ABSTRACT

The Special Theory of Relativity takes us to two results that presently are considered “inexplicable” to many renowned scientists, to know:

- The dilatation of time, and
- The contraction of the Lorentz Length.

The solution to these have driven the author to the development of the Undulating Relativity (UR) theory, where the Temporal variation is due to the differences on the route of the light propagation and the lengths are constants between two landmarks in uniform relative movement.

The Undulating Relativity provides transformations between the two landmarks that differs from the transformations of Lorentz for: Space \((x,y,z)\), Time \((t)\), Speed \((\dot{u})\), Acceleration \((\ddot{a})\), Energy \((E)\), Momentum \((\vec{p})\), Force \((\vec{F})\), Electrical Field \((\vec{E})\), Magnetic Field \((\vec{B})\), Light Frequency \((\nu)\), Electrical Current \((\vec{J})\) and “Electrical Charge” \((\rho)\).

From the analysis of the development of the Undulating Relativity, the following can be synthesized:

- It is a theory with principles completely on physics;
- The transformations are linear;
- Keeps untouched the Euclidian principles;
- Considers the Galileo’s transformation distinct on each referential;
- Ties the Speed of Light and Time to a unique phenomenon;
- The Lorentz force can be attained by two distinct types of Field Forces, and
- With the absence of the spatial contraction of Lorentz, to reach the same classical results of the special relativity rounding is not necessary as concluded on the Doppler effect.

Both, the Undulating Relativity and the Special Relativity of Albert Einstein explain the experience of Michel-Morley, the longitudinal and transversal Doppler effect, and supplies exactly identical formulation to:

\[
\text{Aberration of zenith } \Rightarrow \tan \alpha = \frac{v}{c} \sqrt{1 - \frac{v^2}{c^2}}.
\]

\[
\text{Fresnel’s formula } \Rightarrow c' = \frac{c}{n} + v(l - \frac{l}{n^2}).
\]

\[
\text{Mass } (m) \text{ with velocity } (v) = \text{[resting mass } (m_0)]\sqrt{1 - \frac{v^2}{c^2}}.
\]

\[
E = m.c^2.
\]

\[
\text{Momentum } \Rightarrow \vec{p} = \frac{m_0 \vec{v}}{\sqrt{1 - \frac{v^2}{c^2}}}.
\]

\[
\text{Relation between momentum (p) and Energy (E) } \Rightarrow E = c.\sqrt{m_0^2.c^2 + p^2}.
\]

\[
\text{Relation between the electric field } (\vec{E}) \text{ and the magnetic field } (\vec{B}) \Rightarrow \vec{B} = \frac{\vec{V}}{c^2} \times \vec{E}.
\]

\[
\text{Biot-Savant’s formula } \Rightarrow \vec{B} = \frac{\mu_0 I}{2.\pi. R} \hat{\mu}.
\]

\[
\text{Louis De Broglie’s wave equation } \Rightarrow \psi(x,t) = a.s.i.n \left[ 2.\pi \left( t - \frac{x}{u} \right) \right] u = \frac{c^3}{v}.
\]
Along with the equations of transformations between two references of the UR, we get the invariance of shape to Maxwell’s equations, such as:

\[ \Rightarrow \text{div}\vec{E} = \frac{\rho}{\varepsilon_0} ; \Rightarrow \text{div}\vec{E} = 0. \quad \Rightarrow \text{div}\vec{B} = 0. \quad \Rightarrow \text{Rot}\vec{E} = -\frac{\partial \vec{B}}{\partial t}. \]

\[ \Rightarrow \text{Rot}\vec{B} = \mu_0 \vec{j} + \varepsilon_0 \mu_0 \frac{\partial \vec{E}}{\partial t} ; \Rightarrow \text{Rot}\vec{B} = \varepsilon_0 \mu_0 \frac{\partial \vec{E}}{\partial t}. \]

We also get the invariance of shape to the equation of wave and equation of continuity under differential shape:

Other Works:

§9 Explaining the Sagnac Effect with the Undulating Relativity.
§10 Explaining the experience of Ives-Stilwell with the Undulating Relativity.
§11 Transformation of the power of a luminous ray between two referencials in the Special Theory of Relativity.
§12 Linearity.
§13 Richard C. Tolman.
§14 Velocities composition.
§15 Invariance.
§16 Time and Frequency.
§17 Transformation of H. Lorentz.
§18 The Michelson & Morley experience.
§19 Regression of the perihelion of Mercury of 7,13".
§§19 Advance of Mercury's perihelion of 42.79".
§20 Inertia.
§20 Inertia (clarifications)
§21 Advance of Mercury's perihelion of 42.79" calculated with the Undulating Relativity.
§22 Spatial Deformation.
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§24 Variational Principle.
§25 Logarithmic Spiral.
§26 Mercury Perihelion Advance of 42.99".
§27 Advancement of Perihelion of Mercury of 42.99" "contour Conditions".
§28 Simplified Periellium Advance.
§ 29 Yukawa Potential Energy "Continuation".
§ 30 Energy "continuation".
Undulating Relativity

§ 1 Transformation to space and time

The Undulating Relativity (UR) keep the principle of the relativity and the principle of Constancy of light speed, exactly like Albert Einstein’s Special Relativity Theory defined:

a) The laws, under which the state of physics systems are changed are the same, either when referred to a determined system of coordinates or to any other that has uniform translation movement in relation to the first.

b) Any ray of light moves in the resting coordinates system with a determined velocity c, that is the same, whatever this ray is emitted by a resting body or by a body in movement (which explains the experience of Michel-Morley).

Let’s imagine first that two observers O and O’ (in vacuum), moving in uniform translation movement in relation to each other, that is, the observer don’t rotate relatively to each other. In this way, the observer O together with the axis x, y, and z of a system of a rectangle Cartesian coordinates, sees the observer O’ move with velocity v, on the positive axis x, with the respective parallel axis and sliding along with the x axis while the O’, together with the x’, y’ and z’ axis of a system of a rectangle Cartesian coordinates sees O moving with velocity –v’, in negative direction towards the x’ axis with the respective parallel axis and sliding along with the x’ axis. The observer O measures the time t and the O’ observer measures the time t’ (t ≠ t’).

Let’s admit that both observers set their clocks in such a way that, when the coincidence of the origin of the coordinated system happens t = t’ = zero.

In the instant that t = t’ = 0, a ray of light is projected from the common origin to both observers. After the time interval t the observer O will notice that his ray of light had simultaneously hit the coordinates point A (x, y, z) with the ray of the O’ observer with velocity c and that the origin of the system of the O’ observer has run the distance v t along the positive way of the x axis, concluding that:

\[ x^2 + y^2 + z^2 - c^2 t^2 = 0 \]  \hspace{0.5cm} 1.1

\[ x' = x - v t. \]  \hspace{0.5cm} 1.2

The same way after the time interval t’ the O’ observer will notice that his ray of light simultaneously hit with the observer O the coordinate point A (x’, y’, z’) with velocity c and that the origin of the system for the observer O has run the distance v’ t’ along the negative way of the axis x’, concluding that:

\[ x'^2 + y'^2 + z'^2 - c^2 t'^2 = 0 \]  \hspace{0.5cm} 1.3

\[ x = x' + v' t'. \]  \hspace{0.5cm} 1.4

Making 1.1 equal to 1.3 we have

\[ x^2 + y^2 + z^2 - c^2 t^2 = x'^2 + y'^2 + z'^2 - c^2 t'^2. \]  \hspace{0.5cm} 1.5

Because of the symmetry y = y’ end z = z’, that simplify 1.5 in

\[ x^2 - c^2 t^2 = x'^2 - c^2 t'^2. \]  \hspace{0.5cm} 1.6

To the observer O x’ = x – v t (1.2) that applied in 1.6 supplies

\[ x^2 - c^2 t^2 = (x - v t)^2 - c^2 t^2 \] from where

\[ t' = t \sqrt{1 - \frac{v^2}{c^2}} - \frac{2vx}{c^2} \].  \hspace{0.5cm} 1.7

To the observer O’ x = x’ + v’ t’ (1.4) that applied in 1.6 supplies

\[ (x' + v' t')^2 - c^2 t'^2 = x'^2 - c^2 t'^2 \] from where
\begin{align*}
\frac{t}{t'} &= \sqrt{1 + \frac{v^2}{c^2} + \frac{2v'x'}{c^2t'}}. \quad 1.8 \\
\end{align*}

<table>
<thead>
<tr>
<th>Transformation</th>
<th>1.2</th>
<th>1.4</th>
</tr>
</thead>
</table>
| \(x' = x - vt\) | \(x = x' + v't'\) | \\
| \(y' = y\) | \(y = y'\) | \\
| \(z' = z\) | \(z = z'\) | \\
| \(t' = t\sqrt{1 + \frac{v^2}{c^2} - \frac{2vx}{c^2t}}\) | \(t = t'\sqrt{1 + \frac{v'^2}{c^2} + \frac{2v'x'}{c^2t'}}\) | 1.8 |

From the equation system formed by 1.2 and 1.4 we find

\[ v = v' \quad \text{or} \quad |v| = |v'| \] (considering \(t > 0 \quad \text{and} \quad t' > 0\)) \quad 1.9

what demonstrates the invariance of the space in the Undulatory Relativity.

From the equation system formed by 1.7 and 1.8 we find

\[ \sqrt{1 + \frac{v^2}{c^2} - \frac{2vx}{c^2t}} \cdot \sqrt{1 + \frac{v'^2}{c^2} - \frac{2v'x'}{c^2t'}} = 1. \quad 1.10 \]

If in 1.2 \(x' = 0\) then \(x = vt\), that applied in 1.10 supplies,

\[ \sqrt{1 - \frac{v^2}{c^2}} \cdot \sqrt{1 + \frac{v'^2}{c^2}} = 1. \quad 1.11 \]

If in 1.10 \(x = ct\) and \(x' = c't'\) then

\[ \left(1 - \frac{v}{c}\right)\left(1 + \frac{v'}{c}\right) = 1. \quad 1.12 \]

To the observer \(O\) the principle of light speed constancy guarantees that the components \(ux, uy\) and \(uz\) of the light speed are also constant along its axis, thus

\[ \frac{x}{t} = \frac{dx}{dt} = ux, \quad \frac{y}{t} = \frac{dy}{dt} = uy, \quad \frac{z}{t} = \frac{dz}{dt} = uz \quad 1.13 \]

and then we can write

\[ \sqrt{1 + \frac{v^2}{c^2} - \frac{2vx}{c^2t}} = \sqrt{1 + \frac{v^2}{c^2} - \frac{2vux}{c^2}}. \quad 1.14 \]

With the use of 1.7 and 1.9 and 1.14 we can write

\[ \frac{|v|}{|v'|} \cdot \frac{t'}{t} = \sqrt{1 + \frac{v^2}{c^2} - \frac{2vx}{c^2t}} \cdot \sqrt{1 + \frac{v'^2}{c^2} - \frac{2vux}{c^2}}. \quad 1.15 \]

Differentiating 1.9 with constant \(v\) and \(v'\), or else, only the time varying we have

\[ \frac{|v|}{|v'|} \cdot \frac{dt'}{dt}, \quad 1.16 \]

but from 1.15 \[ \frac{|v|}{|v'|} = \sqrt{1 + \frac{v^2}{c^2} - \frac{2vx}{c^2t}} \] then \(dt = dt' \sqrt{1 + \frac{v^2}{c^2} - \frac{2vux}{c^2}}. \quad 1.17 \]

Being \(v\) and \(v'\) constants, the reasons \[ \frac{|v|}{|v'|} \] and \[ \frac{t'}{t} \] in 1.15 must also be constant because for this the differential of \[ \sqrt{1 + \frac{v^2}{c^2} - \frac{2vx}{c^2t}} \] must be equal to zero from where we conclude \[ \frac{x}{t} = \frac{dx}{dt} = ux \], that is exactly the same as 1.13.
To the observer O’ the principle of Constancy of velocity of light guarantees that the components \( u'x' \), \( u'y' \), and \( u'z' \) of velocity of light are also constant alongside its axis, thus

\[
\begin{align*}
\frac{dx'}{dt'} &= u'x', \\
\frac{dy'}{dt'} &= u'y', \\
\frac{dz'}{dt'} &= u'z',
\end{align*}
\]  
\[1.18\]

and with this we can write,

\[
\sqrt{1 + \frac{v'^2}{c^2} + \frac{2v'x'}{c^2t'}} = \sqrt{1 + \frac{v^2}{c^2} + \frac{2vu'x'}{c^2}}.
\]  
\[1.19\]

With the use of 1.8, 1.9, and 1.19 we can write

\[
\frac{|v'|}{|v|} = \frac{t}{t'} = \sqrt{1 + \frac{v^2}{c^2} + \frac{2vu'x'}{c^2}} = \sqrt{1 + \frac{v^2}{c^2} + \frac{2vu'x'}{c^2}}.
\]  
\[1.20\]

Differentiating 1.9 with \( v' \) and \( v \) constant, that is, only the time varying we have

\[
\frac{|v'|dt'}{|v|dt} = \frac{dt'}{dt},
\]  
\[1.21\]

but from 1.20 \( \frac{|v'|}{|v|} = \sqrt{1 + \frac{v^2}{c^2} + \frac{2vu'x'}{c^2}} \) then \( dt = dt' \sqrt{1 + \frac{v^2}{c^2} + \frac{2vu'x'}{c^2}} \).

\[1.22\]

Being \( v' \) and \( v \) constant the divisions \( \frac{|v'|}{|v|} \) and \( \frac{t}{t'} \) in 1.20 also have to be constant because of this the differential of \( \sqrt{1 + \frac{v^2}{c^2} + \frac{2vu'x'}{c^2}} \) must be equal to zero from where we conclude \( \frac{x'}{t'} = \frac{dx'}{dt'} = u'x' \), that is exactly like to 1.18.

