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Ellipticity Bias in the Global Positioning System

Abstract

In the TI field model, time dilation of a process (e.g., a clock) has only one cause: velocity of the process relative to the TI field. The TI field is directly subject to gravity and is accelerated radially toward the center of mass of a gravitational body (GB). The infall velocity of the TI field at the orbiting satellite produces the time dilation of the satellite clock attributable to gravity. Gravity produces time dilation only indirectly via the mediation of the TI field.

If the satellite is in a perfectly circular orbit the orbital velocity vector and the infall velocity vector of the TI field are orthogonal and the squares of their velocities relative to the satellite clock may be added algebraically. If the satellite is in an elliptical orbit the orbital velocity vector and the infall velocity vector are not orthogonal and must be added vectorially. The difference between these two calculations comprises the ellipticity bias, a bias that exists only if gravitational time dilation is caused by the infall velocity of the TI field and not (directly) by gravity.

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1.0 First Matters

Referring to the ellipticity effects, Neil Ashby writes:

> '... the systematic deviations between theory and experiment tend to occur for one satellite during a pass; this "pass bias" might be removable if we understood better what the cause of it is. As it stands, the agreement between theory and experiment is within about 2.5%.' [1]

> 'To an extremely good approximation in the GPS, however, gravitational effects can be simply added to other effects arising from special relativity.' [2]

2.0 Properties of the Temporal Inertial Field

The fundamental difference between the TI field model of gravity and inertia and the well known Newtonian model resides in how each model mediates gravity and how each model calculates the time dilation of a process.

- In the Newtonian model, matter objects are directly subject to gravity.
- In the Newtonian model, time dilation of a process has two causes: 1) velocity of the process relative to a reference clock, and 2) the strength of the gravitational field at the location of the process.
- In the TI field model, matter objects are not directly subject to gravity.
- In the TI field model, time dilation of a process has only one cause, the velocity of the process relative to the TI field. Gravity affects the time dilation of a process within the gravitational field only through the acceleration of the TI field by gravity.

The numerous properties that support these two principal behaviors are given in Appendix A. No further mention of the Newtonian model will be made.

The properties and behavior of the TI field described in Table 2 and in Appendix A are supported by previous studies. [8] [10]

Table 1. Properties and Behavior of the TI Field Model

Table 1. Properties and Behavior of the TI Field Model
Particles of the TI field are directly subject to gravity.
Matter particles and objects comprising matter particles are not directly subject to gravity.
The acceleration of (particles of) the TI field in response to gravity defines the acceleration profile about a gravitational body (GB).
Acceleration of the TI field relative to a matter object produces the gravitational force on the object. The TI field thus mediates gravity.
The acceleration of a matter object relative to the TI field (in response to an external force) produces the inertial reaction force on the object as expressed by F = ma.
Particles of the TI field permeate space at every scale from subatomic to intergalactic and beyond.
The one and only cause of time dilation of a process is the velocity of that process relative to the TI field. [9] [10]
Acceleration of particles of the TI Field in response to gravity is resisted by the Static field. (See Appendix A.)

3.0 Ellipticity Bias in the Global Positioning System

A bias is introduced in the calculation of time dilation in the Global Positioning System by adding algebraically the contribution of the orbital velocity of a satellite with the contribution of gravity. The orbits of GPS satellites are slightly eccentric. Accordingly, the orbital velocity vector of a satellite is not orthogonal with the gravity vector (except at perigee and apogee). The two vectors must be added vectorially, not algebraically.

Why is this? Shouldn't make any difference? Only if you subscribe to the Newtonian model of gravity. In the TI field model, as described briefly in Table 2 and Appendix A, Section A.6, the only cause of time dilation of the satellite clock is the velocity of the clock relative to the TI field. There are two components to the velocity of the satellite clock relative to the TI field: 1) the orbital velocity of the satellite and 2) the infall velocity of the TI field at the satellite. (The infall velocity of the TI field is the negative of the escape velocity at any point.) In an elliptical orbit, these two velocities are not orthogonal, hence they must be added vectorially for a correct evaluation of their contribution to the time dilation of the satellite clock.

3.1 The Newtonian Model

In the Newtonian Model of gravity time dilation has two causes; velocity of a process relative to the TI field and gravity. Time dilation caused by the orbital velocity vector and time dilation caused by the gravitational vector in the Newtonian model are independent and may be added algebraically. This is not the case for the TI field model of gravity. An elliptical orbit introduces a difference in the calculation of the time dilation for the two different models. It is this difference in the time dilation for these two models that I term ellipticity bias.

3.2 The TI Field Model

In the TI field model, described briefly in Section 2.0 and Appendix A., time dilation of a process (e.g., a clock) has only one cause, velocity of the process relative to the TI field. The TI field is directly subject to gravity and is accelerated radially toward the center of mass of a gravitational body (GB). The infall velocity of the TI field at the orbiting satellite produces the time dilation of the satellite clock attributable to gravity. Gravity produces time dilation only indirectly via the mediation of the TI field. See Appendix A, Section A.6.

If the satellite is in a perfectly circular orbit the orbital velocity vector and the infall velocity vector of the TI field are orthogonal and the squares of their velocities relative to the satellite clock may be added algebraically. If the satellite is in an elliptical orbit the orbital velocity vector and the infall velocity vector are not orthogonal and must be added vectorially. The difference between these two calculations comprises the ellipticity bias. These calculations are shown in Appendix D.

Figures 1 through 3 show the ellipticity bias calculated for a GPS satellite for three different values of eccentricity. Each plot shows the difference in time dilation between using the vector addition of orbital velocity and infall velocity minus the algebraic addition of velocities for one orbit of a GPS satellite.

Does the ellipticity bias contribute to the ‘pass bias’ spoken of in reference [1]?

