Particle Creation and Annihilation through Geometry Fluctuations of a curved Spacetime

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Abstract

We replace the proportionality factor of the Einstein equations by a smooth function, that acts as a potential of a system, of which the scale factor of the metric is a topological solution to. By doing so, we construct gravitational instantons corresponding to tunneling processes of quantum fluctuations of two three-dimensional Euclidean manifolds through a potential well. The classical solution describes a contracting and expanding spacetime with inflationary periods in the vicinity of the bounce. Furthermore, we use the spectral decomposition of the second order fluctuations to construct quantum tensor fields, satisfying a certain CCR when interpreted appropriately. This is possible - not least because we are able to reduce the quantization of the metric tensor to a quantization of the one-dimensional scaling factor. In addition, it was shown how the constructed gravitons create massive particles by mixing positive and negative energy states of a scalar field due to scattering at a Pöschl-Teller potential, which is detected by a late time observer as particle creation.
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1 The Model

We’d like to describe the evolution of spacetime as a trajectory of a three-dimensional Riemannian Manifold \((\Sigma, h)\) with positive, constant scalar curvature equipped with metric tensor \(h_{ij}(x^k)\) in a special kind of double-well potential. We assume a four-dimensional, globally hyperbolic manifold \((M, g)\) is foliated by the family \(\Sigma_t\) under synchronous gauge.

\[N_i = 0\]

\[g_{tt} = -N^2(t)\]

\[g_{\mu\nu} = \begin{pmatrix} g_{tt} & \bar{N}_i \\ N_i & h_{ij} \end{pmatrix},\]

(1)

\[h_{ij}(x^k, t) = \phi(t) \delta_{ij}\]

(2)

where \(N_i\) is called the Shift and \(\phi \in C^{\infty}. \Sigma_t\) is the image of the embedding

\[\mathcal{E}_t : \Sigma \to M\]

so that \((\Sigma, h_t)\) is isomorphic to \(\Sigma_t\). One should be reminded of the absence of any implication, relating the parameter \(t\) to clock readings so far. In this case \(t\) should be understood as a flow parameter. The dynamics of the system are governed by the action of \(\phi(t)\) in Planck-Units times \(\delta_{ij}\).

\[S[\phi] \delta_{ij} = S_{ij}[h_{ij}] = \int dt \frac{1}{2} (\partial_t h_{ij})^2 + R_{ij}(h_{ij})\]

(3)

where \(R_{ij}\) is the Ricci-Tensor, \(\partial_t h_{ij} = -2N K_{ij}\) the second fundamental form of \(\Sigma_t\). A nontrivial solution, for which (3) vanishes is a nontrivial solution of

\[\frac{\partial h_{ij}}{\partial t} = \sqrt{2R_{ij}}\]

(4)

that satisfies the boundary condition

\[h_{ij}(\pm \infty) = \pm v_{ij},\]
where $R_{ij}(\pm u_{ij}) = 0$. In a setting of field theory, equation (4) is called Bogomolny-Equation. Since we want $h_{ij}$ to be a kink, we choose a nontrivial vacuum structure

$$R_{ij} = 2V(\phi)\delta_{ij} = \frac{R_0}{2}(1 - \phi^2)^2\delta_{ij}$$

(5)

with $R_0 = 1/\ell_p^2$ being a real constant, which can be interpreted as the maximum scalar curvature of $h_{ij}$ or equally as the height of the potential-well and with $\delta_{ij}$, the Euclidean metric. Equation (5) is basically a family of Einstein equations, whose proportionality factor is the smooth function $V(\phi)$, so that $\phi$ defines an Einstein three-manifold for every $t$. One may observe that expression (4) and (5) are identical, if $\phi$ fulfils $\dot{\phi} = \sqrt{R_0}(1 - \phi^2)$. Together with (2), (5) then becomes

$$R_{ij} = \frac{1}{2} \left( \frac{\partial h_{ij}}{\partial t} \right)^2$$

(6)

We recognize, that (4) is a generalization of the Einstein-Equations like the Ricci-Flow. Observe that since

$$\sqrt{2R_{ij}} = 2K_{ij}$$

follows from (4), the Riemannian metric $h_{ij}(t)$ satisfies the Hamilton constraint of ADM-formalism

$$H = \int_{\Sigma_t} H\ d^3x = - \int_{\Sigma_t} N\sqrt{h}\left( K^{ij}K_{ij} - K^2 + (^{3}R) \right) d^3x = 0$$