Replacing 1.14 and 1.19 in 1.10 we have

\[
\sqrt{1 + \frac{v^2}{c^2} - \frac{2vux}{c^2}} \cdot \sqrt{1 + \frac{v^2}{c^2} + \frac{2vu'x'}{c^2}} = 1.
\]  
\[1.23\]

To the observer O the vector position of the point A of coordinates (\( x,y,z \)) is

\[\vec{R} = x\hat{i} + y\hat{j} + z\hat{k},\]

\[1.24\]

and the vector position of the origin of the system of the observer O’ is

\[\vec{R}_o' = vt\hat{i} + 0\hat{j} + 0\hat{k} \Rightarrow \vec{R}_o' = vt\hat{i}.
\]  
\[1.25\]

To the observer O’, the vector position of the point A of coordinates (\( x',y',z' \)) is

\[\vec{R}' = x'\hat{i} + y'\hat{j} + z'\hat{k},
\]  
\[1.26\]

and the vector position of the origin of the system of the observer O is

\[\vec{R}_o = -v't\hat{i} + 0\hat{j} + 0\hat{k} \Rightarrow \vec{R}_o = -v't\hat{i}.
\]  
\[1.27\]

Due to 1.9, 1.25, and 1.27 we have, \( \vec{R}_o' = -\vec{R}_o \).

\[1.28\]

As 1.24 is equal to 1.25 plus 1.26 we have

\[\vec{R} = \vec{R}_o + \vec{R}' \Rightarrow \vec{R}' = \vec{R} - \vec{R}_o.
\]  
\[1.29\]

Applying 1.28 in 1.29 we have, \( \vec{R} = \vec{R}' - \vec{R}_o' \).

\[1.30\]
To the observer O the vector velocity of the origin of the system of the observer O' is
\[
\vec{v} = \frac{d\vec{R}'}{dt} = \vec{v}' + 0 \hat{j} + 0 \hat{k} \Rightarrow \vec{v} = \vec{v}'.
\]

To the observer O' the vector velocity of the origin of the system of the observer O is
\[
\vec{v}' = \frac{d\vec{R}'_o}{dt'} = -\vec{v} + 0 \hat{j} + 0 \hat{k} \Rightarrow \vec{v}' = -\vec{v}'.
\]

From 1.15, 1.20, 1.31, and 1.32 we find the following relations between \( \vec{v} \) and \( \vec{v}' \)
\[
\vec{v} = \frac{-\vec{v}'}{\sqrt{1 + \frac{v'^2}{c^2} + \frac{2v'u'x'}{c^2}}}.
\]
\[
\vec{v}' = \frac{-\vec{v}}{\sqrt{1 + \frac{v^2}{c^2} - \frac{2vux}{c^2}}}.
\]

Observation: in the table I the formulas 1.2, 1.2.1, and 1.2.2 are the components of the vector 1.29 and the formulas 1.4, 1.4.1, and 1.4.2 are the components of the vector 1.30.

§2 Law of velocity transformations \( \vec{u} \) and \( \vec{u}' \)

Differentiating 1.29 and dividing it by 1.17 we have
\[
\frac{d\vec{R}'}{dt'} = \frac{d\vec{R} - d\vec{R}'_o}{dt} \Rightarrow \vec{u}' = \frac{\vec{u} - \vec{v}}{\sqrt{K}}.
\]

Differentiating 1.30 and dividing it by 1.22 we have
\[
\frac{d\vec{R}}{dt} = \frac{d\vec{R} - d\vec{R}'_o}{dt'} \Rightarrow \vec{u} = \frac{\vec{u}' - \vec{v}'}{\sqrt{K'}}.
\]

Table 2, Law of velocity transformations \( \vec{u} \) and \( \vec{u}' \)

<table>
<thead>
<tr>
<th>( \vec{u}' ) = \frac{\vec{u} - \vec{v}}{\sqrt{K}}</th>
<th>\ref{2.1}</th>
<th>( \vec{u} = \frac{\vec{u}' - \vec{v}'}{\sqrt{K'}} )</th>
<th>\ref{2.2}</th>
</tr>
</thead>
<tbody>
<tr>
<td>( u'x' = \frac{ux - v}{\sqrt{K}} )</td>
<td>\ref{2.3}</td>
<td>( ux = \frac{u'x' + v'}{\sqrt{K'}} )</td>
<td>\ref{2.4}</td>
</tr>
<tr>
<td>( u'y' = \frac{uy}{\sqrt{K}} )</td>
<td>\ref{2.3.1}</td>
<td>( uy = \frac{u'y'}{\sqrt{K'}} )</td>
<td>\ref{2.4.1}</td>
</tr>
<tr>
<td>( u'z' = \frac{uz}{\sqrt{K}} )</td>
<td>\ref{2.3.2}</td>
<td>( uz = \frac{u'z'}{\sqrt{K'}} )</td>
<td>\ref{2.4.2}</td>
</tr>
<tr>
<td>(</td>
<td>v'</td>
<td>= \frac{</td>
<td>v</td>
</tr>
<tr>
<td>( \sqrt{K} = \sqrt{1 + \frac{v^2}{c^2} - \frac{2vux}{c^2}} )</td>
<td>\ref{2.5}</td>
<td>( \sqrt{K'} = \sqrt{1 + \frac{v'^2}{c^2} + \frac{2v'u'x'}{c^2}} )</td>
<td>\ref{2.6}</td>
</tr>
</tbody>
</table>
Multiplying 2.1 by itself we have
\[ u' = \sqrt{1 + \frac{v'^2}{c^2} + \frac{2v'u'x'}{c^2}} = \frac{c - v}{\sqrt{1 + \frac{v'^2}{c^2} + \frac{2v'u'x'}{c^2}}} \]  \[ u'' \]

If in 2.7 we make \( u = c \) then \( u' = c \) as it is required by the principle of constancy of velocity of light.

Multiplying 2.2 by itself we have
\[ u'' = \sqrt{1 + \frac{v^2}{u'^2} + \frac{2v'u'x'}{u'^2}} = \frac{c + v}{\sqrt{1 + \frac{v^2}{u'^2} + \frac{2v'u'x'}{u'^2}}} \]

If in 2.8 we make \( u' = c \) then \( u = c \) as it is required by the principle of constancy of velocity of light.

If in 2.3 we make \( ux = c \) then \( u'x' = \frac{c - v}{\sqrt{1 + \frac{v^2}{c^2} - \frac{2vc}{c^2}}} \) as it is required by the principle of constancy of velocity of light.

If in 2.4 we make \( u'x' = c \) then \( ux = \frac{c + v'}{\sqrt{1 + \frac{v'^2}{c^2} + \frac{2vc'}{c^2}}} \) as it is required by the principle of constancy of velocity of light.

Remodeling 2.7 and 2.8 we have
\[ \sqrt{1 + \frac{v^2}{c^2} - \frac{2vc}{c^2}} = \sqrt{1 - \frac{u'^2}{c^2}} \]
\[ \sqrt{1 + \frac{v'^2}{c^2} + \frac{2v'u'x'}{c^2}} = \sqrt{1 - \frac{u^2}{c^2}} \]

The direct relations between the times and velocities of two points in space can be obtained with the equalities \( \ddot{u} = 0 \Rightarrow u'x' = 0 \Rightarrow ux = v \) coming from 2.1, that applied in 1.17, 1.22, 1.20, and 1.15 supply
\[ dt' = dt \sqrt{1 + \frac{v^2}{c^2} - \frac{2vv}{c^2}} \Rightarrow dt = \frac{dt'}{\sqrt{1 + \frac{v^2}{c^2}}} \]
\[ dt = dt' \sqrt{1 + \frac{v'^2}{c^2} + \frac{2v'0}{c^2}} \Rightarrow dt' = \frac{dt}{\sqrt{1 + \frac{v'^2}{c^2}}} \]
\[ |v| = \frac{|v'|}{\sqrt{1 + \frac{v^2}{c^2} + \frac{2vv}{c^2}}} \Rightarrow |v'| = \frac{|v|}{\sqrt{1 - \frac{v^2}{c^2}}} \]
\[ |v'| = \frac{|v|}{\sqrt{1 + \frac{v'^2}{c^2} - \frac{2v}{c^2}}} \Rightarrow |v| = \frac{|v'|}{\sqrt{1 + \frac{v'^2}{c^2}}} \]
Aberration of the zenith

To the observer O' along with the star \(u'x' = 0, u'y' = c\) and \(u'z' = 0\), and to the observer O along with the Earth we have the conjunct 2.3

\[
0 = \frac{ux - v}{\sqrt{1 + \frac{v^2}{c^2} - 2uvx/c^2}} \Leftrightarrow ux = v, \quad c = \frac{uy}{\sqrt{1 + \frac{v^2}{c^2} - 2vv/c^2}} \Leftrightarrow uy = c\sqrt{1 - \frac{v^2}{c^2}}, \quad uz = 0,
\]

\[
u = \sqrt{ux^2 + uy^2 + uz^2} = \sqrt{v^2 + \left(c\sqrt{1 - \frac{v^2}{c^2}}\right)^2 + 0} = c \text{ exactly as foreseen by the principle of relativity.}
\]

To the observer O the light propagates in a direction that makes an angle with the vertical axis \(y\) given by

\[
tan\alpha = \frac{ux}{uy} = \frac{v}{c\sqrt{1 - \frac{v^2}{c^2}}} = \frac{v/c}{\sqrt{1 - \frac{v^2}{c^2}}}
\]

that is the aberration formula of the zenith in the special relativity.

If we inverted the observers we would have the conjunct 2.4

\[
0 = \frac{u'x' + v'}{\sqrt{1 + \frac{v'^2}{c^2} + 2v'u'x'/c^2}} \Leftrightarrow u'x' = -v', \quad c = \frac{u'y'}{\sqrt{1 + \frac{v'^2}{c^2} + 2v'(-v'/c^2)}} \Leftrightarrow u'y' = c\sqrt{1 - \frac{v'^2}{c^2}}, \quad u'z' = 0,
\]

\[
u' = \sqrt{u'x'^2 + u'y'^2 + u'z'^2} = \sqrt{(-v')^2 + \left(c\sqrt{1 - \frac{v'^2}{c^2}}\right)^2 + 0^2} = c
\]

\[
tan\alpha = \frac{u'x'}{u'y'} = \frac{-v'/c}{\sqrt{1 - \frac{v'^2}{c^2}}} = \frac{-v'/c}{\sqrt{1 - \frac{v'^2}{c^2}}}
\]

that is equal to 2.15, with the negative sign indicating the contrary direction of the angles.

Fresnel’s formula

Considering in 2.4, \(u'x' = c/n\) the velocity of light relatively to the water, \(v' = v\) the velocity of water in relation to the apparatus then \(ux = c'\) will be the velocity of light relatively to the laboratory

\[
c' = \frac{c/n + v}{\sqrt{1 + \frac{v^2}{c^2} + \frac{2vc/n}{c^2}}} = \frac{c/n + v}{\sqrt{1 + \frac{v^2}{c^2} + \frac{2v}{nc}}} \equiv \frac{c}{n + v} \left(1 + \frac{\sqrt{v^2 + \frac{2v}{nc}}}{2}\right)
\]

Ignoring the term \(v^2/c^2\) we have

\[
c' \equiv \frac{c}{n + v} \left(1 - \frac{v}{nc}\right) \equiv \frac{c}{n} + v - \frac{v}{n^2} - \frac{v^2}{nc}
\]

and ignoring the term \(v^2/nc\) we have the Fresnel’s formula

\[
c' = \frac{c}{n} + v - \frac{v}{n^2} = \frac{c}{n} + v \left(1 - \frac{1}{n^2}\right).
\]
Doppler effect

Making \( r^2 = x^2 + y^2 + z^2 \) and \( r'^2 = x'^2 + y'^2 + z'^2 \) in 1.5 we have \( r^2 - c^2 t^2 = r'^2 - c^2 t'^2 \) or \( (r - ct) = (r' - ct') (r' + ct') \) replacing then \( r = ct \), \( r' = ct' \) and 1.7 we find \( (r - ct) = (r' - ct') \sqrt{1 + \frac{v^2}{c^2} - \frac{2vx}{c^2 t}} \)

as \( c = \frac{w}{k} \) then \( \frac{1}{k} (kr - wt) = \frac{1}{k'} (k'r' - w't') \sqrt{1 + \frac{v^2}{c^2} - \frac{2vx}{c^2 t}} \) where to attend the principle of relativity we will define \( k' = k \sqrt{1 + \frac{v^2}{c^2} - \frac{2vx}{c^2 t}} \).

