ON THE NUMBER OF INTERSECTIONS OF TUBES

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Abstract. In this article we will prove that if the number of δ-tubes is $N = \delta^{1-n}$ and if the δ-tubes intersect on the unit cube, then the number of their intersections of order μ is bounded by $C_n \frac{N^{n/(n-1)}}{\mu}$. This implies that the number of (central) line intersections of order μ is bounded by $C_n \frac{N^{n/(n-1)}}{\mu}$. After making a dyadic decomposition and summing the orders together we will find that the number of (central) line intersections of N lines is bounded by $C_n N^{n/(n-1)}$. Given a finite number of lines we can always assume that they intersect on the unit cube, so we have a essentially sharp bound for the number of line intersections. An extremal case is the standard gris in $\mathbf{R}^n$. Previously this has been studied for special kind of line intersections called joints. Moreover, we will prove a generalized lemma of Córdoba.

1. Introduction

In $\mathbf{R}^n$ a joint is formed by the intersection of n lines whose tangent vectors are linearly independent. It's a fact that the number of joints formed by N lines are bounded by $C_n N^{n/(n-1)}$. This fact has quite an elementary proof [3]. In our paper we control all line or tube intersections in all scales. Our bound for the total line intersections is essentially sharp. An extremal example is a standard grid of N lines. A line l_i is defined as

$$l_i := \{y \in \mathbf{R}^n | \exists a, x \in \mathbf{R}^n \quad \text{and} \quad t \in \mathbf{R} \quad \text{s.t} \quad y = a + xt\}$$

We define the δ-tubes as δ neighbourhoods of lines:

$$T_i := \{x \in \mathbf{R}^n || x - y| \leq \delta, \quad y \in l_i\}.$$

The order of intersection is defined as the number of tubes (lines) intersecting. We use P_μ^δ as the set of δ-tube intersections of order μ and P_μ as the set of line intersections of order P_μ. Moreover, P^δ and P mean the set of δ-tube intersections and the set of line line intersections, respectively. If $\mu > 1$ then

$$P_\mu^\delta := \bigcup_{j=1}^{M_\mu^\delta} \bigcap_{i=1}^{\mu} T_{ij}$$

We define

$$\#(P_\mu^\delta) = M_\mu^\delta$$

and the total number of intersections is

$$\#(P^\delta) = \sum_\mu M_\mu^\delta.$$

In a same way we define $\#(P_\mu)$ and $\#(P)$. Our main theorem is the following:

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Theorem 1.1. *Let $N = \delta^{1-n}$. Given N δ-tubes that intersect on the unit cube, it holds for the number of order $\mu > 1$ intersections that*

$$\#(P_\mu^\delta) \leq C_n \frac{N^{n/(n-1)}}{\mu}. \tag{1.1}$$

Corollary 1.2. Let $N = \delta^{1-n}$. Given N δ-tubes that intersect on the unit cube, it holds for the number intersections that

$$\#(P^\delta) \leq C_n N^{n/(n-1)}. \tag{1.2}$$

Corollary 1.3. Given N lines it holds for the number of intersections of order μ that

$$\#(P_\mu) \leq C_n \frac{N^{n/(n-1)}}{\mu}. \tag{1.3}$$

Corollary 1.4. Given N lines it holds for the number of intersections that

$$\#(P) \leq C_n N^{n/(n-1)}, \tag{1.4}$$

Our other result is the following: a generalization of a lemma of Corbóda.

Lemma 1.5. *[A generalization of a lemma of Corbóda] For tube intersections of order 2^k it holds that*

$$|\bigcap_{i=1}^{2^k} T_i| \lesssim \delta^{n-1} 2^{-k/(n-1)}.$$

It‘s not hard to check that the above bound is essentially tight.

2. Previously known results

We will use the following bound for the pairwise intersections of δ-tubes:

Lemma 2.1 (Corbòda). *For any pair of directions $\omega_i, \omega_j \in S^{n-1}$ and any pair of points $a, b \in \mathbb{R}^n$, we have*

$$|T_{\omega_i}^\delta(a) \cap T_{\omega_j}^\delta(b)| \lesssim \frac{\delta^n}{|\omega_i - \omega_j|}.$$

A proof can be found for example in [2].

For any (spherical) cap $\Omega \subset S^{n-1}, |\Omega| \gtrsim \delta^{n-1}, \delta > 0$, define its δ-entropy $N_\delta(\Omega)$ as the maximum possible cardinality for an δ-separated subset of Ω.

Lemma 2.2. *In the notation just defined*

$$N_\delta(\Omega) \sim \frac{|\Omega|}{\delta^{n-1}}.$$

Again, a proof can essentially be found in [2].

3. A proof of the generalization of the lemma of Corbóda

Let us define

$$E_{2^k} := \{x \in \mathbf{R}^n | 2^k \leq \sum_{i=1}^N 1_{T_i} \leq 2^{k+1}\}.$$

Let us suppose that $2^k = \delta^{-\beta}, 0 < \beta \leq n-1$, and let‘s suppose that tube $T_{\omega'}$ intersecting $T_\omega \cap E_{2^k}$ has it‘s direction outside of a cap of size $\sim \delta^{n-1-\beta}$ on the unit sphere. Then the angle between T_ω and $T_{\omega'}$ is greater than $\sim \delta^{1-\beta/(n-1)}$. Thus by lemma 2.1 the intersection

$$|\bigcap_{i=1}^{2^k} T_i| \leq |T_\omega \cap T_{\omega'} \cap E_{2^k}| \leq |T_\omega \cap T_{\omega'}| \lesssim \delta^{n-1+\beta/(n-1)} = \delta^{n-1} 2^{-k/(n-1)}. \tag{3.1}$$

Thus, we can suppose that the directions in the intersection $E_{2^k} \cap T_\omega \cap T_{\omega'}$ belong to a cap of size $\sim \delta^{n-1+\beta}$. If we δ - separate the cap via lemma 2.2 we get that the cap can contain at most $\sim 2^k$ tube-directions. Thus, for any tube T_ω in the intersection there exists a tube $T_{\omega'}$, such that the angle between T_ω and $T_{\omega'}$ is $\sim \delta^{1-\beta/(n-1)}$ and the inequality (3.1) is valid. Thus we proved the lemma 1.5.