(7)

because

$$K^{ij}K_{ij} - K^2 + (^{3}R) = \frac{3}{4}\dot{\phi}^2 - \frac{9}{4}\dot{\phi}^2 + \frac{3}{2}\phi^2 = 0$$

implying, that

$$ds^2 = g_{\mu\nu}dx^\mu dy^\nu = -dt^2 + h_{ij}(t)dx^i dy^j$$

(8)
defines a gravitational instanton. It should therefore have a representation as a sum of (anti)-self dual instantons in SU(2)-Yang-Mills-Theory [8]. We can rewrite equation (5) in terms of the scalar field \( \phi(t) \).

\[
\frac{\partial h_{ij}}{\partial t} = \sqrt{4V(\phi)}\delta_{ij} .
\]  

(9)

Clearly the minima of the potential \( V(\phi) \) are located at \( \pm 1 \) and they are asymptotically approached by the nontrivial, topological solutions \( \phi(t - t_0) \). More specifically

\[
\phi(t - t_0) = \mp \tanh \left( \sqrt{R_0}(t - t_0) \right) ,
\]

(10)

Figure 1: The instanton interpolating between the two vacuum states

which can be calculated using the Bogomolny-Equation (9). Ref. [1], [2] and [3]. They belong to a group of topological, localised and stable wavelike solutions to nonlinear systems, whose potential satisfies a certain vacuum structure. We conclude, that

\[
h_{ij}(x^k, t - t_0) = \phi(t - t_0)\delta_{ij} = \mp \tanh \left( \sqrt{R_0}(t - t_0) \right) \delta_{ij} .
\]

(11)
are solutions to (4). They interpolate between two Euclidean-Manifolds, which are the minima of the double-well potential \( R_{ij}(h_{ij}) \). They are also stable, because there exist no continuous deformations between \( \phi \) and the vacuums for finite \( t \) due to the different homotopy classes, which they are a part of. Virtually every manifold of the family \( \Sigma_t \) described by a \( h_{ij}(t) \) is almost indistinguishable from a Euclidean manifold, except for those in the close vicinity of the bounce at \( t_0 \). In the following calculations, we restrict to the positive sign and choose \( t_0 = 0 \). The classical mass of \( \phi \) is

\[
S_E = \int_{-\infty}^{\infty} \varepsilon(t) dt = \int_{-\infty}^{\infty} 2V(\phi) dt = \frac{R_0}{2} \int_{-\infty}^{\infty} \text{sech}^4 (\sqrt{R_0}t) dt = \frac{2\sqrt{R_0}}{3} \tag{12}
\]

\[
t \to -\infty \quad t_0 \quad t \to \infty
\]

![Figure 2: 4-d-spacetime connecting two Euclidean Manifolds in the infinite future and past](image)

The solutions (8) generate a spatial coordinate chart

\[
ds^2 = \tanh^2(\sqrt{R_0}t) d\Sigma^2 \tag{13}
\]

with scaling factor \( \phi(t) = \tanh(\sqrt{R_0}t) \). Following from the assumptions made for the Ricci-Tensor, it is now possible to evaluate the dependence of \( R_{ij} \) on the parameter \( t \).
\[ R_{ij}(t) = \frac{1}{2} R_0 \left( \tanh^2(\sqrt{R_0} t) - 1 \right) \delta_{ij} = \frac{1}{2} R_0 \text{sech}^4(\sqrt{R_0} t) \delta_{ij} \]  

(14)

Contraction of \( R_{ij}(t) \) gives the \( t \)-dependent scalar curvature \( R(t) \)

\[ R(t) = 6V(\phi) = \frac{3}{2} \dot{\phi}^2 = \frac{3}{2} R_0 \text{sech}^4(\sqrt{R_0} t). \]  

(15)

A different view on the cosmological constant emerges from the force associated with the trajectory \( \phi(t) \) inside of \( V(\phi) \).

\[ F = m\ddot{\phi} = -2mR_0 \text{tanh}(\sqrt{R_0} t) \text{sech}^2(\sqrt{R_0} t) \]  

(16)

This would induce a negative pressure for positive \( t \) on the family of manifolds \( \Sigma_t \) with surface area \( A(t) \).

\[ p_\Lambda(t) = \frac{F}{A} = -\frac{2mR_0 R(t) \text{tanh}(\sqrt{R_0} t) \text{sech}^2(\sqrt{R_0} t) \text{sech}^4(\sqrt{R_0} t)}{4\pi} = -\frac{mR_0^2}{2\pi} \text{tanh}(\sqrt{R_0} t) \text{sech}^6(\sqrt{R_0} t) \]  