Resulting in the expression \( (kr - wt) = (k'r' - w't') \) symmetric and invariable between the observers.

To the observer O an expression in the formula of \( \psi(r, t) = f(kr - wt) \) represents a curve that propagates in the direction of \( \vec{R} \). To the observer O’ an expression in the formula of \( \psi'(r', t') = f(k'r' - w't') \) represents a curve that propagates in the direction of \( \vec{R}' \).

Applying in 2.18 \( k = \frac{2\pi}{\lambda} \), \( k' = \frac{2\pi}{\lambda'} \), 1.14, 1.19, 1.23, 2.5, and 2.6 we have

\[ \lambda' = \frac{\lambda}{\sqrt{K}}, \quad \lambda = \frac{\lambda'}{\sqrt{K'}} \]

that applied in \( c = y\lambda = y'\lambda' \) supply, \( y' = y\sqrt{K} \) and \( y = y'\sqrt{K'} \).

Considering the relation of Planck-Einstein between energy \( (E) \) and frequency \( (\gamma) \), we have to the observer O \( E = hy \) and to the observer O’ \( E' = hy' \) that replaced in 2.22 supply

\[ E' = E\sqrt{K} \quad \text{and} \quad E = E'\sqrt{K'}. \]

If the observer O that sees the observer O’ moving with velocity \( v \) in a positive way to the axis x, emits waves of frequency \( \gamma \) and velocity c in a positive way to the axis x then, according to 2.22 and \( ux = c \) the observer O’ will measure the waves with velocity c and frequency \( \gamma' = y \left( 1 - \frac{v}{c} \right) \), which is exactly the classic formula of the longitudinal Doppler effect.

If the observer O’ that sees the observer O moving with velocity \( -v' \) in the negative way of the axis x’, emits waves of frequency \( \gamma' \) and velocity c, then the observer O according to 2.22 and \( u'x' = -v' \) will measure waves of frequency \( \gamma \) and velocity c in a perpendicular plane to the movement of O’ given by

\[ \gamma = \gamma' \sqrt{1 - \frac{v'^2}{c^2}} \],

which is exactly the formula of the transversal Doppler effect in the Special Relativity.

§3 Transformations of the accelerations \( \ddot{a} \) and \( \ddot{a}' \)

Differentiating 2.1 and dividing it by 1.17 we have

\[ \frac{du}{dt} = \frac{du}{\sqrt{K}} + (\ddot{u} - \ddot{v}) \frac{v}{c^2} \frac{du}{\sqrt{K}} \quad \Rightarrow \quad \ddot{a} = \ddot{a}' + \left( \ddot{u} - \ddot{v} \right) \frac{v}{c^2} \frac{ax}{K'.K'^2}. \]

Differentiating 2.2 and dividing it by 1.22 we have

\[ \frac{du}{dt} = \frac{du}{\sqrt{K'}} - (\ddot{u}' - \ddot{v}') \frac{v'}{c^2} \frac{du'}{\sqrt{K'}} \quad \Rightarrow \quad \ddot{a} = \ddot{a}' - \left( \ddot{u}' - \ddot{v}' \right) \frac{v'}{c^2} \frac{d'x'}{K'.K'^2}. \]
### Table 3, transformations of the accelerations $\ddot{a}$ and $\ddot{a}'$

<table>
<thead>
<tr>
<th>Expression</th>
<th>Formula</th>
<th>Section</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\ddot{a}' = \frac{\ddot{a}}{K} + (\ddot{u} - \ddot{v}) \frac{v}{c^2} \frac{a_x}{K^2}$</td>
<td>$3.1$</td>
<td></td>
</tr>
<tr>
<td>$\ddot{a}' = \frac{\ddot{a}}{K} - (\dddot{u}' - \dddot{v}') \frac{v'}{c^2} \frac{a'_x}{K'^2}$</td>
<td>$3.2$</td>
<td></td>
</tr>
<tr>
<td>$a'x' = \frac{ax}{K} + (ux - v) \frac{v}{c^2} \frac{a_x}{K^2}$</td>
<td>$3.3$</td>
<td></td>
</tr>
<tr>
<td>$a'y' = \frac{ay}{K} + uy \frac{v}{c^2} \frac{a_x}{K^2}$</td>
<td>$3.3.1$</td>
<td></td>
</tr>
<tr>
<td>$a'z' = \frac{az}{K} + uz \frac{v}{c^2} \frac{a_x}{K^2}$</td>
<td>$3.3.2$</td>
<td></td>
</tr>
<tr>
<td>$a' = \frac{a}{K}$</td>
<td>$3.8$</td>
<td></td>
</tr>
<tr>
<td>$K' = \frac{1}{1 + \frac{\gamma}{c^2} - \frac{2vux}{c^2}}$</td>
<td>$3.5$</td>
<td></td>
</tr>
<tr>
<td>$K' = \frac{1}{1 + \frac{\gamma^2}{c^2} + \frac{2v'y'x'}{c^2}}$</td>
<td>$3.6$</td>
<td></td>
</tr>
</tbody>
</table>

From the tables 2 and 3 we can conclude that if to the observer $O$ $\ddot{u}, \dddot{a} = \text{zero}$ and $c^2 = ux^2 + uy^2 + uz^2$, then it is also to the observer $O'$ $\ddot{u'}, \dddot{a} = \text{zero}$ and $c^2 = u'x'^2 + u'y'^2 + u'z'^2$, thus $\dddot{u}$ is perpendicular to $\dddot{a}$ and $\dddot{u'}$ is perpendicular to $\dddot{a'}$ as the vectors theory requires.

Differentiating 1.9 with the velocities and the times changing we have, $td\ddot{v} + vdt = t'd\dddot{v'} + v'dt'$, but considering 1.16 we have, $vdt = v'dt' \Rightarrow t\ddot{v} + vdt = t\ddot{v'} + v'dt'$. Where replacing 1.15 and dividing it by 1.17 we have, $\frac{dv'}{dt'} = \frac{dv}{dtK'}$ or $a' = \frac{a}{K'}$. Where we can also replace 1.20 in 3.7 and divide it by 1.22 deducing $\frac{dv}{dt} = \frac{dv'}{dt'K'}$ or $a = \frac{a'}{K'}$.

The direct relations between the modules of the accelerations $a$ and $a'$ of two points in space can be obtained with the $\dddot{u} = 0 \Rightarrow u'x' = 0 \Rightarrow a'x' = 0 \Rightarrow \dddot{u} = \ddot{v} \Rightarrow ux = v$ coming from 2.1, that applied in 3.8 and 3.9 supply

$$a' = \frac{a}{1 + \frac{\gamma v^2}{c^2} - \frac{2vux}{c^2}} \text{ and } a = \frac{a'}{1 + \frac{\gamma^2 v^2}{c^2} + \frac{2v'y'x'}{c^2}} = \frac{a'}{1 + \frac{\gamma^2 v^2}{c^2}}.$$  

That can also be reduced from 3.1 and 3.2 if we use the same equalities

$\dddot{u} = 0 \Rightarrow u'x' = 0 \Rightarrow a'x' = 0 \Rightarrow \dddot{u} = \dddot{v} \Rightarrow ux = v$ coming from 2.1.

## §4 Transformations of the Moments $\ddot{p}$ and $\ddot{p'}$

Defined as $\ddot{p} = m(u)\dddot{u}$ and $\ddot{p'} = m'(u')\dddot{u'}$, where $m(u)$ and $m'(u')$ symbolizes the function masses of the modules of velocities $u = \|\dddot{u}\|$ and $u' = \|\dddot{u'}\|$.

We will have the relations between $m(u)$ and $m'(u')$ and the resting mass $m_0$, analyzing the elastic collision in a plane between the sphere $s$ that for the observer $o$ moves alongside the axis $y$ with velocity $uy = w$ and the sphere $s'$ that for the observer $O'$ moves alongside the axis $y'$ with velocity $u'y' = -w$. The spheres while observed in relative resting are identical and have the mass $m_0$. The considered collision is symmetric in relation to a parallel line to the axis $y$ and $y'$ passing by the center of the spheres in the moment of Collision.

Before and after the collision the spheres have velocities observed by O and O' according to the following table gotten from table 2.
becomes

\[ u'x's = -v', \quad u'y's = w\sqrt{1 - \frac{v'^2}{c^2}} \]

To the observer O, the principle of conservation of moments establishes that the moments \( px = m(u)ux \) and \( py = m(u)uy \), of the spheres \( s \) and \( s' \) in relation to the axis x and y, remain constant before and after the collision thus for the axis x we have

\[ m\left(\sqrt{uxs^2 + uys^2}\right)uxs + m\left(\sqrt{uxs'^2 + uys'^2}\right)uxs' = m\left(\sqrt{uxs^2 + uys^2}\right)uxs + m\left(\sqrt{uxs'^2 + uys'^2}\right)uxs', \]

where replacing the values of the table we have

\[ m\left(\sqrt{v^2 + \left(-w\sqrt{1 - \frac{v^2}{c^2}}\right)^2}\right)v = m\left(\sqrt{v^2 + \left(\bar{w}\sqrt{1 - \frac{v^2}{c^2}}\right)^2}\right)v \quad \text{from where we conclude that} \quad \bar{w} = w, \]

and for the axis y

\[ m\left(\sqrt{uxs^2 + uys^2}\right)uys + m\left(\sqrt{uxs'^2 + uys'^2}\right)uys' = m\left(\sqrt{uxs^2 + uys^2}\right)uys + m\left(\sqrt{uxs'^2 + uys'^2}\right)uys', \]

where replacing the values of the table we have

\[ m(w)w - m\left(\sqrt{v^2 + \left(-w\sqrt{1 - \frac{v^2}{c^2}}\right)^2}\right)w\sqrt{1 - \frac{v^2}{c^2}} = -m(\bar{w})\bar{w} + m\left(\sqrt{v^2 + \left(\bar{w}\sqrt{1 - \frac{v^2}{c^2}}\right)^2}\right)\bar{w}\sqrt{1 - \frac{v^2}{c^2}}, \]

simplifying we have

\[ m(w) = m\left(\sqrt{v^2 + w^2\left(1 - \frac{v^2}{c^2}\right)}\right)\sqrt{1 - \frac{v^2}{c^2}}, \quad \text{where when} \quad w \to 0 \quad \text{becomes} \]

\[ m(0) = m\left(\sqrt{v^2 + 0^2\left(1 - \frac{v^2}{c^2}\right)}\right)\sqrt{1 - \frac{v^2}{c^2}} \Rightarrow m(0) = m(v)\sqrt{1 - \frac{v^2}{c^2}} \Rightarrow m(v) = \frac{m(0)}{\sqrt{1 - \frac{v^2}{c^2}}}, \]

but \( m(0) \) is equal to the resting mass \( m_0 \) thus

\[ m(v) = \frac{m_0}{\sqrt{1 - \frac{v^2}{c^2}}}, \quad \text{with a relative velocity} \quad v = u \Rightarrow m(u) = \frac{m_0}{\sqrt{1 - \frac{u^2}{c^2}}}. \]

that applied in 4.1 supplies \( \vec{p} = m(u)\vec{u} = \frac{m_0\vec{u}}{\sqrt{1 - \frac{u^2}{c^2}}} \).

With the same procedures we would have for the O’ observer
\[ m'(u') = \frac{m_0}{\sqrt{1 - \frac{u'^2}{c^2}}} \]

and \[ \vec{p}' = m'(u')\vec{u}' = \frac{m_0\vec{u}'}{\sqrt{1 - \frac{u'^2}{c^2}}} \].

Simplifying the simbology we will adopt \[ m = m(u) = \frac{m_0}{\sqrt{1 - \frac{u^2}{c^2}}} \]

and \[ m' = m'(u') = \frac{m_0}{\sqrt{1 - \frac{u'^2}{c^2}}} \]

that simplify the moments in \[ \vec{p} = m\vec{u} \] and \[ \vec{p}' = m'\vec{u}' \].

Applying 4.2 and 4.3 in 2.9 and 2.10 we have

\[ \vec{F} = \frac{d\vec{p}}{dt} = \frac{d(m\vec{u})}{dt} \] and \[ \vec{F}' = \frac{d\vec{p}'}{dt'} = \frac{d(m'\vec{u}')}{dt'} \], with this we can define then kinetic energy \( (E_k, E'_k) \) as

\[ E_k = \int_0^u \vec{F}.d\vec{R} = \int_0^u \frac{d(m\vec{u})}{dt}.d\vec{R} = \int_0^u (u^2 dm + mdu), \]

and \[ E'_k = \int_0^u \vec{F}'.d\vec{R}' = \int_0^u \frac{d(m'\vec{u}')}{dt'}.d\vec{R}' = \int_0^u (u'^2 dm' + m'u'du'). \]

Remodeling 4.2 and 4.3 and differentiating we have \[ m^2c^2 - m^2u^2 = m_0^2c^2 \Rightarrow u^2 dm + mdu = c^2 dm \] and \[ m'^2c^2 - m'^2u'^2 = m_0^2c^2 \Rightarrow u'^2 dm' + m'u'du' = c^2 dm' \], that applying in the formulas of kinetic energy supplies \[ E_k = \int_m^{m_0} c^2 dm = mc^2 - m_0c^2 = E - E_0 \] and \[ E'_k = \int_{m_0}^m c^2 dm' = m'^2c^2 - m_0c^2 = E' - E_0 \],

where \( E = mc^2 \) and \( E' = m'c^2 \)

are the total energies as in the special relativity and \( E_o = m_0c^2 \)

the resting energy.