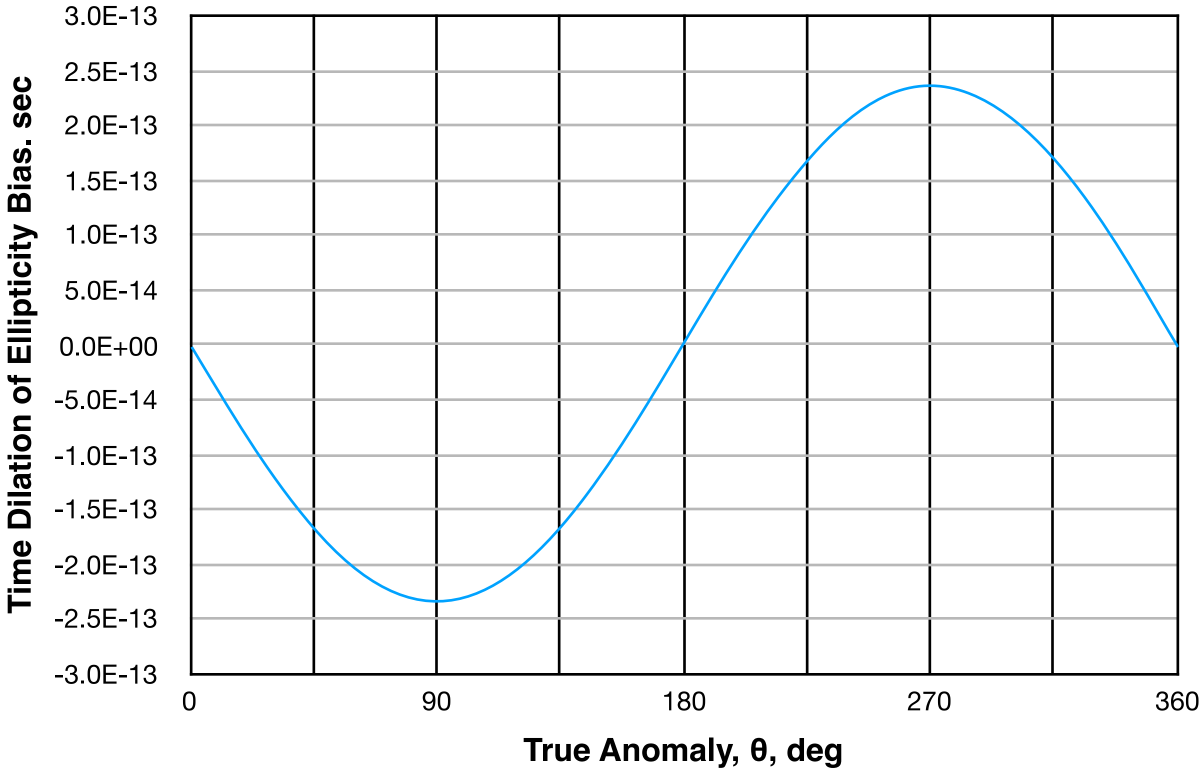

Figure 1. Time Dilation Difference, Vector Minus Algebraic Calculation with Ellipticity of 0.001

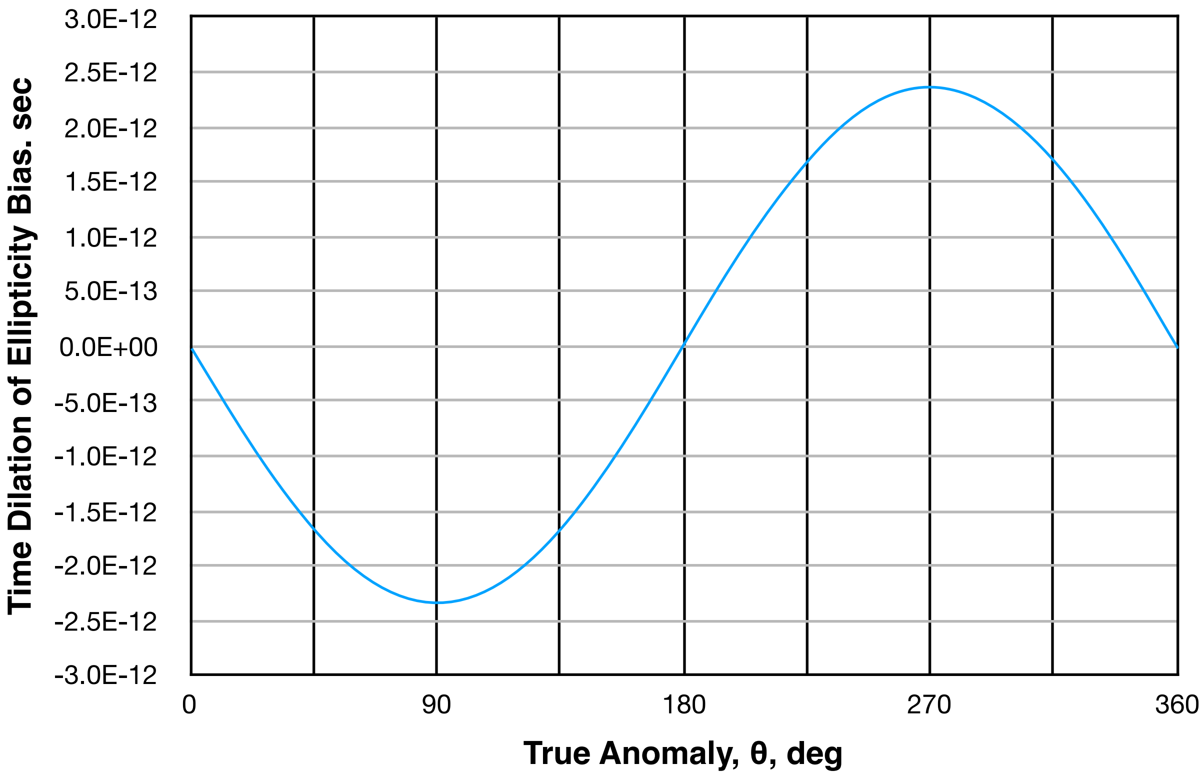

Figure 2. Time Dilation Difference, Vector Minus Algebraic Calculation with Ellipticity of 0.01