(17)

Figure 3: Plot of \( p_\Lambda \) as a function of \( t \) for \( R_0 = 1 \) and \( m=1 \)

converging exponentially towards zero. However, matter would manifest itself as friction, dampening the increase of the negative pressure \( p_\Lambda \). It should be emphasised, that although quantum corrections are missing in the previous calculations, quantum fluctuations about

\[ h_{ij}(x^k, t) = \text{tanh}(\sqrt{R_0} t) \delta_{ij} \]  

(18)

are described by the fluctuation operator
\[
\frac{\delta^2 S_E}{\delta \phi(t) \delta \phi(t')} = \left[ -\partial^2_t + V''(\phi) \right] \delta(t - t'),
\]

(19)

where \( S_E \) is the Euclidean action. The freedom of an arbitrary \( t_0 \) causes \( h_{ij}(t - t_0) \) to form an equipotential curve in field configuration space. For every \( t_0 \) the action is minimized, so that along the curve the fluctuation operator vanishes, concluding that the ground state of the quantum system, is the derivative of \( h_{ij} \) with respect to \( t \) up to a normalization-constant.

\[
\psi_{0,ij} = \psi_0 \delta_{ij} = \frac{\partial \phi}{\partial t} \delta_{ij} = \sqrt{R_0} sech^2(\sqrt{R_0} t) \delta_{ij}
\]

(20)

This mode is non quantum. Additionally, for the family \( h_{ij}(t - t_0) \) there exists a period of inflation for every \( t_0 \).

2 Tunneling-Amplitude

In the path-integral approach to quantum gravity it is proposed that one only needs to specify the metric on the boundary \( \Sigma_t \).

\[
Z(h_f, h_i, t) = \langle h_f | e^{-iHt} | h_i \rangle = N \int \mathcal{D}[h_{ij}] e^{iS[h_{ij}]}
\]

(21)

A Gibbons-Hawking-York boundary term ensures the composition rule

\[
\langle h_{n+2}, \Sigma_{n+2} | h_n, \Sigma_n \rangle \sum_{h_{n+1}} \langle h_{n+2}, \Sigma_{n+2} | h_{n+1}, \Sigma_{n+1} \rangle \langle h_{n+1}, \Sigma_{n+1} | h_n, \Sigma_n \rangle
\]

(22)

to hold, so that the amplitude is obtained by summing over all states on the intermediate surface. In the presented model \( h_{ij} \) can be viewed as an instanton, connecting to Euclidean manifolds, which are the vacuums of the double-well potential. The analytically continued generating functional includes the determinant of the fluctuation operator (19). For a detailed derivation we refer to [4]. The tunnelling solution satisfies the boundary conditions

\[
h_{ij} \left(-\frac{\tau}{2}\right) = h_i \hspace{1cm} h_{ij} \left(\frac{\tau}{2}\right) = h_f
\]

(23)
for $\tau \to \infty$ and the propagator in Euclidean time can be expressed as

$$Z(\delta_{ij}, -\delta_{ij}) = \mathcal{N} e^{-S_E[h_{ij}]} \sqrt{\frac{S_E[h_{ij}]}{2\pi} \tau (det' F[h_{ij}])^{-\frac{1}{2}}}$$  \hspace{1cm} (24)$$

In the present model this is equal to

$$Z(1, -1) = \mathcal{N} e^{-S_E[\phi]} \sqrt{\frac{6S_E[\phi]}{\pi} \tau (det' F[\phi])^{-\frac{1}{2}}}$$  \hspace{1cm} (25)$$

where the zero mode of the fluctuation operator is of course dropped in the calculation of the determinant. Evaluating the components of the expression leads to the tunnelling propagator up to $O(\hbar)$ in semiclassical approximation for large $\tau$.

$$Z(1, -1) = \sqrt{\frac{2\sqrt{R_0}}{\pi}} e^{-\sqrt{R_0} \tau} \sqrt{R_0 \tau} e^{-S_E[\phi]} \int_{\frac{\tau}{2}}^{\tau} \prod_{n=1}^{N} \int_{\tau_{0,n-1}}^{\tau_{0,n}} d\tau_0, d\tau_0, n_1, n_2}$$  \hspace{1cm} (26)$$