Applying 4.6 in 4.4 we have exactly 2.23.

From 4.6, 4.2, 4.3, and 4.1 we find

\[ E = c\sqrt{m_0^2c^2 + p^2} \] and \[ E' = c\sqrt{m_0^2c^2 + p'^2} \]

identical relations to the Special Relativity.

Multiplying 2.1 and 2.2 by \( m_0 \) we get
\[ \frac{m \ddot{u}'}{\sqrt{1 - \frac{u'^2}{c^2}}} = \frac{m \ddot{u}}{\sqrt{1 - \frac{u^2}{c^2}}} - \frac{m \ddot{v}}{\sqrt{1 - \frac{u^2}{c^2}}} \Rightarrow m' \ddot{u}' = m \ddot{u} - m \ddot{v} \Rightarrow \ddot{p}' = \ddot{\bar{p}} - \frac{E}{c^2} \dddot{v} \]  
\[ \text{4.9} \]

and \[ \frac{m \ddot{u}'}{\sqrt{1 - \frac{u'^2}{c^2}}} \sqrt{1 - \frac{u'^2}{c^2}} \Rightarrow m' \ddot{u}' - m' \ddot{v}' \Rightarrow \ddot{p}' = \dddot{\bar{p}} - \frac{E'}{c^2} \dddot{v}' . \]  
\[ \text{4.10} \]

Table 4, transformations of moments \( \ddot{\bar{p}} \) and \( \ddot{\bar{p}}' \)

<table>
<thead>
<tr>
<th>Transformation</th>
<th>Formula</th>
<th>Notes</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \ddot{p}' = \ddot{\bar{p}} - \frac{E}{c^2} \dddot{v} )</td>
<td>4.9</td>
<td></td>
</tr>
<tr>
<td>( p' x' = px - \frac{E}{c^2} v )</td>
<td>4.11</td>
<td></td>
</tr>
<tr>
<td>( p' y' = py )</td>
<td>4.11.1</td>
<td></td>
</tr>
<tr>
<td>( p' z' = pz )</td>
<td>4.11.2</td>
<td></td>
</tr>
<tr>
<td>( E' = E \sqrt{K} )</td>
<td>2.23</td>
<td></td>
</tr>
<tr>
<td>( m = m(u) = \frac{m_0}{\sqrt{1 - \frac{u^2}{c^2}}} )</td>
<td>4.2</td>
<td>From ( \text{4.2} ) where ( u = v )</td>
</tr>
<tr>
<td>( m' = m'(u') = \frac{m_0}{\sqrt{1 - \frac{u'^2}{c^2}}} )</td>
<td>4.3</td>
<td></td>
</tr>
<tr>
<td>( E_k = E - E_o )</td>
<td>4.5</td>
<td></td>
</tr>
<tr>
<td>( E'_k = E' - E_o )</td>
<td>4.5</td>
<td>From ( \text{4.5} )</td>
</tr>
<tr>
<td>( E = mc^2 )</td>
<td>4.6</td>
<td></td>
</tr>
<tr>
<td>( E' = m'c^2 )</td>
<td>4.6</td>
<td>From ( \text{4.6} )</td>
</tr>
<tr>
<td>( E_o = m_o c^2 )</td>
<td>4.7</td>
<td></td>
</tr>
<tr>
<td>( E' = m_o c^2 )</td>
<td>4.7</td>
<td>From ( \text{4.7} )</td>
</tr>
<tr>
<td>( E = c \sqrt{m_o^2 c^4 + p^2} )</td>
<td>4.8</td>
<td></td>
</tr>
<tr>
<td>( E' = c \sqrt{m_o^2 c^4 + p'^2} )</td>
<td>4.8</td>
<td>From ( \text{4.8} )</td>
</tr>
</tbody>
</table>

Wave equation of Louis de Broglie

The observer \( O' \) associates to a resting particle in its origin the following properties:

- Resting mass \( m_o \)
- Time \( t' = t_o \)
- Resting Energy \( E_o = m_o c^2 \)
- Frequency \( \nu_o = \frac{E_o}{h} = \frac{m_o c^2}{h} \)
- Wave function \( \psi_o = a \text{sen} 2\pi \nu_o t_o \) with \( a = \text{constant} \).

The observer \( O \) associates to a particle with velocity \( v \) the following:

- Mass \( m = m(v) = \frac{m_o}{\sqrt{1 - \frac{v^2}{c^2}}} \) (from \( \text{4.2} \) where \( u = v \))
- Time \( t = \frac{t_o}{\sqrt{1 + \frac{v^2}{c^2} - \frac{2vv}{c^2}}} = \frac{t_o}{\sqrt{1 - \frac{v^2}{c^2}}} \) (from \( \text{1.7} \) with \( ux = v \) and \( t' = t_o \))
- Energy \( E = \frac{E_o}{\sqrt{1 - \frac{v^2}{c^2}}} = \frac{m_o c^2}{\sqrt{1 - \frac{v^2}{c^2}}} \) (from \( \text{2.23} \) with \( ux = v \) and \( E' = E_o \))
- Frequency \( \nu = \frac{y_o}{\sqrt{1 - \frac{v^2}{c^2}}} \) (from 2.22 with \( u_x = v \) and \( y' = y_o \))

- Distance \( x = vt \) (from 1.2 with \( x' = 0 \))

- Wave function \( \psi = \text{asen} 2\pi \sqrt{1 - \frac{v^2}{c^2}}t \sqrt{1 - \frac{v^2}{c^2}} = \text{asen} 2\pi y \left( \frac{t - x}{u} \right) \) with \( u = \frac{c^2}{v} \)

- Wave length \( \lambda = \frac{E}{h} = \frac{y h}{\nu} \rightarrow \lambda = \frac{h}{p} \) (from 4.9 with \( \tilde{p}' = \tilde{p}_o = 0 \))

To go back to the O' observer referential where \( \tilde{u}' = 0 \Rightarrow u' x' = 0 \), we will consider the following variables:

- Distance \( x = v't' \) (from 1.4 with \( x' = 0 \))

- Time \( t = t' \sqrt{1 + \frac{v'^2}{c^2}} \) (from 1.8 with \( u' x' = 0 \))

- Frequency \( \nu = \nu' \sqrt{1 + \frac{v'^2}{c^2}} \) (from 2.22 with \( u' x' = 0 \))

- Velocity \( v = \frac{v'}{c^2} \) (de 2.13)

that applied to the wave function supplies

\[
\psi' = \text{asen} 2\pi y \left( \frac{t - v x}{c^2} \right) = \text{asen} 2\pi y' \left( \frac{t' - \frac{v'^2}{c^2} t'}{c^2} \right) = \text{asen} 2\pi y' t',
\]

but as \( t' = t_o \) and \( y' = y_o \) then \( \psi' = \psi_o \).

§5 Transformations of the Forces \( \tilde{F} \) and \( \tilde{F}' \)

Differentiating 4.9 and dividing by 1.17 we have

\[
\frac{d\tilde{p}'}{dt'} = \frac{d\tilde{p}}{dt} - \frac{dE}{dt \sqrt{K}} \frac{\tilde{v}}{c^2} \Rightarrow \tilde{F}' = \frac{1}{\sqrt{K}} \left[ \tilde{F} - \frac{dE}{dt} \tilde{v} \right] \Rightarrow \tilde{F}' = \frac{1}{\sqrt{K}} \left[ \tilde{F} - \tilde{(\tilde{F}, \tilde{u})} \frac{\tilde{v}}{c^2} \right]. \tag{5.1}
\]

Differentiating 4.10 and dividing by 1.22 we have

\[
\frac{d\tilde{p}}{dt} = \frac{d\tilde{p}'}{dt' \sqrt{K'}} - \frac{dE'}{dt' \sqrt{K'}} \frac{\tilde{v}'}{c^2} \Rightarrow \tilde{F} = \frac{1}{\sqrt{K'}} \left[ \tilde{F}' - \frac{dE'}{dt'} \tilde{v}' \right] \Rightarrow \tilde{F} = \frac{1}{\sqrt{K'}} \left[ \tilde{F}' - \tilde{(\tilde{F}', \tilde{u}') \frac{\tilde{v}'}{c^2}} \right]. \tag{5.2}
\]

From the system formed by 5.1 and 5.2 we have

\[
\frac{dE}{dt} = \frac{dE'}{dt'} \text{ or } \tilde{F}, \tilde{u} = \tilde{F}', \tilde{u}', \tag{5.3}
\]

that is an invariant between the observers in the Undulating Relativity.
Table 5, transformations of the Forces \( \vec{F} \) and \( \vec{F}' \)

<table>
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<tr>
<th>Formula</th>
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<tbody>
<tr>
<td>( \vec{F}' = \frac{1}{\sqrt{K}} \left[ \vec{F} - (\vec{F} \cdot \vec{u}) \vec{v} \right] )</td>
<td>5.1</td>
</tr>
<tr>
<td>( \vec{F} = \frac{1}{\sqrt{K'}} \left[ \vec{F}' - (\vec{F}' \cdot \vec{u}') \vec{v}' \right] )</td>
<td>5.2</td>
</tr>
<tr>
<td>( F' x' = \frac{1}{\sqrt{K}} \left[ F x - (\vec{F} \cdot \vec{u}) v \right] )</td>
<td>5.4</td>
</tr>
<tr>
<td>( F x = \frac{1}{\sqrt{K'}} \left[ F' x' + (\vec{F}' \cdot \vec{u}') v' \right] )</td>
<td>5.5</td>
</tr>
<tr>
<td>( F' y' = F y / \sqrt{K} )</td>
<td>5.4.1</td>
</tr>
<tr>
<td>( F y = F' y' / \sqrt{K'} )</td>
<td>5.5.1</td>
</tr>
<tr>
<td>( F' z' = F z / \sqrt{K} )</td>
<td>5.4.2</td>
</tr>
<tr>
<td>( F z = F' z' / \sqrt{K'} )</td>
<td>5.5.2</td>
</tr>
<tr>
<td>( \frac{dE'}{dt'} = \frac{dE}{dt} )</td>
<td>5.3</td>
</tr>
<tr>
<td>( \vec{F} \cdot \vec{u} = \vec{F}' \cdot \vec{u}' )</td>
<td>5.3</td>
</tr>
</tbody>
</table>

§6 Transformations of the density of charge \( \rho \), \( \rho' \) and density of current \( \vec{J} \) and \( \vec{J}' \)

Multiplying 2.1 and 2.2 by the density of the resting electric charge defined as \( \rho_o = \frac{dq}{dv_o} \), we have

\[
\frac{\rho_o \ddot{u}'}{\sqrt{1 - \frac{u'^2}{c^2}}} - \frac{\rho_o \dot{v}}{\sqrt{1 - \frac{u^2}{c^2}}} \Rightarrow \rho' \ddot{u}' = \rho \ddot{u} - \rho \ddot{v} \Rightarrow \vec{J}' = \vec{J} - \rho \ddot{v} \]

6.1

and

\[
\frac{\rho_o \ddot{u}}{\sqrt{1 - \frac{u'^2}{c^2}}} - \frac{\rho_o \dot{v}'}{\sqrt{1 - \frac{u'^2}{c^2}}} \Rightarrow \rho' \ddot{u}' = \rho' \ddot{u}' - \rho' \ddot{v}' \Rightarrow \vec{J}' = \vec{J} - \rho' \ddot{v}'.
\]

6.2

Table 6, transformations of the density of charges \( \rho \), \( \rho' \) and density of current \( \vec{J} \) and \( \vec{J}' \)

<table>
<thead>
<tr>
<th>Formula</th>
<th>Page</th>
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<tbody>
<tr>
<td>( \vec{J}' = \vec{J} - \rho \ddot{v} )</td>
<td>6.1</td>
</tr>
<tr>
<td>( \vec{J}' = \vec{J} - \rho' \ddot{v}' )</td>
<td>6.2</td>
</tr>
<tr>
<td>( J' x' = J x - \rho v )</td>
<td>6.3</td>
</tr>
<tr>
<td>( J x = J' x' + \rho \dot{v}' )</td>
<td>6.4</td>
</tr>
<tr>
<td>( J' y' = J y )</td>
<td>6.3.1</td>
</tr>
<tr>
<td>( J y = J' y' )</td>
<td>6.4.1</td>
</tr>
<tr>
<td>( J' z' = J z )</td>
<td>6.3.2</td>
</tr>
<tr>
<td>( J z = J' z' )</td>
<td>6.4.2</td>
</tr>
<tr>
<td>( \vec{J} = \rho \ddot{u} )</td>
<td>6.5</td>
</tr>
<tr>
<td>( \vec{J}' = \rho' \ddot{u}' )</td>
<td>6.6</td>
</tr>
<tr>
<td>( \rho = \frac{\rho_o}{\sqrt{1 - \frac{u^2}{c^2}}} )</td>
<td>6.7</td>
</tr>
<tr>
<td>( \rho' = \frac{\rho_o}{\sqrt{1 - \frac{u'^2}{c^2}}} )</td>
<td>6.8</td>
</tr>
<tr>
<td>( \rho' = \rho \sqrt{K'} )</td>
<td>6.9</td>
</tr>
<tr>
<td>( \rho = \rho' \sqrt{K'} )</td>
<td>6.10</td>
</tr>
</tbody>
</table>

From the system formed by 6.1 and 6.2 we had 6.9 and 6.10.