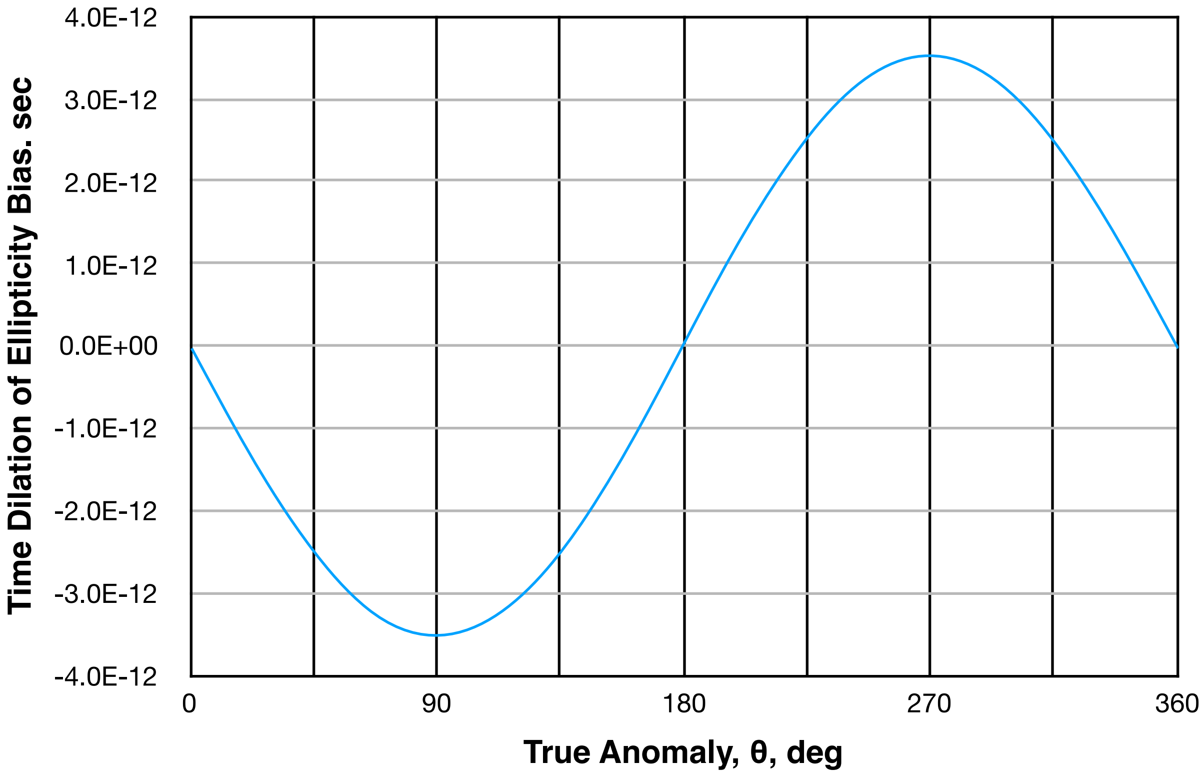

Figure 3. Time Dilation Difference, Vector Minus Algebraic Calculation with Ellipticity of 0.015

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Appendix A

Properties of the Temporal Inertial Field in Brief

A.1 Definitions of Mass

A brief description follows of the forms of mass existent in the two models of gravity described in this paper. I paraphrase the three definitions of mass offered by Wikipedia [10] for the Newtonian model and modify those definitions where appropriate for the TI field model.

Table A.1	Definitions of Mass
Mass in the Newtonian Model	**Definition**
Active gravitational mass of a matter object	A measure of the gravitational force exerted by a matter object.
Passive gravitational mass of a matter object	A measure of the gravitational force experienced by a matter object in a known gravitational field.
Inertial mass of a matter object	A measure of a matter object's resistance to being accelerated by a gravitational or non-gravitational force.
Mass in the TI Field Model	**Definition**
Active gravitational mass of a particle of the TI field	Particles of the TI field do not possess active gravitational mass.
Active gravitational mass of a matter object	A measure of the gravitational force exerted by a matter object.
Passive gravitational mass of a particle of the TI field	A measure of the gravitational force experienced by a particle of the TI field in a known gravitational field.
Passive gravitational mass of a matter object	Matter objects do not possess passive gravitational mass.

Table A.1	Definitions of Mass
Inertial mass of a particle of the TI field	A measure of the resistance of a particle of the TI field to being accelerated by the force of gravity.
Inertial mass of a matter object	A measure of a matter object's resistance to being accelerated by a non-gravitational force.

A.2 Mass Properties of the Newtonian and TI Field Models of Gravity

The mass properties of matter objects and particles of the TI field depend on the model of gravity and inertia.

Table A.2 Mass Properties of the Newtonian and TI Field Models of Gravity

Gravitational Model	Active Gravitational Mass	Passive Gravitational Mass	Inertial Mass
Matter Objects in the Newtonian Model	Yes, matter objects assert the gravitational force.	Yes, matter objects are directly subject to gravity.	Yes, matter objects resist acceleration relative to the TI field.
Matter Objects in the TI Field Model	Yes, matter objects assert the gravitational force.	No, matter objects are not directly subject to gravity.	Yes, matter objects resist acceleration relative to the TI field.
Particles of the TI Field Model	No, particles of the TI field do not assert the gravitational force.	Yes, particles of the TI field are directly subject to gravity.	Yes, particles of the TI field resist acceleration relative to the static field.

A.3 Properties and Behavior of Objects in the TI Field Model of Gravity

The properties and behavior of objects in the TI model of gravity and inertia depend on the properties of mass of objects in this model.

Table A.3 Properties and Behavior of Objects in the TI Field Model of Gravity
Objects Possess Active Gravitational Mass
Objects exert the gravitational force by the emission of gravitons.
The rate of emission of gravitons by an object is proportional to the active gravitational mass of the object.
Objects Do Not Possess Passive Gravitational Mass
Objects are not directly subject to gravity.
Objects respond to the gravitational force indirectly through the intermediation of the TI field. See Table A.4 below.
Objects Possess Inertial Mass
The inertial mass of an object is a measure of the coupling between the object and the TI field.
An object resists the application of a non-gravitational force. An object resists acceleration relative to the TI field in the object's response to a non-gravitational force.
The resistance of an object to the acceleration caused by the application of a non-gravitational force is proportional to both the inertial mass of the object and to the acceleration of the object relative to the TI field. ($F = ma$).

A.4 Properties and Behavior of the TI Field in the TI Field Model of Gravity

The properties and behavior of the TI field itself in the TI model of gravity and inertia depend on the properties of mass of particles of the TI field in this model.