The derivation includes only saddle points of a single instanton, but we need to cover all additional saddle points, because they contribute to the amplitude for large $\tau$. The fluctuations about an ordered superposition of instantons are of the form

$$\psi(\tau) = \psi_0(\tau) + \sum_{n=1}^{N} \psi_n(\tau)$$  \hspace{1cm} (27)$$

where $\psi_0(\tau)$ represents fluctuations about the trivial solutions $\phi_0 = \pm 1$. For the multi-instanton propagator, we have to take all possible values of the position $\tau_0$ into account by integrating the contribution of a single instanton to the propagator (31) over all values for $N$ well separated (anti-) instantons, where $\tau_{0,n-1} < \tau_{0,n} < \frac{\tau}{2}$.

$$Z_I = \sqrt{\frac{6S_E[\phi]}{\pi} e^{-S_E[\phi]}} \int_{\frac{\tau}{2}}^{\tau} \prod_{n=1}^{N} \int_{\tau_{0,n}}^{\frac{\tau}{2}} d\tau_0, n_1, n_2}$$  \hspace{1cm} (28)$$
This yields

\[ Z_N \approx Z_0 \left( \frac{Z_1 \tau}{N!} \right)^N \]  

(29)

\[ Z_0 \] is the determinant of the fluctuations about the true vacuums. Furthermore

\[ Z_0(\pm 1, \pm 1) = \mathcal{N} \left( \det \hat{F}_0[\phi] \right)^{1/2} \delta_{ij} = \sqrt{\frac{2\sqrt{R_0}}{\pi}} e^{-\sqrt{R_0} \tau} \]  

(30)

and

\[ Z_1' = 2\sqrt{R_0} \sqrt{\frac{6S_E[\phi]}{\pi}} e^{-s_E[\phi]} \delta_{ij} = 4 \sqrt{\frac{3}{R_0^2}} \frac{\sqrt{2}}{\pi} e^{-\sqrt{R_0} \tau} \]  

(31)

The final expression is obtained by summing over all odd \( N \). This is required due to the fact, that only for even \( N \) the instantons satisfy the boundary conditions (23).

\[ Z(1, -1) = Z_0 \sum_{N \text{ odd}} \left( \frac{Z_1 \tau}{N!} \right)^N \]  

(32)

\[ = \sqrt{\frac{2\sqrt{R_0}}{\pi}} e^{-\sqrt{R_0} \tau} \left( e^{\tau Z_1'} - e^{-\tau Z_1'} \right) \]  

(33)

This yields the tunnelling propagator in WKB-approximation and Euclidean time

\[ Z(1, -1) = \sqrt{\frac{2\sqrt{R_0}}{\pi}} e^{-\sqrt{R_0} \tau} \sinh \left( 4\tau \sqrt{\frac{R_0^2}{3}} e^{\frac{2\sqrt{R_0}}{\pi}} \right) \]  

(34)

and

\[ Z(-1, -1) = \sqrt{\frac{2\sqrt{R_0}}{\pi}} e^{-\sqrt{R_0} \tau} \cosh \left( 4\tau \sqrt{\frac{R_0^2}{3}} e^{\frac{2\sqrt{R_0}}{\pi}} \right) \]
for even $N$. It can be understood as a propagator of the system

$$\mathcal{H}\Psi = 0$$  \hspace{1cm} (35)

on minisuperspace which is called the Wheeler-DeWitt equation. Ref [5] [6, Chapter 1.2]. Rotated back to real time, (34) oscillates in the vicinity of the Planck-frequency.

3 Hamilton Quantization

It is relatively rare to know the spectral properties of operators describing quantum fluctuations inside systems with a nontrivial vacuum structure. Nevertheless, they are well known concerning the model discussed, not least because of the SUSY structure those Schrödinger-like operators admit. From that one is able, to build a representation in terms of operator valued distributional ladder operators on a fitting Fock space via constructive QFT. In a Hamilton quantization of the vacuum sector [7], we want the operators

$$\hat{h}_{ij} = (\phi_0 + \varphi(t))\delta_{ij}$$
$$\hat{\pi}_{ij} = \pi(t)\delta_{ij}$$