§7 Transformation of the electric fields \( \vec{E} \), \( \vec{E}' \) and magnetic fields \( \vec{B} \), \( \vec{B}' \)

Applying the forces of Lorentz \( \vec{F} = q(\vec{E} + \vec{u} \times \vec{B}) \) and \( \vec{F}' = q(\vec{E}' + \vec{u}' \times \vec{B}') \) in 5.1 and 5.2 we have

\[
q(\vec{E}' + \ddot{u}' \times \vec{B}') = \frac{1}{\sqrt{K}} \left[ q(\vec{E} + \vec{u} \times \vec{B}) - q(\vec{E} + \vec{u} \times \vec{B}) \frac{\vec{v}}{c^2} \right]
\]

and

\[
q(\vec{E} + \ddot{u} \times \vec{B}) = \frac{1}{\sqrt{K'}} \left[ q(\vec{E}' + \vec{u}' \times \vec{B}') - q(\vec{E}' + \vec{u}' \times \vec{B}') \frac{\vec{v}'}{c^2} \right],
\]

that simplified become

\[
(\vec{E}' + \ddot{u}' \times \vec{B}') = \frac{1}{\sqrt{K}} \left[ (\vec{E} + \ddot{u} \times \vec{B}) - (\vec{E} \cdot \vec{u}) \frac{\vec{v}}{c^2} \right]
\]

and

\[
(\vec{E} + \ddot{u} \times \vec{B}) = \frac{1}{\sqrt{K'}} \left[ (\vec{E}' + \ddot{u}' \times \vec{B}') - (\vec{E}' \cdot \vec{u}') \frac{\vec{v}'}{c^2} \right]
\]

from where we get the invariance of \( \vec{E} \cdot \vec{u} = \vec{E}' \cdot \vec{u}' \) between the observers as a consequence of 5.3 and the following components of each axis
\[ E' \times u' y' B' z' - u' z' B' y' = \frac{1}{\sqrt{K}} \left[ E_x + u y B_z - u z B_y - \frac{E_{\text{uvv}}}{c^2} - \frac{E_{\text{yyv}}}{c^2} - \frac{E_{\text{uzv}}}{c^2} \right] \]

\[ E' y' + u' z' B' x' - u' x' B' z' = \frac{1}{\sqrt{K}} \left[ E_y + u z B x - u x B z \right] \]

\[ E' z' + u' x' B' y' - u' y' B' x' = \frac{1}{\sqrt{K}} \left[ E_z + u x B y - u y B x \right] \]

\[ \text{Ex} + u y B z - u z B y = \frac{1}{\sqrt{K}} \left[ E' x' + u' y' B' z' - u' z' B' y' + \frac{E' x' u' x' v'}{c^2} + \frac{E' y' u' y' v'}{c^2} + \frac{E' z' u' z' v'}{c^2} \right] \]

\[ E y + u z B x - u x B z = \frac{1}{\sqrt{K'}} \left[ E' y' + u' z' B' x' - u' x' B' z' \right] \]

\[ E z + u x B y - u y B x = \frac{1}{\sqrt{K'}} \left[ E' z' + u' x' B' y' - u' y' B' x' \right] \]

To the conjunct 7.1 and 7.2 we have two solutions described in the tables 7 and 8.

| Table 7, transformations of the electric fields $\vec{E}$, $\vec{E}'$ and magnetic fields $\vec{B}$ e $\vec{B}'$ |
|---------------------------------------------|---------------------------------------------|---------------------------------------------|
| $E' x'$ = $\frac{\text{Ex}}{\sqrt{K}} \left( 1 - \frac{\text{vux}}{c^2} \right)$ | $E' y'$ = $\frac{\text{Ey}}{\sqrt{K}} \left( 1 + \frac{\text{vux}}{c^2} \right) - \frac{\text{vBz}}{\sqrt{K}}$ | $E' z'$ = $\frac{\text{Ez}}{\sqrt{K}} \left( 1 + \frac{\text{vux}}{c^2} \right) + \frac{\text{vBy}}{\sqrt{K}}$ |
| $\text{Ex} = \frac{E' x'}{\sqrt{K'}} \left( 1 + \frac{\text{v'ux'}}{c^2} \right)$ | $E y$ = $\frac{E' y'}{\sqrt{K'}} \left( 1 + \frac{\text{v'ux'}}{c^2} \right) + \frac{\text{v'z'}}{\sqrt{K'}}$ | $E z$ = $\frac{E' z'}{\sqrt{K'}} \left( 1 + \frac{\text{v'ux'}}{c^2} \right) - \frac{\text{v'B'y'}}{\sqrt{K'}}$ |

| $B' x'$ = $Bx$ | $B' y'$ = $By + \frac{v}{c^2} Ez$ | $B' z'$ = $Bz - \frac{v}{c^2} Ey$ |
| $Bx = B' x'$ | $By = B' y' - \frac{v'}{c^2} E' z'$ | $Bz = B' z' + \frac{v'}{c^2} E' y'$ |

| $E' y'$ = $E y \sqrt{K}$ | $E y$ = $E' y' \sqrt{K'}$ | $E z'$ = $E z' \sqrt{K'}$ |
| $By = -\frac{ux}{c^2} Ez$ | $B' y'$ = $-\frac{u'x'}{c^2} E' z'$ | $Bz = \frac{ux}{c^2} Ey$ |

| $B' x'$ = $Bx$ | $B' y'$ = $By$ | $B' z'$ = $Bz$ |

| Table 8, transformations of the electric fields $\vec{E}$, $\vec{E}'$ and magnetic fields $\vec{B}$ e $\vec{B}'$ |
|-----------------|-----------------|-----------------|
| $E' x'$ = $\frac{1}{\sqrt{K}} \left[ E_x - (\vec{E} \cdot \vec{u'}) \frac{v}{c^2} \right]$ | $E' y'$ = $\frac{1}{\sqrt{K}} \left( E_y - v B z \right)$ | $E' z'$ = $\frac{1}{\sqrt{K}} \left( E_z + v B y \right)$ |
| $\text{Ex} = \frac{1}{\sqrt{K'}} \left[ E' x' + (\vec{E}' \cdot \vec{u'}) \frac{v'}{c^2} \right]$ | $\text{Ey} = \frac{1}{\sqrt{K'}} \left( E' y' + v' B' z' \right)$ | $\text{Ez} = \frac{1}{\sqrt{K'}} \left( E' z' - v' B' y' \right)$ |

| $B' x'$ = $Bx$ | $B' y'$ = $By$ | $B' z'$ = $Bz$ |
Relation between the electric field and magnetic field

If an electric-magnetic field has to the observer $O'$ the naught magnetic component $\vec{B}' = \text{zero}$ and the electric component $\vec{E}'$. To the observer $O$ this field is represented with both components, being the magnetic field described by the conjunct 7.5 and has as components

$$B_x = \text{zero}, \quad B_y = -\frac{vE_z}{c^2}, \quad B_z = \frac{vE_y}{c^2}, \quad 7.15$$

that are equivalent to $\vec{B} = \frac{I}{c^2} \vec{v} \times \vec{E}$. \quad 7.16

Formula of Biot-Savart

The observer $O'$ associates to a resting electric charge, uniformly distributed alongside its axis $x'$ the following electric-magnetic properties:

- Linear density of resting electric charge $\rho_o = \frac{dq}{dx'}$
- Naught electric current $I' = \text{zero}$
- Naught magnetic field $\vec{B}' = \text{zero} \Rightarrow \vec{u}' = \text{zero}$
- Radial electrical field of module $E' = \sqrt{E'_x^2 + E'_y^2 + E'_z^2}$ at any point of radius $R = \sqrt{y'^2 + z'^2}$ with the component $E'_x = \text{zero}$.

To the observer $O$ it relates to an electric charge uniformly distributed alongside its axis with velocity $ux = v$ to which it associates the following electric-magnetic properties:

- Linear density of the electric charge $\rho = \frac{\rho_o}{\sqrt{1 - \frac{v^2}{c^2}}}$ (from 6.7 with $u = v$)
- Electric current $I = \rho v = \frac{\rho_o v}{\sqrt{1 - \frac{v^2}{c^2}}}$
- Radial electrical field of module $E = \frac{E'}{\sqrt{1 - \frac{v^2}{c^2}}}$ (according to the conjuncts 7.3 and 7.5 with $\vec{B}' = \text{zero} \Rightarrow \vec{u}' = \text{zero}$ and $ux = v$)
- Magnetic field of components $B_x = \text{zero}, \quad B_y = -\frac{vE_z}{c^2}, \quad B_z = \frac{vE_y}{c^2}$ and module
  $$B = \frac{vE}{c^2} = \frac{vE'}{c^2} \frac{E'}{c^2} = \frac{v}{c^2} \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}} \frac{\rho_o}{\sqrt{2 \pi \epsilon_o R}} = \frac{\mu_o I}{2 \pi R} \quad \text{where} \quad \mu_o = \frac{1}{\epsilon_o c^2}, \quad \text{being in the vectorial form}$$
  $$\vec{B} = \frac{\mu_o I}{2 \pi R} \vec{u} \quad 7.17$$

where $\vec{u}$ is a unitary vector perpendicular to the electrical field $\vec{E}$ and tangent to the circumference that passes by the point of radius $R = \sqrt{y^2 + z^2}$ because from the conjunct 7.4 and 7.6 $\vec{E} \times \vec{B} = \text{zero}$. 

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§8 Transformations of the differential operators

Table 9, differential operators

<table>
<thead>
<tr>
<th>Operator</th>
<th>Formula</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \frac{\partial}{\partial x'} )</td>
<td>( = \frac{\partial}{\partial x} + \frac{v}{c^2} \frac{\partial}{\partial t} )</td>
</tr>
<tr>
<td>( \frac{\partial}{\partial y'} )</td>
<td>( = \frac{\partial}{\partial y} )</td>
</tr>
<tr>
<td>( \frac{\partial}{\partial z'} )</td>
<td>( = \frac{\partial}{\partial z} )</td>
</tr>
</tbody>
</table>

\[ \begin{align*}
\frac{\partial}{\partial t'} &= \frac{\partial}{\partial t} + \frac{x/t}{c^2} \frac{\partial}{\partial t} \\
\frac{\partial}{\partial t'} &= \frac{\partial}{\partial t} + \frac{x'/t'}{c^2} \frac{\partial}{\partial t'}
\end{align*} \]  

8.1

From the system formed by 8.1, 8.2, 8.3, and 8.4 and with 1.15 and 1.20 we only find the solutions

\[ \begin{align*}
\frac{\partial}{\partial x} + \frac{x/t}{c^2} \frac{\partial}{\partial t} &= 0 \quad \text{and} \quad \frac{\partial}{\partial x'} + \frac{x'/t'}{c^2} \frac{\partial}{\partial t'} &= 0.
\end{align*} \]  

8.5

From where we conclude that only the functions \( \psi \) (2.19) and \( \psi' \) (2.20) that supply the conditions

\[ \begin{align*}
\frac{\partial \psi}{\partial x} + \frac{x/t}{c^2} \frac{\partial \psi}{\partial t} &= 0 \quad \text{and} \quad \frac{\partial \psi'}{\partial x'} + \frac{x'/t'}{c^2} \frac{\partial \psi'}{\partial t'} &= 0,
\end{align*} \]  

8.6

can represent the propagation with velocity \( c \) in the Undulating Relativity indicating that the field propagates with definite velocity and without distortion being applied to 1.13 and 1.18. Because of symmetry we can also write to the other axis

\[ \begin{align*}
\frac{\partial \psi}{\partial y} + \frac{y/t}{c^2} \frac{\partial \psi}{\partial t} &= 0, \quad \frac{\partial \psi'}{\partial y'} + \frac{y'/t'}{c^2} \frac{\partial \psi'}{\partial t'} &= 0, \quad \frac{\partial \psi}{\partial z} + \frac{z/t}{c^2} \frac{\partial \psi}{\partial t} &= 0, \quad \frac{\partial \psi'}{\partial z'} + \frac{z'/t'}{c^2} \frac{\partial \psi'}{\partial t'} &= 0.
\end{align*} \]  

8.7

From the transformations of space and time of the Undulatory Relativity we get to Jacob’s theorem

\[ \begin{align*}
J &= \frac{\partial (x', y', z', t')}{\partial (x, y, z, t)} = \frac{1 + \frac{vu}{c^2}}{\sqrt{K}} \quad \text{and} \quad J' &= \frac{\partial (x', y', z', t')}{\partial (x, y, z, t)} = \frac{1 + \frac{v' u' x'}{c^2}}{\sqrt{K'}},
\end{align*} \]  

8.8

variables with \( ux \) and \( u'x' \) as a consequence of the principle of contancy of the light velocity but are equal as \( J = J' \) and will be equal to one \( J = J' = 1 \) when \( ux = u'x' = c \).