4. On the number of intersections of given order

Define the following set

$$E_\mu := \{x \in \mathbf{R}^n | \sum_{i=1}^{N} 1_{T_i} = \mu\}. \tag{4.1}$$

So that

$$\begin{aligned} \mu|E_\mu| = \int_{[-1,1]^n \cap E_{2^k}} \sum_{i=1}^{N} 1_{T_i} = \sum_{i=1}^{N} \int_{[-1,1]^n \cap E_{2^k}} 1_{T_i} \\ \leq 2^n \delta^{n-1} N |B(1,0)| = \delta^{n-1} C_n N. \end{aligned} \tag{4.2}$$

We define an intersection I_{jk} of order $\mu > 1$ as

$$I_{kj} := \bigcap_{i=1}^{\mu} T_{ij}.$$

So that

$$E_\mu = \bigcup I_{j\mu}$$

and

$$|E_\mu| = \sum_{j=1}^{M_\mu} |I_{j\mu}|.$$

Now, let us scale δ to 2δ. Define the scaled versions $I_{j\mu}$ and E_μ as $I'_{j\mu}$ and E'_μ, respectively. It holds that $I'_{j\mu} \cap [-1,1]^n$ contains a δ-ball. So that

$$\delta^n |B(0,1)| \leq |I'_{j\mu}| \tag{4.3}$$

We define M'_u as the number of intersection of order μ of 2δ-tubes. Clearly

$$\#(P^\delta_\mu) = M_u \leq M_{u'} = \#(P^{\delta}_\mu{}'). \tag{4.4}$$

It follows from above (4.3), (4.4) and from (4.2) that

$$\mu \delta^n |B(0,1)||P^\delta_\mu| \leq \mu \sum_{j=1}^{M'_\mu} |I'_{j\mu}| \leq \mu |E'_\mu| \leq \delta^{n-1} 2^n N.$$

Thus,

$$\mu \delta^n |P^\delta_\mu| \leq \delta^{n-1} C_n N,$$

which is equivalent to

$$\delta |P^\delta_\mu| \leq C_n \frac{N}{\mu}. \tag{4.5}$$

We assumed in our theorem 1.1 that

$$N = \delta^{1-n},$$

so that

$$N^{-1/(1-n)} = \delta \tag{4.6}$$

Thus, it follows from (4.5) and (4.6) that

$$N^{-1/(n-1)}|P_\mu| \le C_n \frac{N}{\mu}.$$

which is equivalent to

$$|P_\mu| \le C_n \frac{N^{n/(n-1)}}{\mu}.$$

The above implies our main theorem 1.1.
If we have N lines that intersect on $[-R, R]^n$ then we can scale $\mathbf{R}^n$ s.t the lines intersect in $[-1, 1]^n$. Then we choose $\delta^{1-n} = N$. Now, the number of central line intersections is less than the number of tube intersections. So from 1.1 it follows 1.3.
In order to prove 1.2 we will take μ dyadically. So that we have

$$E_{2^k} := \{x \in \mathbf{R}^n | 2^k \le \sum_{i=1}^{N} 1_{T_i} \le 2^{k+1}\}. \tag{4.7}$$

The set of intersections are now defined as

$$P^\delta_{2^k} := \bigcup_{j=1}^{M^\delta_{2^k}} \bigcap_{i=1}^{\mu} T_{ij},$$

$\#(p^\delta_{2^k}) = M_{2^k}$, and

$$|E_{2^k}| = \sum_j |I_{j2^k}|.$$

So we have

$$2^k \delta^n |B(0,1)||P^\delta_{2^k}| \le 2^k \sum_j |I'_{j2^k}| \le 2^k |E'_{2^k}| \le \delta^{n-1}|C_n N,$$

where I'_{j2^k} and E'_{2^k} are scaled versions of I_{j2^k} and E_{2^k}, respectively. Thus, like before it follows that

$$|P^\delta_{2^k}| \le C_n \frac{N^{n/(n-1)}}{2^k}.$$

But if we sum above over k we have

$$|P^\delta| = \sum_{k \ne 0} |P^\delta_{2^k}| \le \sum_{k \ne 0} C_n \frac{N^{n/(n-1)}}{2^k} \le C_n N^{n/(n-1)} \sum_{k=1}^{\infty} \frac{1}{2^k} = C_n N^{n/(n-1)}.$$

This proves 1.3. And again if we have N lines that intersect on $[-R, R]^n$ then we can scale $\mathbf{R}^n$ s.t the lines intersect in $[-1, 1]^n$. Then we choose $\delta^{1-n} = N$ and put the lines as central lines of the tubes. So 1.4 follows from 1.3.

References

[1] A. Còrdoba, *The Kakeya Maximal Function and the Spherical Summation Multipliers*, American Journal of Mathematics 99 (1977), 1-22.

[2] E.Kroc, *The Kakeya problem*, available at
http://ekroc.weebly.com/uploads/2/1/6/3/21633182/mscessay-final.pdf

[3] R. Quilodrán, The Joints Problem in $\mathbb{R}^n$, SIAM Journal on Discrete Mathematics 23(4)(2010),2211-2213.