(36)

to satisfy the canonical commutation relation

$$[\hat{h}_{ij}, \hat{\pi}_{ij}] = [(\phi_0 + \varphi(s))\delta_{ij}, \pi(t)\delta_{ij}] = i\delta(s - t)\delta_{ij}$$  \hspace{1cm} (37)

where

$$\varphi(t) = \frac{1}{\sqrt{2\pi}} \int \frac{dk}{\sqrt{2\omega_k}} (a_k e^{-ikt} + a_k^* e^{ikt})$$  \hspace{1cm} (38)

and

$$\pi(t) = \int \frac{dk}{\sqrt{2\pi}} (-i)^\frac{\sqrt{\omega_k}}{2} (a_k e^{-ikt} - a_k^* e^{ikt})$$

are operator valued distributions and $\omega_k = \sqrt{k^2 + 4}$. Annihilation and creation operators of the vacuum sector are acting on elements of the Fock space
\[ \mathfrak{g}_0 = \bigoplus_{n = 0}^{\infty} \text{Sym}^n(L^2(\mathbb{R}, dk)). \]  

The spin-2 field of linearized quantum fluctuations about the Euclidean metrics can be expressed as

\[
\varphi_{ij}(x^k, t) = \begin{pmatrix}
\frac{1}{\sqrt{2\pi}} \int \frac{dk}{\sqrt{2\omega_k}} (a_k e^{-ikt} + a_k^* e^{ikt}) & 0 & 0 \\
0 & \frac{1}{\sqrt{2\pi}} \int \frac{dk}{\sqrt{2\omega_k}} (a_k e^{-ikt} + a_k^* e^{ikt}) & 0 \\
0 & 0 & \frac{1}{\sqrt{2\pi}} \int \frac{dk}{\sqrt{2\omega_k}} (a_k e^{-ikt} + a_k^* e^{ikt})
\end{pmatrix}
\]

so that fluctuations about the scaling factor induce fluctuations of the metric \( \delta_{ij} \). The Hamiltonian is

\[
H_0 = \frac{1}{2} \int : \pi^2 + \varphi F_0 \varphi : dt = \int \omega_k a_k^* a_k dk
\]

with domain

\[
\text{Dom}(H_0) = \left\{ \Psi \in \bigoplus_{n = 0}^{\infty} \text{Sym}^n(L^2(\mathbb{R})): \sum_n \left( \sum_{i=1}^{n} \omega_k \psi_n(k_1, \ldots, k_n) \right)^2 < \infty \right\},
\]

where \( F_0 \) is the fluctuation operator of the vacuum sector. In the instanton sector we have

\[
\hat{h}_{ij} = (\phi + \varphi(t)) \delta_{ij}
\]

where

\[
\varphi(t) = -\sqrt{S_0} X \psi_0(t) + \frac{1}{\sqrt{2\omega_m}} (a_m \psi_1(t) + a_m^* \psi_1(t)) + \int \frac{1}{2\sqrt{\pi\omega_k}} (a_k Y_k(t) + a_k^* \overline{Y}_k(t)) dk
\]

and

\[
\pi(t) = -\frac{\mathcal{P}}{\sqrt{S_0}} \psi_0(t) - i \sqrt{\frac{\omega_m}{2}} (a_m \psi_1(t) - a_m^* \psi_1(t)) - \int i \frac{\sqrt{\omega_k}}{2\sqrt{\pi}} (a_k Y_k(t) - a_k^* \overline{Y}_k(t)) dk
\]
The \(a_m, a_m^\dagger\) and \(a_k, a_k^\dagger\) destroy and create states

\[
a_m = b(\psi_1), \quad a_m^\dagger = b^\dagger(\psi_1),
\]

\[
a_k = b\left(\frac{1}{\sqrt{2\pi}} Y_k(t)\right), \quad a_k^\dagger = b^\dagger\left(\frac{1}{\sqrt{2\pi}} Y_k(t)\right)
\]

respectively. \(b\) and \(b^\dagger\) act on the Fock space

\[
\mathcal{F} = \bigoplus \text{Sym}^n \mathbb{P}_0 \perp (L^2(\mathbb{R}))
\]

where \(\mathbb{P}_0 \perp\) is the orthogonal projector onto the space, that is orthogonal to the linear eigenspace of \(\psi_0\). \(Y_k(t) \in L^\infty(\mathbb{R}) \cap C^\infty(\mathbb{R})\) are generalized eigenfunctions of the fluctuation operator in the instanton sector, representing radiation modes in a reflectionless Pöschl-Teller potential, which is part of the SUSY-Chain \(F_{N-} = -\partial^2_t + N^2 - N(N+1) \text{sech}^2(t)\).