Invariance of the wave equation

The wave equation to the observer \( O' \) is

\[ \frac{\partial^2}{\partial x'^2} + \frac{\partial^2}{\partial y'^2} + \frac{\partial^2}{\partial z'^2} - \frac{1}{c^2} \frac{\partial^2}{\partial t'^2} = \text{zero} \]

where applying to the formulas of tables 9 and 1.13 we get

\[ \left( \frac{\partial}{\partial x} + \frac{v}{c^2} \frac{\partial}{\partial t} \right)^2 \frac{\partial^2}{\partial y'^2} + \frac{\partial^2}{\partial z'^2} - \frac{1}{c^2} \frac{\partial^2}{\partial t'^2} = \frac{v}{\sqrt{K}} \frac{\partial}{\partial x} + \frac{1}{\sqrt{K}} \left( 1 + \frac{vu}{c^2} \right) \frac{\partial}{\partial t} \]

from where we find
\[ K \frac{\partial^2}{\partial x^2} + K \frac{\partial^2}{\partial y^2} + K \frac{\partial^2}{\partial z^2} - \frac{1}{c^2} \frac{\partial^2}{\partial t^2} + 2v \frac{\partial^2}{\partial x^2} + 2v^2 \frac{\partial^2}{\partial x^2 \partial t^2} - 4v^2 \frac{\partial^2}{c^4 \partial t^4} + v^2 \frac{\partial^2}{\partial t^2} + v^2 \frac{\partial^2}{\partial t^2} - 2v'^{\prime} \frac{\partial^2}{\partial t^2} = 0 \]

that simplifying supplies

\[ K \frac{\partial^2}{\partial x^2} + K \frac{\partial^2}{\partial y^2} + K \frac{\partial^2}{\partial z^2} - \frac{1}{c^2} + 2v \frac{\partial^2}{\partial x^2} + 2v^2 \frac{\partial^2}{\partial x^2 \partial t^2} - \frac{v^2}{c^4} + \frac{v^2}{c^4} \frac{\partial^2}{\partial t^2} - \frac{v^2}{c^6} \frac{\partial^2}{\partial t^2} = 0 \]

where reordering the terms we find

\[ K \frac{\partial^2}{\partial x^2} + K \frac{\partial^2}{\partial y^2} + K \frac{\partial^2}{\partial z^2} + \left( 1 + \frac{v^2}{c^2} \right) \frac{\partial^2}{\partial t^2} \frac{v^2}{c^2} \frac{\partial^2}{\partial x^2} + \frac{v^2}{c^4} \frac{\partial^2}{\partial t^4} + \frac{v^2}{c^4} \frac{\partial^2}{\partial t^2} + \frac{2ux}{c^2} \frac{\partial^2}{\partial x^2} + \frac{c^2}{c^4} \frac{\partial^2}{\partial t^2} = 0 \]

where 8.9 and 1.13 we have

\[ \frac{\partial}{\partial x} \frac{x}{c^2} \frac{\partial t}{\partial t} = o \Rightarrow \left( \frac{\partial}{\partial x} + \frac{ux}{c^2} \frac{\partial t}{\partial t} \right)^2 = \frac{\partial^2}{\partial x^2} + \frac{2ux}{c^2} \frac{\partial^2}{\partial x^2} + \frac{ux^2}{c^4} \frac{\partial^2}{\partial t^2} = 0 \]

that applied in 8.9 supplies the wave equation to the observer \( O \)

\[ \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} - \frac{1}{c^2} \frac{\partial^2}{\partial t^2} = 0 \]

To return to the referential of the observer \( O' \) we will apply 8.10 to the formulas of tables 9 and 1.18, getting

\[ \left( \frac{\partial}{\partial x' \partial^2} - \frac{c^2}{c^2} + \frac{\partial t'}{\partial t'} \right)^2 + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} - \frac{1}{c^2} \left[ - \frac{\partial^2}{\partial x^2} \frac{\partial x^2}{\partial t} + \frac{1}{\sqrt{K}} \frac{\partial}{\partial x'} \frac{\partial x^2}{\partial t} \right] = 0 \]

from where we find

\[ K' \frac{\partial^2}{\partial x^2} + K' \frac{\partial^2}{\partial y^2} + K' \frac{\partial^2}{\partial z^2} - \frac{1}{c^2} \frac{\partial^2}{\partial t^2} - \frac{2v'}{c^2} \frac{\partial^2}{\partial x^2} - \frac{2v'}{c^4} \frac{\partial^2}{\partial x^2 \partial t^2} + \frac{4v^2}{c^4} \frac{\partial^2}{\partial x^2 \partial t^2} + \frac{v^2}{c^4} \frac{\partial^2}{\partial t^2} + \frac{v^2}{c^4} \frac{\partial^2}{\partial t^2} + \frac{2v'}{c^2} \frac{\partial^2}{c^2} \frac{\partial^2}{\partial x^2 \partial t^2} + \frac{v^2}{c^6} \frac{\partial^2}{\partial t^2} \]

that simplifying supplies

\[ K' \frac{\partial^2}{\partial x^2} + K' \frac{\partial^2}{\partial y^2} + K' \frac{\partial^2}{\partial z^2} - \frac{1}{c^2} \frac{\partial^2}{\partial t^2} - \frac{2v'}{c^2} \frac{\partial^2}{\partial x^2} - \frac{2v'}{c^4} \frac{\partial^2}{\partial x^2 \partial t^2} + \frac{4v^2}{c^4} \frac{\partial^2}{\partial x^2 \partial t^2} + \frac{v^2}{c^4} \frac{\partial^2}{\partial t^2} + \frac{v^2}{c^4} \frac{\partial^2}{\partial t^2} + \frac{2v'}{c^6} \frac{\partial^2}{\partial t^2} + \frac{v^2}{c^6} \frac{\partial^2}{\partial t^2} = 0 \]

where reordering the terms we find

\[ K' \frac{\partial^2}{\partial x^2} + K' \frac{\partial^2}{\partial y^2} + K' \frac{\partial^2}{\partial z^2} - \left( 1 + \frac{v'x'}{c^2} \frac{\partial^2}{\partial t^2} \right) \frac{1}{\sqrt{K}} \frac{\partial^2}{\partial x^2} + \frac{2v'x'}{c^4} \frac{\partial^2}{\partial x^2} + \frac{u'x'^2}{c^6} \frac{\partial^2}{\partial t^2} = 0 \]

but from 8.5 and 1.18 we have

\[ \frac{\partial}{\partial x} \frac{x}{c^2} \frac{\partial t}{\partial t} = 0 \Rightarrow \left( \frac{\partial}{\partial x} + \frac{u'x'}{c^2} \frac{\partial t}{\partial t} \right)^2 = \frac{\partial^2}{\partial x^2} + \frac{2u'x'}{c^2} \frac{\partial^2}{\partial x^2} + \frac{u'x'^2}{c^4} \frac{\partial^2}{\partial t^2} = 0 \]

that replaced in the reordered equation supplies the wave equation to the observer \( O' \).

**Invariance of the Continuity equation**

The continuity equation in the differential form to the observer \( O' \) is

\[ \frac{\partial p'}{\partial t'} + \nabla \cdot j' = 0 \Rightarrow \frac{\partial p'}{\partial t'} + \frac{\partial Jx'}{\partial x'} + \frac{\partial Jy'}{\partial y'} + \frac{\partial Jz'}{\partial z'} = 0 \]

where replacing the formulas of tables 6, 9, and 1.13 we get

\[ \left( \frac{v}{\sqrt{K}} \frac{\partial}{\partial x} + \frac{1}{\sqrt{K}} \left( 1 + \frac{v^2}{c^2} - \frac{vux}{c^2} \right) \frac{\partial}{\partial t} \right) \rho \sqrt{K} + \left( \frac{\partial}{\partial x} + \frac{v}{c^2} \frac{\partial}{\partial t} \right) (Jx - \rho v) + \frac{\partial Jy}{\partial y} + \frac{\partial Jz}{\partial z} = 0 \]
making the operations we find
\[
\frac{v \partial \rho}{\partial t} + \frac{v^2 \partial \rho}{c^2 \partial t} + \frac{v x \partial \rho}{c^2 \partial t} + \frac{\partial J_x}{c^2 \partial t} + \frac{v \partial J_x}{c^2 \partial t} - \frac{v \partial \rho}{c^2 \partial t} + \frac{\partial J_y}{\partial y} + \frac{\partial J_z}{\partial z} = 0
\]
that simplifying supplies
\[
\frac{\partial \rho}{\partial t} - \frac{vux \partial \rho}{c^2 \partial t} + \frac{\partial J_x}{\partial x} + \frac{\partial J_y}{\partial y} + \frac{\partial J_z}{\partial z} = 0
\]

where applying \(J_x = \rho u_x\) with \(u_x\) constant we get
\[
\frac{\partial \rho}{\partial t} - \frac{vux \partial \rho}{c^2 \partial t} + \frac{\partial J_x}{\partial x} + \frac{\partial J_y}{\partial y} + \frac{\partial J_z}{\partial z} = 0 \Rightarrow \frac{\partial \rho}{\partial t} + \frac{\partial J_x}{\partial x} + \frac{\partial J_y}{\partial y} + \frac{\partial J_z}{\partial z} = 0 \quad 8.12
\]
that is the continuity equation in the differential form to the observer O.

To get again the continuity equation in the differential form to the observer O’ we will replace the formulas of tables 6, 9, and 1.18 in 8.12 getting
\[
\left( -\frac{\partial}{\partial t'} + \frac{1}{\sqrt{K'}} \frac{\partial}{\partial x'} \right) \rho' + \frac{v'}{\sqrt{K'}} \left( \frac{\partial}{\partial x'} - \frac{v'}{c^2} \frac{\partial}{\partial t'} \right) (J'x' + \rho'v') + \frac{\partial J'y'}{\partial y'} + \frac{\partial J'z'}{\partial z'} = 0
\]

making the operations we find
\[
-\frac{\partial}{\partial t'} + \frac{v'u'x'}{c^2} \frac{\partial}{\partial t'} + \frac{\partial J'x'}{\partial x'} - \frac{v'}{c^2} \frac{\partial}{\partial t'} + \frac{\partial J'y'}{\partial y'} + \frac{\partial J'z'}{\partial z'} = 0
\]
that simplifying supplies
\[
\frac{\partial \rho'}{\partial t'} - \frac{v'u'x'}{c^2} \frac{\partial \rho'}{\partial t'} + \frac{\partial J'x'}{\partial x'} - \frac{v'}{c^2} \frac{\partial J'x'}{\partial t'} + \frac{\partial J'y'}{\partial y'} + \frac{\partial J'z'}{\partial z'} = 0
\]
where applying \(J'x' = \rho'u'u'x'\) with \(u'u'\) constant we get
\[
\frac{\partial \rho'}{\partial t'} - \frac{v'u'x'}{c^2} \frac{\partial \rho'}{\partial t'} + \frac{\partial J'x'}{\partial x'} - \frac{v'}{c^2} \frac{\partial \rho'}{\partial t'} + \frac{\partial J'y'}{\partial y'} + \frac{\partial J'z'}{\partial z'} = 0 \Rightarrow \frac{\partial \rho'}{\partial t'} + \frac{\partial J'x'}{\partial x'} + \frac{\partial J'y'}{\partial y'} + \frac{\partial J'z'}{\partial z'} = 0
\]
that is the continuity equation in the differential form to the observer O’.