Table A.4 Properties and Behavior of the TI Field in the TI Field Model of Gravity
Particles of the TI Field Do Not Possess Active Gravitational Mass
Particles of the TI field do not exert the gravitational force.
Particles of the TI Field Possess Passive Gravitational Mass
Particles of the TI field experience the gravitational force through their interaction with gravitons.
The gravitational force experienced by a particle of the TI field is proportional to the passive gravitational mass of the particle.
The acceleration of a particle of the TI field is proportional to the graviton flux at the particle.
Particles of the TI Field Possess Inertial Mass
Particles of the TI field resist the application of the gravitational force.
The resistance of a particle of the TI field to the application of a gravitational force is proportional to the inertial mass of the particle.
Interaction of Objects with the TI Field
The inertial mass of an object is a measure of its coupling with the TI field.
The acceleration of the TI field in its response to gravity applies a force to any object within the TI field. This force causes the object to accelerate at the same rate as particles of the TI field at the location of the object.

A.5 Properties and Behavior of the Static Field in the TI Field Model of Gravity

The static field is a conjecture of this author that is required to resist the acceleration of particles of the TI field in their response to gravity. Absent such resistance, the acceleration of particles of the TI field would be unlimited.

Table A.5 Properties and Behavior of the Static Field in the TI Field Model of Gravity
Particles of the Static Field Do Not Possess Active Gravitational Mass
Particles of the static field do not exert the gravitational force.
Particles of the Static Field Do Not Possess Passive Gravitational Mass
Particles of the static field do not experience the gravitational force.
Whether Or Not Particles of the Static Field Possess Inertial Mass Is Undefined
The static field resists the acceleration of particles of the TI field in the response of the TI field to gravity.

A.6 Time Dilation Comparison - Newtonian Model vs TI Field Model

I speak of the effect of time dilation on a process. A process is an action that takes time. A process can be atomic, chemical, biological or mechanical. In the TI field model, a process that moves relative to the TI field takes longer than a process that moves more slowly relative to the TI field. Time does not slow down, processes take longer when moving relative to the TI field. [7]

In the Newtonian model of gravity, time dilation of a process has two causes: 1) velocity of the process relative to a reference clock, and 2) the gravitational field at the process. In the TI field model, the only cause of time dilation of a process is the velocity of that process relative to the TI field. [7] Table A.6 lists the equations pertinent to time dilation caused by velocity. Table A.7 lists the equations pertinent to time dilation caused directly of indirectly by gravity.

Table A.6	**Time Dilation Caused by Velocity**
Newtonian Model	**TI Field Model**
Time dilation is caused by velocity of a process (clock) relative to a reference clock.	Time dilation is caused by the velocity of a process (clock) relative to the TI field.
$t_v / t_0 = 1 / (1 - v^2 / c^2)^{1/2}$	$t_v / t_0 = 1 / (1 - v^2 / c^2)^{1/2}$
v is the velocity of moving clock relative to the reference clock. t_v is the period of the moving clock. t_0 is the period of the reference clock. c is the velocity of light in vacuo.	v is the velocity of moving clock relative to the TI field. t_v is the period of the moving clock. t_0 is the period of the reference clock at rest relative to the TI field. c is the velocity of light in vacuo.

Table A.7	**Time Dilation Caused Directly or Indirectly by Gravity**
Newtonian Model	**TI Field Model**
Gravitational time dilation of a process (clock) is caused directly by the gravitational field at the process.	Gravitational time dilation of a process (clock) is caused by the velocity of the process relative to the TI field.
$t_v / t_0 = (1 - 2GM / (r c^2))^{1/2}$	$t_v / t_0 = 1 / (1 - v^2 / c^2)^{1/2}$ $v_{in}^2 = (2GM / r)$ $t_v / t_0 = 1 / (1 - 2GM / (r c^2))^{1/2}$
t_v is the period of the clock located a distance of r from the center of mass of a gravitational body. t_0 is the period of the reference clock located at infinity. GM is the standard gravitational parameter. r is the distance of the measuring clock from the center of mass of a gravitational body. c is the velocity of light in vacuo.	t_v is the period of the clock located a distance of r from the center of mass of a gravitational body. t_0 is the period of the reference clock located at infinity. v_{in} is the infall velocity of the TI field at the location of the measuring clock. GM is the standard gravitational parameter. r is the distance of the measuring clock from the center of mass of a gravitational body. c is the velocity of light in vacuo.

Appendix B

Gravitational Time Dilation

B.1 Introduction

In the TI field model of gravity and inertia, time dilation of a process has only one cause; the velocity of the process relative to the TI field. Gravitational time dilation is not caused directly by gravity, but indirectly by the acceleration of the TI field toward the center of mass of the gravitational body (GB). Particles of the TI field are subject to gravity and flow toward the center of mass of the GB. Matter objects are not directly subject to gravity, but are accelerated at the same rate as the TI field. The infall velocity of the TI field at a given distance from the GB is the negative of the escape velocity at that distance. Accordingly, the time dilation of, say an orbiting satellite clock, is a function of the vectorial sum of the infall velocity of the TI field at the clock and the orbital velocity of the clock. If these velocities are added algebraically, an error is introduced in the result, an error I term the ellipticity bias.

B.2 Escape Velocity from a Gravitational Body

The infall velocity of the TI field is the negative of the escape velocity from a gravitational body (GB) as given by:

$$v_{in} = - v_e = - (2 GM / r)^{1/2} \tag{B-1}$$

where

v_{in} is the infall velocity of the TI field at the radius r from the center of mass of the GB. The infall velocity is directed toward the center of mass of the GB.

v_e is the escape velocity at the radius r from the center of mass of the GB.

GM is the standard gravitational parameter.

M is the active gravitational mass of the gravitational body.

r is the radius of from the center of mass of the GB at which measurements are taken.

B.3 Gravitational Time Dilation

Clocks run more slowly, their period is longer, in a strong gravitational field than in a lesser field. The basic equations for this behavior are shown below. According to the TI field model of gravity that I lay out in References [9] and [10], gravitational time dilation is caused by the infall velocity of the TI field in its response to gravity, not by gravity directly.

The period of a clock within a gravitational field compared with that of an identical clock at infinity is given by Eq (B-2). Equation (B-2) shows the period of a clock in a gravitational field as a function of the infall velocity at the clock. The clock is stationary relative to the gravitational body (GB).

$$t_v / t_0 = 1 / (1 - v_{in}^2 / c^2)^{1/2} \qquad \text{(B-2)}$$

where

t_v is the period of a clock locate a distance r from the center of mass of a GB.

t_0 is the period of an identical clock located, hypothetically, at infinity.

v_{in} is the infall velocity passed the clock.

c is the velocity of light in vacuo.