\[
Y_k(t) = \frac{(-k^2 - 3ik \tanh(\sqrt{R_0} t) + 2 - 3 \text{sech}^2(\sqrt{R_0} t))e^{-ikt}}{\sqrt{(k^2 + 1)(k^2 + 4)}} e^{i\delta_k}, \quad k \geq 0
\]

\[
Y_k(t) = \frac{(-k^2 - 3ik \tanh(t\sqrt{R_0}) + 2 - 3 \text{sech}^2(t\sqrt{R_0}))e^{-ikt}}{\sqrt{(k^2 + 1)(k^2 + 4)}} e^{-i\delta_k}, \quad k < 0
\]

where

\[
e^{\pm i\delta_k} = \frac{-k^2 \mp 3ik + 4}{\sqrt{(k^2 + 1)(k^2 + 4)}}
\]

\(\psi_0(t), \psi_1(t) \in \mathcal{S}(\mathbb{R})\) are the zero mode, vibrational mode, respectively. \(\mathcal{X}\) is a multiplication operator characterizing fluctuations about the position of the instanton with commutation relation \([\mathcal{X}, \mathcal{P}] = i\). The eigenfunctions of the fluctuation operator satisfy

\[
\int Y_m(t)Y_n(t)dt = 2\pi\delta(m - n) \quad \forall m, n \in \mathbb{R}
\]

\[
\int \psi_a(t)\psi_b(t)dt = \delta_{ab} \quad a, b \in \{0,1\}
\]

\[
\int \psi_a(t)Y_n(t)dt = 0.
\]
By this construction the relation

\[ [\hat{h}_{ij}, \hat{r}_{ij}] = [(\phi + \varphi(s))\delta_{ij}, \pi(t)\delta_{ij}] = i\delta(s - t)\delta_{ij} \]  

(51)

holds. The quantized instanton sector describes bosons of the vacuum sector moving in a background potential created by the instanton itself. By introducing a distorted Fourier transform, the bosons in the kink sector are mapped onto the free ones via scattering theory. The tensor field associated with the instanton fluctuations is

\[ \varphi_{ij}(x^k, t) = \begin{pmatrix} \varphi(t) & 0 & 0 \\ 0 & \varphi(t) & 0 \\ 0 & 0 & \varphi(t) \end{pmatrix} \]

where \( \varphi(t) \) is defined as in (43). Quantum fluctuations about the scaling factor, both in the vacuum and instanton sector, are the diagonal elements of a tensor field embodying quantum fluctuations of the geometry of the underlying manifolds. The Hamiltonian of the kink sector is

\[ \phi H_0^{\text{kink}} \phi : \frac{1}{2} \int \pi^2 + \varphi F\varphi : dt \]  

(52)

\[ \phi H_0^{\text{kink}} \phi : = \frac{p^2}{2S_0} + \omega_m a_m^+ a_m + \int \omega_k a_k^+ a_k \, dk \]  

(53)

Triple-points show the normal ordering with respect to the kink.

\[ \text{Dom}(\phi H_0^{\text{kink}} \phi : ) = \left\{ \Psi : \sum_v \left( \left\| (w \omega_m + \sum_{i=1}^v \omega_{k_i}) \Psi_{v,w} \right\|_L^2(w, dXdk) \right)^2 \right\} < \infty \]  

(54)

where \( \Psi \in \mathcal{F} \).
4 Particle creation and annihilation from scale factor fluctuations

An interesting feature of this theory is, that the time development of a massive scalar field defined on the manifold admits the same modes as the quantum fluctuations about the scaling factor, if there is a specific relation between the individual momenta, as will become clear. A conformally coupled, massive scalar field $\Phi(x, t)$ on $(M, g)$ satisfies the equation of motion

$$\left[g^{\mu\nu}\nabla_\mu \nabla_\nu + m^2 + \xi R\right] \Phi(x, t) = 0 \quad (55)$$

for $\xi = \frac{1}{6}$. The set of solutions in conformal time may be written as

$$f_\kappa(x, \eta) = \frac{e^{i\kappa x}}{\phi(\eta)\sqrt{(2\pi)^3}} \zeta_\kappa(\eta) \quad (56)$$

where $\zeta_\kappa(\eta)$ are solutions of

$$\left[\frac{\partial^2}{\partial \eta^2} + |\kappa|^2 + m^2\phi^2(\eta)\right] \zeta_\kappa(\eta) = \left[-\frac{\partial^2}{\partial \eta^2} - |\kappa|^2 - m^2 \tanh^2(\eta)\right] \zeta_\kappa(\eta) = 0 \quad (57)$$

with $\kappa = |\kappa|$ and the conformal scale defined by

$$ds^2 = \phi^2(t)\left(\frac{-dt^2}{\phi(t)^2} + dx^2\right) = \phi^2(\eta)(-d\eta^2 + dx^2) \quad (58)$$