**Invariance of Maxwell’s equations**

That in the differential form are written this way

<table>
<thead>
<tr>
<th>With the electrical charge</th>
<th>To the observer O</th>
<th>To the observer O’</th>
</tr>
</thead>
<tbody>
<tr>
<td>(\frac{\partial E_x}{\partial t} + \frac{\partial E_y}{\partial y} + \frac{\partial E_z}{\partial z} = \frac{\rho}{\varepsilon_o})</td>
<td>(8.13)</td>
<td>(\frac{\partial E'_x}{\partial t'} + \frac{\partial E'_y}{\partial y'} + \frac{\partial E'_z}{\partial z'} = \frac{\rho'}{\varepsilon_o})</td>
</tr>
<tr>
<td>(\frac{\partial B_x}{\partial t} + \frac{\partial B_y}{\partial y} + \frac{\partial B_z}{\partial z} = 0)</td>
<td>(8.15)</td>
<td>(\frac{\partial B'_x}{\partial t'} + \frac{\partial B'_y}{\partial y'} + \frac{\partial B'_z}{\partial z'} = 0)</td>
</tr>
<tr>
<td>(\frac{\partial E_y}{\partial y} - \frac{\partial E_x}{\partial x} = \frac{\partial B_z}{\partial t})</td>
<td>(8.17)</td>
<td>(\frac{\partial E'_y}{\partial y'} - \frac{\partial E'_x}{\partial x'} = -\frac{\partial B'_z}{\partial t'})</td>
</tr>
<tr>
<td>(\frac{\partial E_z}{\partial z} - \frac{\partial E_y}{\partial y} = \frac{\partial B_x}{\partial t})</td>
<td>(8.19)</td>
<td>(\frac{\partial E'_z}{\partial z'} - \frac{\partial E'_y}{\partial y'} = -\frac{\partial B'_x}{\partial t'})</td>
</tr>
<tr>
<td>(\frac{\partial E_x}{\partial x} - \frac{\partial E_z}{\partial z} = \frac{\partial B_y}{\partial t})</td>
<td>(8.21)</td>
<td>(\frac{\partial E'_x}{\partial x'} - \frac{\partial E'_z}{\partial z'} = -\frac{\partial B'_y}{\partial t'})</td>
</tr>
<tr>
<td>(\frac{\partial B_y}{\partial y} - \frac{\partial B_x}{\partial x} = \mu_o J_z + \varepsilon_o \mu_o \frac{\partial E_z}{\partial t})</td>
<td>(8.23)</td>
<td>(\frac{\partial B'_y}{\partial y'} - \frac{\partial B'_x}{\partial x'} = \mu_o J'_z + \varepsilon_o \mu_o \frac{\partial E'_z}{\partial t'})</td>
</tr>
</tbody>
</table>
We demonstrate the invariance of the Law of Gauss in the differential form that for the observer $O'$ is
\[
\frac{\partial E'_x}{\partial x'} + \frac{\partial E'_y'}{\partial y'} + \frac{\partial E'_z'}{\partial z'} = \frac{p'}{\varepsilon_o}
\]  
8.14

where replacing the formulas from the tables 6, 7, 9, and 1.18, and considering $u'x'$ constant, we get

\[
\frac{\partial E}{\partial x} = \varepsilon_o \mu_o \frac{\partial E}{\partial t}
\]
8.29
\[
\frac{\partial E_y}{\partial y} + \frac{\partial E_z}{\partial z} = 0
\]
8.30
\[
\frac{\partial B_x}{\partial x} + \frac{\partial B_y}{\partial y} + \frac{\partial B_z}{\partial z} = 0
\]
8.31
\[
\frac{\partial E}{\partial x} - \frac{\partial E_y}{\partial y} = \frac{\partial E_z}{\partial z}
\]
8.32
\[
\frac{\partial E}{\partial x} - \frac{\partial E_z}{\partial z} = \frac{\partial E_y}{\partial y}
\]
8.33
\[
\frac{\partial B}{\partial x} - \frac{\partial B_y}{\partial y} = \frac{\partial B_z}{\partial z}
\]
8.34
\[
\frac{\partial B}{\partial x} - \frac{\partial B_z}{\partial z} = \frac{\partial B_y}{\partial y}
\]
8.35
\[
\frac{\partial B}{\partial y} - \frac{\partial B_z}{\partial z} = \frac{\partial B_x}{\partial x}
\]
8.36
\[
\frac{\partial B}{\partial x} + \frac{\partial B}{\partial z} = \frac{\partial B_y}{\partial y}
\]
8.37
\[
\frac{\partial B}{\partial y} - \frac{\partial B}{\partial z} = \frac{\partial B_x}{\partial x}
\]
8.38
\[
\frac{\partial B}{\partial y} - \frac{\partial B}{\partial z} = \frac{\partial B_x}{\partial x}
\]
8.39
\[
\frac{\partial B}{\partial y} - \frac{\partial B}{\partial z} = \frac{\partial B_x}{\partial x}
\]
8.40
\[
\frac{\partial B}{\partial y} - \frac{\partial B}{\partial z} = \frac{\partial B_x}{\partial x}
\]
8.41
\[
\frac{\partial B}{\partial y} - \frac{\partial B}{\partial z} = \frac{\partial B_x}{\partial x}
\]
8.42
\[
\frac{\partial B}{\partial y} - \frac{\partial B}{\partial z} = \frac{\partial B_x}{\partial x}
\]
8.43
\[
\frac{\partial E}{\partial x} = \frac{1}{\varepsilon_o}
\]
8.44
where the first parentheses is 8.5 and because of this equal to zero, the second blank is equal to 
\(- v(\mu_x J x) = - v \mu_x \rho u x = - \frac{v \mu x}{\varepsilon_0 c^2}\) gotten from 8.25 and 8.45 resulting in

\[
\left( \frac{\partial E x}{\partial x} + \frac{\partial E y}{\partial y} + \frac{\partial E z}{\partial z} \right) \left( 1 + \frac{v^2}{c^2} - \frac{v u x}{\varepsilon_0 c^2} \right) = \frac{\rho}{\varepsilon_0} \left( 1 + \frac{v^2}{c^2} - \frac{v u x}{\varepsilon_0 c^2} \right) - \frac{\rho}{\varepsilon_0} \frac{v u x}{\varepsilon_0 c^2} + \frac{\rho}{\varepsilon_0} \frac{v u x}{\varepsilon_0 c^2}
\]

from where we get
\[
\frac{\partial E x}{\partial x} + \frac{\partial E y}{\partial y} + \frac{\partial E z}{\partial z} = \frac{\rho}{\varepsilon_0}
\]

that is the Law of Gauss in the differential form to the observer O.

To make the inverse we will replace in 8.13 the formulas of the tables 6, 7, 9, and 1.13, and considering \(u x\) constant, we get

\[
\frac{\partial}{\partial x} \left[ \frac{v^2}{c^2} \frac{\partial E x'}{\partial x'} \right] - \frac{v^2}{c^2} \frac{\partial E x'}{\partial t'} + \frac{v^2 u x'}{c^2} \frac{\partial E x'}{\partial x'} + \frac{v^2 u x'}{c^2} \frac{\partial E x'}{\partial y'} + \frac{v^2 u x'}{c^2} \frac{\partial E x'}{\partial z'} + \frac{v^2}{c^2} \frac{\partial E x'}{\partial t'} + \frac{v^2}{c^2} \frac{\partial E x'}{\partial y'} + \frac{v^2}{c^2} \frac{\partial E x'}{\partial z'} = \frac{\rho'}{\varepsilon_0}
\]

making the products, adding and subtracting the term \(\frac{v^2}{c^2} \frac{\partial E x'}{\partial x'}\), we get

\[
\frac{\partial E x'}{\partial x'} c^{-2} \frac{\partial E x'}{\partial t'} + \frac{v^2}{c^2} \frac{\partial E x'}{\partial x'} + \frac{v^2}{c^2} \frac{\partial E x'}{\partial y'} + \frac{v^2}{c^2} \frac{\partial E x'}{\partial z'} + \frac{v^2}{c^2} \frac{\partial E x'}{\partial x'} + \frac{v^2}{c^2} \frac{\partial E x'}{\partial y'} + \frac{v^2}{c^2} \frac{\partial E x'}{\partial z'} - \frac{v^2}{c^2} \frac{\partial E x'}{\partial x'} = \frac{\rho'}{\varepsilon_0}
\]

that reordering results in

\[
- \frac{v^2}{c^2} \left( \frac{\partial E x'}{\partial x'} + \frac{\partial E y'}{\partial y'} + \frac{\partial E z'}{\partial z'} \right) + v^2 \left( \frac{\partial B'}{\partial y'} + \frac{\partial B'}{\partial z'} - \frac{\partial B'}{\partial x'} \right) = \frac{\rho'}{\varepsilon_0}
\]

where the first blank is 8.5 and because of this equals to zero, the second blank is equal to 
\(v' (\mu_x J' x') = v' \mu_x \rho' u x' = \frac{v' \rho' u x'}{\varepsilon_0 c^2}\) gotten from 8.26 and 8.45 resulting in

\[
\left( \frac{\partial E x'}{\partial x'} + \frac{\partial E y'}{\partial y'} + \frac{\partial E z'}{\partial z'} \right) \left( 1 + \frac{v^2}{c^2} - \frac{v u x'}{\varepsilon_0 c^2} \right) = \frac{\rho'}{\varepsilon_0} \left( 1 + \frac{v^2}{c^2} - \frac{v u x'}{\varepsilon_0 c^2} \right) - \frac{\rho}{\varepsilon_0} \frac{v u x'}{\varepsilon_0 c^2} - \frac{\rho}{\varepsilon_0} \frac{v u x'}{\varepsilon_0 c^2}
\]

from where we get
\[
\frac{\partial E x'}{\partial x'} + \frac{\partial E y'}{\partial y'} + \frac{\partial E z'}{\partial z'} = \frac{\rho'}{\varepsilon_0}
\]

that is the Law of Gauss in the differential form to the observer O.

Proceeding this way we can prove the invariance of form for all the other equations of Maxwell.

§ 9 Explaining the Sagnac Effect with the Undulating Relativity

We must transform the straight movement of the two observers O and O’ used in the deduction of the Undulating Relativity in a plain circular movement with a constant radius. Let’s imagine that the observer O sees the observer O’ turning around with a tangential speed v in a clockwise way (C) equals to the positive course of the axis x of UR and that the observer O’ sees the observer O turning around with a tangential speed v in an unclockwise way (U) equals to the negative course of the axis x of the UR.

In the moment \(t = t' = 0\), the observer O emits two rays of light from the common origin to both observers, one in an unclockwise way of arc \(c t_0\) and another in a clockwise way of arc \(c t_C\), therefore \(c t_0 = c t_C\) and \(t_0 = t_C\), because c is the speed of the constant light, and \(t_0\) and \(t_C\) the time.

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In the moment $t = t' = 0$ the observer $O'$ also emits two rays of light from the common origin to both observers, one in a clockwise way (useless) of arc $ct'_U$ and another one in a clockwise way of arc $ct'_C$, thus $ct'_U = ct'_C$ and $t'_U = t'_C$ because $c$ is the speed of the constant light, and $t'_U$ and $t'_C$ the time.

Rewriting the equations 1.15 and 1.20 of the Undulating Relativity (UR):

$$\frac{v}{v'} = \frac{t'}{t} = \sqrt{1 + \frac{v^2}{c^2} - \frac{2vux}{c^2}}. \quad 1.15$$

$$\frac{v'}{v} = \frac{t}{t'} = \sqrt{1 + \frac{v'^2}{c^2} + \frac{2v'u'x'}{c^2}}. \quad 1.20$$

Making $ux = u'x' = c$ (ray of light projected alongside the positive axis $x$) and splitting the equations we have:

$$t' = t \left(1 - \frac{v}{c}\right) \quad 9.1 \quad t = t' \left(1 + \frac{v'}{c}\right) \quad 9.2$$

$$v' = \frac{v}{\left(1 - \frac{v}{c}\right)} \quad 9.3 \quad v = \frac{v'}{\left(1 + \frac{v'}{c}\right)} \quad 9.4$$

When the origin of the observer $O'$ detects the unclockwise ray of the observer $O$, will be at the distance $vt'_C = v' t'_U$ of the observer $O$ and simultaneously will detect its clockwise ray of light at the same point of the observer $O$, in a symmetric position to the diameter that goes through the observer $O$ because $ct'_U = ct'_C \Rightarrow t'_U = t'_C$ and $ct'_U = ct'_C \Rightarrow t'_U = t'_C$, following the four equations above we have:

$$ct'_U + vt'_C = 2\pi R \Rightarrow t'_C = \frac{2\pi R}{c + v} \quad 9.5$$

$$ct'_C + 2v't'_U = 2\pi R \Rightarrow t'_C = \frac{2\pi R}{c + 2v'} \quad 9.6$$

When the origin of the observer $O'$ detects the clockwise ray of the observer $O$, simultaneously will detect its own clockwise ray and will be at the distance $vt'_2C = v't'_2U$ of the observer $O$, then following the equations 1,2,3 and 4 above we have:

$$ct'_2C = 2\pi R + vt'_2C \Rightarrow t'_2C = \frac{2\pi R}{c - v} \quad 9.7$$

$$ct'_2C = 2\pi R \Rightarrow t'_2C = \frac{2\pi R}{c} \quad 9.8$$

The time difference to the observer $O$ is:

$$\Delta t = t'_2C - t'_C = \frac{2\pi R}{c - v} - \frac{2\pi R}{c + v} = \frac{4\pi Rv}{c^2 - v^2} \quad 9.9$$

The time difference to the observer $O'$ is:

$$\Delta t' = t'_2C - t'_2C = \frac{2\pi R}{c} - \frac{2\pi R}{c + 2v'} = \frac{4\pi Rv'}{(c + 2v')c} \quad 9.10$$

Replacing the equations 5 to 10 in 1 to 4 we prove that they confirm the transformations of the Undulating Relativity.
§10 Explaining the experience of Ives-Stilwell with the Undulating Relativity

We should rewrite the equations (2.21) to the wave length in the Undulating Relativity:

\[ \lambda' = \frac{\lambda}{\sqrt{1 + \frac{v^2}{c^2} - 2\frac{vu}{c^2}}} \quad \text{and} \quad \lambda = \frac{\lambda'}{\sqrt{1 + \frac{v'^2}{c^2} + 2\frac{v'u'}{c^2}}} \]

Making \( u'x' = c \) (Ray of light projected alongside the positive axis \( x \)), we have the equations:

\[ \lambda' = \frac{\lambda}{1 - \frac{v}{c}} \quad \text{and} \quad \lambda = \frac{\lambda'}{1 + \frac{v'}{c}} \]

If the observer \( O \), who sees the observer \( O' \) going away with the velocity \( v \) in the positive way of the axis \( x \), emits waves, provenient of a resting source in its origin with velocity \( c \) and wave length \( \lambda_F \) in the positive way of the axis \( x \), then according to the equation 10.1 the observer \( O' \) will measure the waves with velocity \( c \) and the wave length \( \lambda'_F \) according to the formulas:

\[ \lambda'_F = \frac{\lambda_F}{1 - \frac{v}{c}} \quad \text{and} \quad \lambda_F = \frac{\lambda'_F}{1 + \frac{v'}{c}} \]

If the observer \( O' \), who sees the observer \( O \) going away with velocity \( v' \) in the negative way of the axis \( x \), emits waves, provenient of a resting source in its origin with velocity \( c \) and the wave length \( \lambda'_F \) in the positive way of the axis \( x \), then according to the equation 10.1 the observer \( O \) will measure waves with velocity \( c \) and wave length \( \lambda_A \) according to the formulas:

\[ \lambda'_F = \frac{\lambda_A}{1 - \frac{v}{c}} \quad \text{and} \quad \lambda_A = \frac{\lambda'_F}{1 + \frac{v'}{c}} \]

The resting sources in the origin of the observers \( O \) and \( O' \) are identical thus \( \lambda_F = \lambda'_F \).