Equation (B-1) showed that the square of the infall velocity of the TI field is

$$v_{in}^2 = (2 GM / r) \qquad \text{(B-3)}$$

Substituting this value for v_{in}^2 into Eq (B-2) yields Eq (B-4) which shows the period of the clock as a function of the strength of the gravitational field. Again, the equation compares the period of the clock with an identical clock located at infinity.

$$t_v / t_0 = 1 / (1 - 2GM / (r c^2)^{1/2} \qquad \text{(B-4)}$$

where

t_v is the period of a clock located a distance r from the center of mass of a gravitational body (GB) of mass M.

t_0 is the period of an identical reference clock located, hypothetically, at infinity.

GM is the standard gravitational parameter.

r is the distance of the measuring clock from the center of mass of the GB.

c is the velocity of light in vacuo.

Equation (B-4) is the usual form for gravitational time dilation written as a function of gravity rather than as a function of the infall velocity of the TI field. (Actually the usual form shows the inverse of Eq (B-4) because that form of the equation is intended to show the time measured by the clock, not the period of the clock. [6] I find that form confusing, so I write the equation to show the period of the clock.)

In a gravitational field a clock runs more slowly, its period is greater, than its counterpart at infinity. Equation (B-4) shows the period t_v of the measuring clock to be greater than the period t_0 of the reference clock.

B.4 Simplifying the Equation for Gravitational Time Dilation

If the term $2GM / (r c^2) << 1$, as it is in the GPS, Eq (B-4) simplifies to Eq (B-5).

$$t_v / t_0 = 1 / [1 - GM / (r c^2)] \qquad \text{(B-5)}$$

Now multiply both numerator and denominator of Eq (B-5) by $[1 + GM / (r c^2)]$ and simplify further to yield Eq (B-6).

$$t_v / t_0 = 1 + GM / (r c^2) \qquad \text{(B-6)}$$

B.5 Summary

Table B.1	Gravitational Time Dilation Equations
The infall velocity of the TI field is the negative of the escape velocity of the TI field.	$v_{in} = - v_e = - (2 GM / r)^{1/2}$
Gravitational time dilation as a function of the infall velocity of the TI field.	$t_v / t_0 = 1 / (1 - v_{in}^2 / c^2)^{1/2}$
Gravitational time dilation as a function of gravity.	$t_v / t_0 = 1 / (1 - 2GM / (r c^2))^{1/2}$
Simplified equation for gravitational time dilation as a function of gravity.	$t_v / t_0 = 1 + GM / (r c^2)$
Parameters	t_v is the period of the clock located a distance of r from the center of mass of a gravitational body. t_0 is the period of the reference clock located, hypothetically, at infinity. v_{in} is the infall velocity of the TI field at the location of the measuring clock. v_e is the escape velocity at a distance of r from the center of mass of the gravitational body. GM is the standard gravitational parameter. r is the distance of the measuring clock from the center of mass of the gravitational body. c is the velocity of light in vacuo.

Appendix C

Elliptical Orbit Calculations

C.1 Equation for an Ellipse

Plot the elliptical path of a satellite on a Cartesian coordinate system with the origin at the left-hand focus of the ellipse. The X-axis is parallel with the major axis of the ellipse. Both axes pass through the focus of the ellipse as shown in Figure B.1. The velocity vector of the satellite is tangent to the ellipse at a point in the first quadrant of the figure at a distance r from the focus of the ellipse. This tangent forms an angle β with the positive X-axis. The angle β is the direction of the velocity vector of the satellite relative to the positive X-axis and is measured counterclockwise from a line parallel with the X-axis to the velocity vector. The radial r from the focus to the satellite forms an angle θ with the X-axis. The angle θ is measured counterclockwise from the X-axis to the radial r.

The velocity vector of the satellite forms an angle α with the radial r from the focus. The angle alpha is measured counterclockwise from the velocity vector of the satellite to the radial r from the focus. The infall velocity of the TI field at the satellite is directed from the satellite along the radial r to the center of mass of the GB located at one focus of the ellipse.

The object of this appendix is to calculate both the direction β of the velocity vector of the satellite and the angle α between the velocity vector and the radial r from the focus. These two angles are needed to calculate the ellipticity bias in the Global Positioning System.

The ellipse's equation is given by Eq (C-1). [4]

$$r = a\,(1 - e^2) / (1 - e\cos\theta) \qquad \text{(C-1)}$$

where

r is the polar coordinate measured from the origin to a point on the ellipse.

θ is the angular coordinate.

θ is measured CCW from the positive X-axis to the intersection of the polar coordinate r with its intersection with the ellipse.

a is the semi-major axis of the ellipse.

e is the eccentricity of the ellipse.

The sign in the denominator is negative if the reference direction $\theta = 0$ points toward the center (as illustrated in the Figure B.1) and positive if that direction points away from the center.