The $f_\kappa$ form a complete orthonormal system with inner product

$$\langle f_1^*, f_2 \rangle \equiv \int d\Sigma^\mu f_1^* i \partial_\mu f_2 \quad (59)$$

where $d\Sigma^\mu = d\Sigma n^\mu$, with $d\Sigma$ being the volume element on a given hypersurface and $n^\mu$ the timelike unit normal vector. The product is invariant under the choice of hypersurface. Now if we make the assumptions

$$|\kappa|^2 = \omega_\kappa^2 - 10 - (m^2 - 6) \tanh^2(\eta) \quad (60)$$

equation (57) is equal to
\[-\partial^2_\eta + 4 - 6 \tanh^2(\eta) + 6] \zeta_\kappa(\eta) = \left[-\partial^2_\eta + 4 - 6 \text{sech}^2(\eta)\right] \zeta_\kappa(\eta) = \omega_\kappa^2 \zeta_\kappa(\eta) \quad (61)\]

which is the fluctuation operator of conformal scale factor $\phi(\eta)$ in the kink sector of the theory. It has two localised modes with eigenvalues $\omega_0^2 = 0$, $\omega_1^2 = 3$ and delocalized modes corresponding to massive particles, where $\omega_\kappa^2 = q^2 + 4$. The time dependent frequency $q$ is related to $\kappa$, so that the $\kappa$th mode has energy

$$\omega_\kappa^2 = q^2 + 4 = |\kappa|^2 + 10 + (m^2 - 6) \tanh^2(\eta) \quad (62)$$

$$q = \sqrt{|\kappa|^2 + 6 + (m^2 - 6) \tanh^2(\eta)} = \sqrt{|\kappa|^2 + m^2 \tanh^2(\eta) + 6 \text{sech}^2(\eta)}$$

Now in the special case of $m^2 = 6$, the frequency $q$ is not time dependent and we have $q = \sqrt{|\kappa|^2 + 6}$. Observe, that this is also the case for $\eta = 0$. The associated generalized eigenfunctions are

$$\zeta_\kappa(\eta) = \frac{(-q^2 - 3iq \tanh(\eta) + 2 - 3 \text{sech}^2(\eta)) e^{-iq\eta} e^{i\delta q}}{\sqrt{(q^2 + 1)(q^2 + 4)}} \quad (63)$$

Plugging this into (56) we see that the modes $f_\kappa$ can be expressed by

$$f_\kappa(x, \eta) = \frac{e^{i(\kappa x - \eta q + \delta q)}}{\tanh(\eta) \sqrt{(2\pi)^3}} \frac{(-q^2 - 3iq \tanh(\eta) + 2 - 3 \text{sech}^2(\eta))}{\sqrt{(q^2 + 1)(q^2 + 4)}} \quad (64)$$

as well as in terms of scale factor fluctuations, because $Y_q(t) = \phi(\eta)^{-1} \zeta_\kappa(\eta)$

$$f_\kappa(x, t) = \frac{e^{i\kappa x}}{\sqrt{(2\pi)^3}} Y_q(t) \quad (65)$$

so that $Y_q(t)$ describe time development of $f_\kappa(x, t)$ if $q = \sqrt{|\kappa|^2 + m^2 \tanh^2(t) + 6 \text{sech}^2(t)}$. They are normalized in such a way that

$$f_\kappa(x, t) = \frac{e^{i(\kappa x - qt + \delta q)}}{\sqrt{(2\pi)^3}} + O(e^{-|t|}) \quad (66)$$
as $t \to -\infty$ and

$$f_\kappa(x, t) = \frac{e^{i(\kappa x - qt + \delta_\kappa)}}{\sqrt{(2\pi)^3}} + O(e^{-|t|})$$

while as $t \to +\infty$. The asymptotic behaviour of $q(\kappa, t)$ is

$$q(\kappa, t \to \pm \infty) \to \sqrt{\kappa^2 + m^2}$$

For a massless field we have

$$q = \sqrt{|\kappa|^2 + 6 - 6 \tanh^2(t)} = \sqrt{|\kappa|^2 + 6 \text{sech}^2(t)}$$

