \frac{\overline{\rho_0}}{1-\rho}\right) = 0$$

$$\overline{\left(1-\frac{\overline{\rho_0}}{\rho_2}\right)\left(1-\frac{\overline{\rho_0}}{1-\rho_2}\right)} = 0$$ for some $\rho_2 \in \mathbb{C}$ s.t $\operatorname{Im} \rho_2 > 0$ & $\operatorname{Re} \rho_2 > 1/2$, $\rho_2 \neq \rho_0$

$\rho_2 = \overline{\rho_0}$ or $\overline{\rho_0} = 1-\rho_2$

let $\rho_0 = \sigma_0 + it_0$

~~Bef~~ & $\rho_2 = \sigma_2 + it_2$

$\operatorname{Im} \rho_2 > 0$ or $\sigma_0 - it_0 = 1-\sigma_2 - it_2$

$\Rightarrow \operatorname{Im} \overline{\rho_0} > 0$ or $\sigma_0 = 1-\sigma_2$

$-\operatorname{Im} \rho_0 > 0$ or $\operatorname{Re} \rho_0 > 1/2$

$\operatorname{Im} \rho_0 < 0$ or $\Rightarrow \sigma_0 > 1/2$

Contradicts $\operatorname{Im} \rho_0 > 0$

$\Rightarrow 1-\sigma_2 > 1/2$

$\Rightarrow \sigma_2 < 1/2$.

~~So~~ $\operatorname{Re} \rho_2 < 1/2$

Contradicts $\operatorname{Re} \rho > 1/2$.

So, in both the cases we get a Contradiction.

$\therefore$ Our Assumption that $\xi(\rho_0) = 0$ is wrong.

$\therefore \operatorname{Re}(\rho) \neq \frac{1}{2} \Rightarrow \xi(\rho) \neq 0$

$\therefore \xi(\rho) = 0 \Rightarrow \operatorname{Re}(\rho) = \frac{1}{2}$.