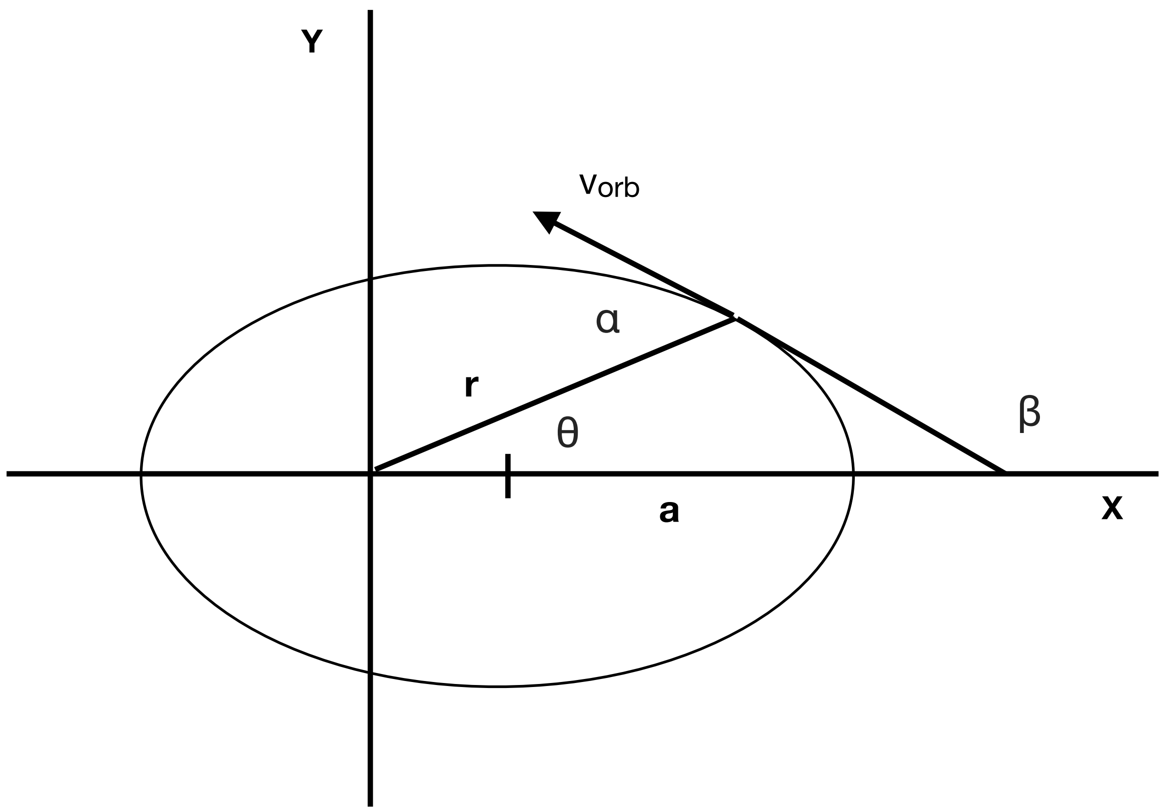

Figure B.1 Elliptical Orbit Parameters
Eccentricity Exaggerated

C.2 Calculate the Direction β of the Velocity Vector of the Satellite

Assuming a top-down view of the elliptical path of the satellite, the slope or direction of the velocity vector of the satellite is the angle between the X-axis and the velocity vector of the satellite as shown in Figure B.1.

Let $K = a (1 - e^2)$, so

$$r = K / (1 - e \cos \theta) \tag{C-2}$$

Assume a rectangular coordinate system with one focus of the ellipse at the origin. The xy coordinates of the satellite are then:

$$x = r \cos \theta = K \cos \theta / (1 - e \cos \theta) \tag{C-3}$$

$$y = r \sin \theta = K \sin \theta / (1 - e \cos \theta)$$

or

$$y = a (1 - e^2) \sin \theta / (1 - e \cos \theta) \tag{C-4}$$

The slope of the velocity vector at any point is given by $(dy / dr) / (dx / dr)$.

$$dy / dr = [K \sin \theta / (1 - e \cos \theta)]' \tag{C-5}$$

The derivative of a quotient $(f / g)'$ is $(f' g - f g') / g^2$ (C-6)

For brevity, let $f = K \sin \theta$ and $g = (1 - e \cos \theta)$.

so

$$dy / dr = [K \cos \theta (1 - e \cos \theta) - K \sin \theta (e \sin \theta)] / g^2 \tag{C-7}$$

$$dy / dr = [K \cos \theta - K e (\cos^2 \theta + \sin^2 \theta)] / g^2 \tag{C-8}$$

Lastly,

$$dy / dr = K (\cos \theta - e) / (1 - e \cos \theta)^2 \tag{C-9}$$

Now,

$$dx / dr = [K \cos \theta / (1 - e \cos \theta)] ' \tag{C-10}$$

$$dx / dr = [- K \sin \theta (1 - e \cos \theta) - K \cos \theta (e \sin \theta)] / g^2 \tag{C-11}$$

$$dx / dr = - K \sin \theta / g^2 \tag{C-12}$$

or

$$dx / dr = - K \sin \theta / (1 - e \cos \theta)^2 \tag{C-13}$$

and finally

$$dy / dx = (e - \cos \theta) / \sin \theta \tag{C-14}$$

The calculation of the direction angle β depends on the quadrant of the orbit. In the first and second quadrants, the direction angle is given in Eq (C-15). In the third and fourth quadrants, the directions in given in Eq (C-16). The calculations are summarized in Table C.1.

$$\beta = \pi + \arctan (e - \cos \theta) / \sin \theta \tag{C-15}$$

$$\beta = \arctan (e - \cos \theta) / \sin \theta \tag{C-16}$$

C.3 Calculate the Direction β of the Velocity Vector of the Satellite for Each Quadrant of the Flight Path

Assuming a top-down view of the elliptical path of the satellite, the direction of the satellite is the angle between the X-axis and the velocity vector of the satellite. The calculation of this angle depends on the quadrant 'flown' by the satellite. Table C.1 lists the calculations for the direction of the velocity vector for the four quadrants.

Table C.1		**Calculation of the Direction β of the Velocity Vector of the Satellite**
Quadrant	**Formula**	**Numbers Equation**
All	dy / dx	(e - cos θ) / sin θ
1	π + arctan (dy / dx)	IF (y > 0, π + ATAN (dy / dx), ATAN (dy / dx))
2	π + arctan (dy / dx)	IF (y > 0, π + ATAN (dy / dx), ATAN (dy / dx))
3	arctan (dy / dx)	IF (y > 0, π + ATAN (dy / dx), ATAN (dy / dx))
4	arctan (dy / dx)	IF (y > 0, π + ATAN (dy / dx), ATAN (dy / dx))

C.4 Calculate the Angle Alpha Between the Orbital Velocity Vector and the Infall Velocity Vector of the TI Field

The angle alpha between the orbital velocity vector or the satellite and the infall velocity vector can be determined from inspection of Figure B.1.

$$\alpha = \pi + (\theta - \beta) \qquad \text{(C-17)}$$

Appendix D

Separating the Orbital and Gravitational Time Dilation for an Elliptical Orbit

D.1 Orbital Velocity of a Body in an Elliptical Orbit [5]

Under standard assumptions the orbital speed v_{orb} of a body traveling along an elliptic orbit can be computed from the Vis-viva equation as:

$$v_{orb} = [\ GM\ (\ 2\ /\ r - 1\ /\ a\ )\ ]^{1/2} \qquad \text{(D-1)}$$

where

v_{orb} is the orbital velocity of the satellite

GM is the standard gravitational parameter.