$$q(\kappa, t \to \pm \infty) \to \kappa$$

which indicates that massive fields behave like massless ones at $t = 0$. The exponential decaying extra term is generated by the time dependent curvature and therefore by the gravitational field. The long-time behaviour is in harmony with the asymptotic behaviour of $(M, g)$. Since the manifold approaches Minkowski spaces for $t \to \pm \infty$, the scalar field $\Phi(x, t)$ satisfies asymptotically the free field equations in this regime, corresponding to the vanishing of scalar curvature in equation (55) and the unique determination of the particles notion through the asymptotic Poincaré symmetry. Yet there occurs a phase shift $\delta_\kappa$ due to the scattering at the Pöschl-Teller potential. However, these definitions of particles will not agree in the intermediate time-dependent phase i.e. there is a disagreement about the particle content because an early-time Heisenberg state wouldn’t be interpreted as a vacuum state by a late time observer. Instead it would be seen, as particle generation that is caused by the time dependent gravitational field. Inside the dense subset of hypersurfaces $\Sigma_t$ in the vicinity of $t_0$, where $R$ is large, fluctuations of geometry become significant in comparison to the classical scale. Spacetime development of $f_\kappa(x, t)$, which is of oscillatory nature, is determined by quantum fluctuations of the scale factor $\phi(t)$. A mode of scale fluctuation with momentum $q$ is equivalent to a scalar particle with energy $\sqrt{|\kappa|^2 + m^2 \tanh^2(t)} + 6 \text{sech}^2(t)$. We conclude that the gravitons established in section three create massive particles detected by a late time observer. This means, the creation operators of fluctuations of the scale factor, which are diagonal matrix elements of the graviton field, produce a mixing of positive and negative frequencies. These are then interpreted as particle content by an inertial late time observer. Hence, the field operator for $\Phi(x, t)$ can be expressed in terms of scale factor fluctuations.

$$\varphi(x, t) = \sum_\kappa a_\kappa f_\kappa + a_\kappa^\dagger f_\kappa^\dagger = \frac{1}{\sqrt{(2\pi)^3}} \int \frac{d^3q}{\sqrt{2q(\kappa)}} \left( a_q Y_q(t)e^{i\kappa x} + a_q^\dagger Y_q(t)e^{-i\kappa x} \right)$$ (67)
\[ \varphi(x, t \to \pm \infty) = \frac{1}{\sqrt{(2\pi)^3}} \int \frac{d^3q}{\sqrt{2q(\kappa)}} \left( a_q e^{i(\kappa x - qt + \delta_q)} + a_q^* e^{-i(\kappa x - qt + \delta_q)} \right) \]

where \( a_q, a_q^* \) are defined as in section three for. An interesting feature is that since in-vacuum and out-vacuum states are identical, the particles created during expansion, have to be annihilated during contraction of spacetime [20]. The modes created and destroyed in between the asymptotically Minkowskian metrics are the \( Y_q(t) \).

Further Discussion

We assumed that the proportionality factor of a one-parameter family of three-dimensional Einstein-Manifolds to their Ricci-Tensor is a smooth function, playing the role of a potential of a quartic system for that the cosmic scaling factor is a nontrivial solution to. By doing so we identified scalar curvature as potential energy and were able to find asymptotically flat solutions to the free Einstein-Equations. Moreover, this simplified the problem of quantization to a much more trivial one-dimensional one. We found, that classically \( h_{ij}(t - t_0) \) is a nontrivial family of trajectories inside a double-well potential (5), representing contracting and expanding spacetimes as well as inflationary periods. From a quantum point of view, these solutions describe tunnelling processes of metric fluctuations of Euclidean metrics through a potential well with hight \( R_0 \). Basically, we reduced the problem of the quantization of the metric \( h_{ij} \) to a quantization of the scaling factor. Classic spacetime becomes a connection between the three-dimensional, flat infinite future- and infinite past-hypersurface, corresponding to quantum fluctuations tunnelling through a potential-well. The two Ricci-flat manifolds are trivial vacuums of the potential of curvature \( R_{ij} \). For the model to produce these results, the well in between is to be of finite energy density to allow tunnelling and the existence of a family of nontrivial, topological solution. This also eliminates the problem of the Big-Bang-Singularity. The finite energy density of the kink is centered around \( t_0 \). Furthermore, due to tunnelling, the true quantum vacuum is an antisymmetric linear combination of to single-well ground states, which manifests as a nonzero probability amplitude in the classically forbidden region and is also known as level splitting. In the last part it was shown how the constructed gravitons create massive particles by mixing positive and negative energy states of a scalar field on the manifold, which is detected by a late time observer as particle creation.
Bibliography

[20] J. Tóthova, M. Hudák, Ondrej Hudák, Generation and annihilation of scalar particles due to a curved expanding and contracting space-time, 2019