M is the active gravitational mass of the major gravitational body.

r is the distance between the major gravitational body and the orbiting body.

a is the length of the semi-major axis of the elliptical path of the orbiting body.

D.2 Calculate the Time Dilation in a Satellite in an Elliptical Orbit Caused by the Combination of Orbital Velocity and Infall Velocity of the TI Field

I use the polar form of an ellipse where the origin of the coordinate system is at one of the foci of the ellipse as shown in Figure D.1. The equations are derived using this figure for the first quadrant of the ellipse, although the derivations for the other three quadrants yield the same equations.

Both the infall velocity vector of the TI field and one focus of the elliptical orbit of the GPS satellite are located at the center of mass of the Earth.

In an elliptical orbit the orbital velocity vector and the infall velocity vector are not orthogonal. Accordingly, these vectors must be added vectorially as detailed below.

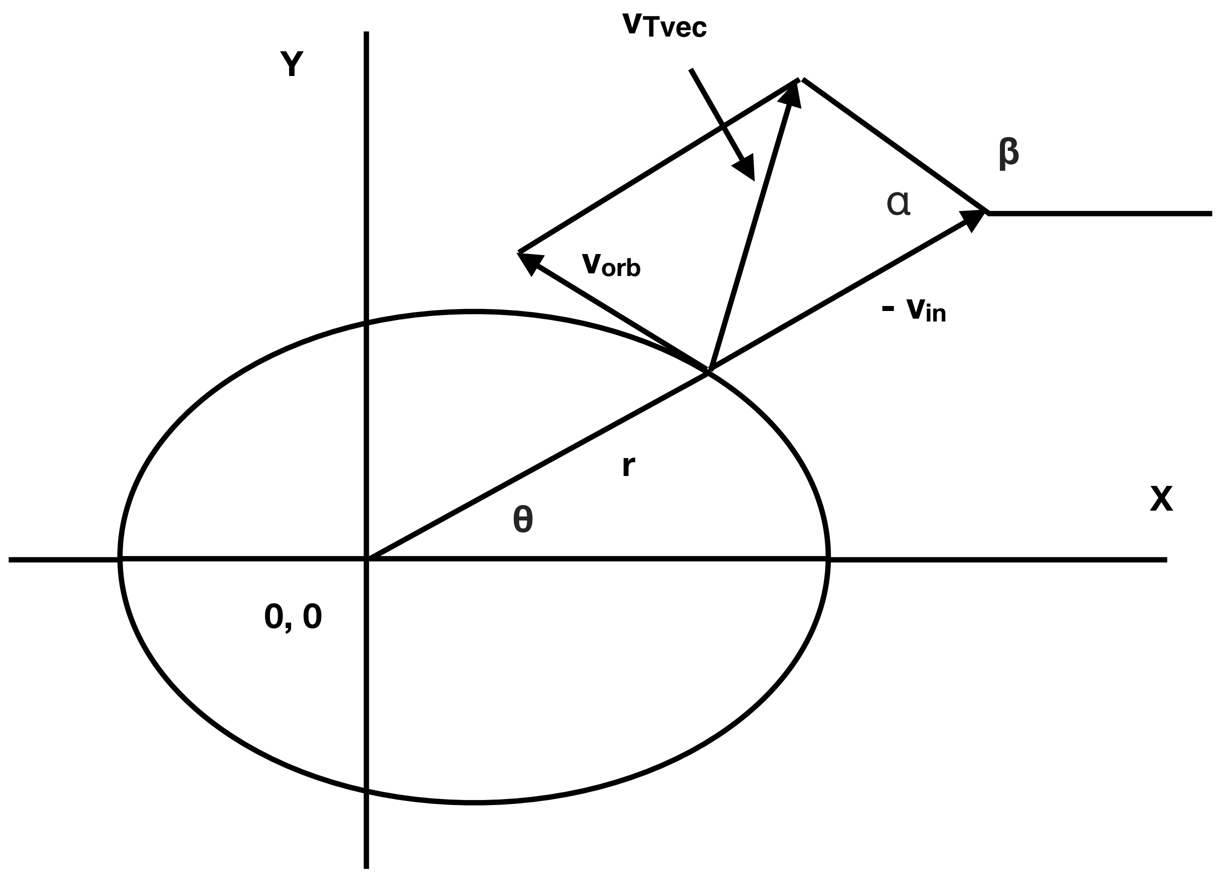

Figure D.1 Elliptical Orbit of GPS Satellite - Ellipticity Exaggerated

Figure D.1 shows an ellipse representing the orbit of a GPS satellite with the eccentricity exaggerated to show more clearly the relationship of the orbital velocity vector and the infall velocity vector of the TI field.

In Figure D.1 we see a parallelogram representing the orbital velocity vector v_{orb}, the infall velocity v_{in} of the TI field and the total velocity of the TI field v_T relative to the orbiting space vehicle. The total velocity v_T is the short diagonal of the parallelogram of velocity vectors. Its value is given by

$$v_{Tvec} = (v_{in}^2 + v_{orb}^2 - 2\, v_{in}\, v_{orb} \cos \alpha)^{1/2} \qquad \text{(D-2)}$$

where

v_{in} is the infall velocity of the TI field toward the center of the Earth. See Eq (B-1).

v_{orb} is the orbital velocity of the GPS satellite.

v_{Tvec} is the total velocity of the GPS satellite relative to the TI field using vector addition of the velocities v_{in} and v_{orb}.

r is the radius from the center of mass of the Earth to the orbiting satellite.

α is the angle between the infall velocity and orbital velocity vectors. For the nearly circular orbits of the GPS satellites, this angle is close to 90 deg.

π is the angle 180 deg in radians.

β is the slope of the orbital velocity vector relative to the positive X-axis.

θ is the true anomaly of the point and is measured from apogee along the orbit to the satellite's instantaneous position [4].

The effect on time dilation is given by

$$t_{vec} / t_0 = 1 / (1 - v_{Tvec}^2 / c^2)^{1/2} \qquad \text{(D-3)}$$

where

t_{vec} is the period of the satellite clock.

t_0 is the period of an identical clock located, hypothetically, at infinity.

v_{Tvec} is the total velocity of the GPS satellite relative to the TI field using vector addition of velocities.

c is the velocity of light in vacuo.

The value of α can be determined by inspection of Figure D.1.

$$\alpha = \pi + (\theta - \beta) \tag{D-4}$$

If $v_{Tvec}^2 / c^2 << 1$, which it is in the GPS, Eq (D-3) can be simplified to yield Eq (D-5).

$$t_{vec} / t_0 = 1 / [1 - v_{Tvec}^2 / (2 c^2)] \tag{D-5}$$

Multiply both numerator and denominator of Eq (D-5) by $[(1 + v_{Tvec}^2 / (2 c^2)]$ and simplify using the assumption that $v_{Tvec}^4 / (4 c^4) << 1$. This results in:

$$t_{vec} / t_0 = 1 + v_{Tvec}^2 / (2 c^2) \tag{D-6}$$

Substitute the value of v_{Tvec}^2 from Eq (D-2) into Eq (D-6).

$$\Delta t_{vec} / t_0 = 1 + (v_{in}^2 + v_{orb}^2 - 2 v_{in} v_{orb} \cos \alpha) / (2 c^2) \tag{D-7}$$

D.3 Calculate the Difference Between the Time Dilation Using Vector Addition of Orbital and Infall Velocities and Algebraic Addition of Velocities

Using algebraic addition of the infall and orbital velocities:

$$v_{Talg} = (v_{orb}^2 + v_{in}^2)^{1/2} \tag{D-8}$$

Using the same simplifications used for Eq (D-5) the time dilation calculated using algebraic addition of velocities is then:

$$\Delta t_{alg} / t_0 = 1 + (v_{in}^2 + v_{orb}^2) / (2 c^2) \qquad \text{(D-9)}$$

The difference in time dilation between the calculation using vector addition of velocities and the algebraic addition of velocities is then:

$$\Delta t_{vec} / t_0 - \Delta t_{alg} / t_0 = (- v_{in} v_{orb} \cos \alpha) / c^2 \qquad \text{(D-10)}$$

or

$$\text{Eccentricity Bias} = (- v_{in} v_{orb} \cos \alpha) / c^2 \qquad \text{(D-11)}$$

Equation (D-10) and Eq (D-11) are each expressed as a function of α, the angle between the orbital velocity vector of the satellite and the infall velocity vector of the TI field at the satellite. The angle α is measured CCW from the satellite's orbital velocity vector to the infall velocity vector of the TI field.

A summary of the equations of Appendix D is shown in Table E.1 of Appendix E.

Appendix E

Summary of the Calculations of the Ellipticity Bias

Table E.1 Summary of Calculations of the Ellipticity Bias

Table E.1		Summary of Calculations of the Ellipticity Bias
Term or Description	**Symbol or Equation #**	**Equation or Value**
Earth-centered inertial coordinate system [3]	ECI	A right-handed Cartesian coordinate system with the origin at the center of mass of the Earth. In this analysis, the XY-plane is the plane of the satellite's orbit. The X-axis is coincident with the major axis of the orbit of the satellite. The Y-axis extends from the origin through the geographic North Pole. The coordinate system does not rotate with the Earth.
Standard gravitational parameter for the Earth [12]	GM	3.9860044 E5 km^3 / s^2
The Semi-major axis of an orbiting GPS satellite	a	26,561.75 km
The velocity of light in vacuo [11]	c	2.99792458 E5 km / s
Ellipticity of the satellite orbit	e	approximately 0.001 for GPS satellites

Table E.1		Summary of Calculations of the Ellipticity Bias
Term or Description	**Symbol or Equation #**	**Equation or Value**
θ is the true anomaly of the point and is measured from apogee along the orbit to the satellite's instantaneous position [4].	θ	θ is the independent variable in our calculation of the time dilation in orbit.
y position of satellite	Eq (C-4)	$a (1 - e^2) \sin \theta / (1 - e \cos \theta)$
dy / dx	Eq (C-14)	$(e - \cos \theta) / \sin \theta$
β is the slope of a tangent to the orbit at the location of the satellite. β is measured CCW from a line originating at the satellite and extending parallel to the x-axis to the direction of the velocity of the orbiting satellite.	Table C.1	IF (y > 0, π + ATAN (dy / dx), ATAN (dy / dx))
α is the angle between the infall velocity vector at the satellite and the orbital velocity vector of the satellite. α is measured CCW from the satellite's orbital velocity vector to the infall velocity vector of the TI field.	Eq (C-17)	$\alpha = \pi + (\theta - \beta)$
The radius from the focus of the elliptical orbit to the position of the satellite	Eq (C-1)	$r = a (1 - e^2) / (1 - e \cos \theta)$
Orbital velocity of the satellite	Eq (D-1)	$v_{orb} = [GM (2 / r - 1 / a)]^{1/2}$
Infall velocity of the TI field at the location of the satellite	Eq (B-1)	$v_{in} = (2 GM / r)^{1/2}$

Table E.1		Summary of Calculations of the Ellipticity Bias
Term or Description	**Symbol or Equation #**	**Equation or Value**
The vector sum of the orbital velocity of the satellite and the infall velocity of the TI field at the location of the satellite	Eq (D-2)	$v_{Tvec} =$ $(v_{in}^2 + v_{orb}^2 - 2\, v_{in}\, v_{orb} \cos \alpha)^{1/2}$
Vector calculation of time dilation	Eq (D-7)	$\Delta t_{vec} / t_0 = 1 + (v_{in}^2 + v_{orb}^2$ $- 2\, v_{in}\, v_{orb} \cos \alpha) / (2 c^2)$
Algebraic calculation of time dilation	Eq (D-9)	$\Delta t_{alg} / t_0 =$ $1 + (v_{in}^2 + v_{orb}^2) / (2 c^2)$
The eccentricity bias is the difference in time dilation between the calculation using algebraic addition of velocities and vector addition of velocities.	Eq (D-10) Eq (D-11)	$\Delta t_{vec} / t_0 - \Delta t_{alg} / t_0 =$ $(- v_{in}\, v_{orb} \cos \alpha) / c^2$ or Ellipticity Bias = $(- v_{in}\, v_{orb} \cos \alpha) / c^2$