We calculate the average wave length \( \lambda \) of the measured waves \( (\lambda_A, \lambda'_F) \) using the equations 10.2 and 10.3, the left side in each equation:

\[ \lambda = \frac{\lambda'_D + \lambda_A}{2} = \frac{1}{2} \left[ \frac{\lambda_F}{1 - \frac{v}{c}} + \lambda'_F \left( 1 - \frac{v}{c} \right) \right] \Rightarrow \lambda = \frac{\lambda'_D + \lambda_A}{2} = \frac{\lambda'_F}{2 \left( 1 - \frac{v}{c} \right)} \left[ 1 + \left( 1 - \frac{v}{c} \right)^2 \right] \]

We calculate the difference between the average wave length \( \lambda \) and the emitted wave length by the sources \( \Delta \lambda = \lambda - \lambda_F \):

\[ \Delta \lambda = \frac{\lambda'_F}{2 \left( 1 - \frac{v}{c} \right)} \left[ 1 + \left( 1 - \frac{v}{c} \right)^2 \right] - \lambda_F \]

\[ \Delta \lambda = \frac{\lambda'_F}{2 \left( 1 - \frac{v}{c} \right)} \left[ 1 + \left( 1 - \frac{v}{c} \right)^2 \right] - \lambda_F \]

\[ \Delta \lambda = \frac{\lambda'_F}{2 \left( 1 - \frac{v}{c} \right)} \left[ 1 + \left( 1 - \frac{v}{c} \right)^2 \right] - \lambda_F \]

\[ \Delta \lambda = \frac{\lambda'_F}{2 \left( 1 - \frac{v}{c} \right)} \left[ 1 + \left( 1 - \frac{v}{c} \right)^2 \right] - \lambda_F \]
\[ \Delta \lambda = \frac{\lambda_F}{2(1-\frac{v}{c})^2} \left[ 1 + 1 - 2 \frac{v}{c} + \frac{v^2}{c^2} - 2 + 2 \frac{v}{c} \right] \]

\[ \Delta \lambda = \frac{\lambda_F \cdot \frac{v^2}{c^2}}{2} \]

10.4

Reference
http://www.wbabin.net.physics/faraj7.htm

§10 Ives-Stilwell (continuation)

The Doppler’s effect transversal to the Undulating Relativity was obtained in the §2 as follows:

If the observer O’, that sees the observer O, moves with the speed \(-v’\) in a negative way to the axis x’, emits waves with the frequency \(y’\) and the speed c then the observer O according to 2.22 and \(u’x’ = -v’\) will measure waves of frequency \(y\) and speed c in a perpendicular plane to the movement of O’ given by

\[ y = y' \sqrt{1 - \frac{v^2}{c^2}} \]

2.25

For \(u'x' = -v'\) we will have \(ux = zero\) and \(\sqrt{1 - \frac{v^2}{c^2}} \sqrt{1 + \frac{v^2}{c^2}} = 1\) with this we can write the relation between the transversal frequency \(y = y_i\) and the source frequency \(y' = y'_F\) like this

\[ y_i = \frac{y'_F}{\sqrt{1 + \frac{v^2}{c^2}}} \]

10.5

With \(c = y'_F \lambda_i = y'_F \lambda'_F\) we have the relation between the length of the transversal wave \(\lambda_i\) and the length of the source wave \(\lambda'_F\)

\[ \lambda_i = \lambda'_F \sqrt{1 + \frac{v^2}{c^2}} \]

10.6

The variation of the length of the transversal wave in the relation to the length of the source wave is:

\[ \Delta \lambda_i = \lambda_i - \lambda'_F = \lambda'_F \sqrt{1 + \frac{v^2}{c^2}} - \lambda'_F = \lambda'_F \left( \sqrt{1 + \frac{v^2}{c^2}} - 1 \right) \approx \lambda'_F \left( 1 + \frac{v^2}{2c^2} - 1 \right) = \frac{\lambda'_F v^2}{2c^2} \]

10.7

that is the same value gotten in the Theory of Special Relativity.

Applying 10.7 in 10.4 we have

\[ \Delta \lambda = \frac{\Delta \lambda_i}{1 - \frac{v}{c}} \]

10.8

With the equations 10.2 and 10.3 we can get the relations 10.9, 10.10, and 10.11 described as follows

\[ \lambda_i = \lambda'_D \left( 1 - \frac{v}{c} \right)^2 \]

10.9

And from this we have the formula of speed \(\frac{v}{c} = 1 - \sqrt{\frac{\lambda_i}{\lambda'_D}}\)

10.10

\[ \lambda_F = \lambda'_F = \sqrt{\lambda_i \lambda'_D} \]

10.11

Applying 10.10 and 10.11 in 10.6 we have

\[ \lambda_i = \sqrt{\lambda_i \lambda'_D} \sqrt{1 + \left( 1 - \frac{\lambda_i}{\sqrt{\lambda'_D}} \right)^2} \]

10.12
From 10.8 and 10.12 we conclude that \( \lambda_4 \leq \lambda_F \leq \lambda_i \leq \lambda_D' \).

So that we the values of \( \lambda_4 \) and \( \lambda_D' \) obtained from the Ives-Stiwell experience we can evaluate \( \lambda_i, \lambda_F \), \( \frac{v}{c} \) and conclude whether there is or not the space deformation predicted in the Theory of Special Relativity.

§11 Transformation of the power of a luminous ray between two referencials in the Special Theory of Relativity

The relationship within the power developed by the forces between two referencials is written in the Special Theory of the Relativity in the following way:

\[
\tilde{F}'\tilde{u}' = \frac{\tilde{F}\tilde{u} - vFx}{\left(1 - \frac{vux}{c^2}\right)}
\]

The definition of the component of the force along the axis x is:

\[
F_x = \frac{dp}{dt} = \frac{d(mux)}{dt} = \frac{dm}{dt}ux + m\frac{dux}{dt}
\]

For a luminous ray, the principle of light speed constancy guarantees that the component \( ux \) of the light speed is also constant along its axis, thus

\[
\frac{x}{t} = \frac{dx}{dt} = ux = \text{constant}, \text{ demonstrating that in two \( \frac{dux}{dt} = \text{zero} \) and} \quad F_x = \frac{dm}{dt}ux
\]

The formula of energy is \( E = mc^2 \) from where we have \( \frac{dm}{dt} = \frac{1}{c^2}\frac{dE}{dt} \)

From the definition of energy we have \( \frac{dE}{dt} = \tilde{F}\tilde{u} \) that applying in 4 and 3 we have \( F_x = \tilde{F}\tilde{u}\frac{ux}{c^2} \)

Applying 5 in 1 we have:

\[
\tilde{F}'\tilde{u}' = \frac{\tilde{F}\tilde{u} - \left(\tilde{F}\tilde{u}\right)^{vux}}{\left(1 - \frac{vux}{c^2}\right)}
\]

From where we find that \( \tilde{F}'\tilde{u}' = \tilde{F}\tilde{u} \) or \( \frac{dE'}{dt'} = \frac{dE}{dt} \)

A result equal to 5.3 of the Undulating Relativity that can be experimentally proven, considering the ‘Sun’ as the source.

§12 Linearity

The Theory of Undulating Relativity has as its fundamental axiom the necessity that inertial referencials be named exclusively as those ones in which a ray of light emitted in any direction from its origin spreads in a straight line, what is mathematically described by the formulae (1.13, 1.18, 8.6 e 8.7) of the Undulating Relativity:

\[
x = \frac{dx}{dt} = ux, \quad y = \frac{dy}{dt} = uy, \quad z = \frac{dz}{dt} = uz
\]

\[
x' = \frac{dx'}{dt'} = u'x', \quad y' = \frac{dy'}{dt'} = u'y', \quad z' = \frac{dz'}{dt'} = u'z'
\]

Woldemar Voigt wrote in 1.887 the linear transformation between the referencials os the observers O e O’ in the following way:
\[ x = Ax' + Bt' \]
\[ t = Ex' + Ft' \]

With the respective inverted equations:
\[ x' = \frac{F}{AF - BE} x + \frac{-B}{AF - BE} t \]
\[ t' = \frac{-E}{AF - BE} x + \frac{A}{AF - BE} t \]

Where \( A, B, E \) and \( F \) are constants and because of the symmetry we don’t consider the terms with \( y, z \) and \( y', z' \).

We know that \( x \) and \( x' \) are projections of the two rays of lights \( ct \) and \( ct' \) that spread with Constant speed \( c \) (due to the constancy principle of the Ray of light), emitted in any direction from the origin of the respective inertials referential at the moment in which the origins are coincident and at the moment where:
\[ t = t' = 0 \]

because of this in the equation 12.2 at the moment where \( t' = 0 \) we must have \( E = 0 \) so that we also have \( t = 0 \), we can’t assume that when \( t' = 0 \), \( x' \) also be equal to zero, because if the spreading happens in the plane \( y'z' \) we will have \( x' = 0 \) plus \( t' \neq 0 \).

We should rewrite the corrected equations (\( E = 0 \)):
\[ x = Ax' + Bt' \]
\[ t = Ft' \]

With the respective corrected inverted equations:
\[ x' = \frac{x}{A} - \frac{Bt}{AF} \]
\[ t' = \frac{t}{F} \]

If the spreading happens in the plane \( y'z' \) we have \( x' = 0 \) and dividing 12.6 by 12.7 we have:
\[ \frac{x}{t} = \frac{B}{F} = v \]

where \( v \) is the module of the speed in which the observer \( O \) sees the referential of the observer \( O' \) moving alongside the \( x \) axis in the positive way because the sign of the equation is positive.

If the spreading happens in the plane \( yz \) we have \( x = 0 \) and dividing 12.8 by 12.9 we have:
\[ \frac{x}{t'} = -\frac{B}{A} = -v' \quad \text{or} \quad \frac{B}{A} = v' \]

where \( v' \) is the module of the speed in which the observer \( O' \) sees the referential of the observer \( O \) moving alongside the \( x' \) axis in the negative way because the signal of the equation is negative.

The equation 1.6 describes the constancy principle of the speed of light that must be assumed by the equations 12.6 to 12.9:
\[ x^2 - c^2 t^2 = x'^2 - c^2 t'^2 \]

Applying 12.6 and 12.7 in 1.6 we have:
\[ (Ax' + Bt')^2 - c^2 F^2 t'^2 = x'^2 - c^2 t'^2 \]

From where we have:
\[
\left( A^2 x'^2 \right) - c^2 t'^2 \left[ F^2 - \frac{B^2}{c^2} - \frac{2ABx'}{c^2 t'} \right] = x'^2 - c^2 t'^2
\]

where making \( A^2 = 1 \) in the brackets in arc and \( \left[ F^2 - \frac{B^2}{c^2} - \frac{2ABx'}{c^2 t'} \right] = l \) in the straight brackets we have the equality between both sides of the equal signal of the equation.

Applying \( A = 1 \) in \( \left[ F^2 - \frac{B^2}{c^2} - \frac{2ABx'}{c^2 t'} \right] = l \) we have \( F^2 = l + \frac{B^2}{c^2} + \frac{2Bx'}{c^2 t'} \) \( 12.12 \)

Applying \( A = 1 \) in \( 12.11 \) we have \( \frac{B}{A} = \frac{B}{l} = B = v' \) \( 12.11 \)

That applied in \( 12.12 \) suplies:
\[
F = \sqrt{l + \frac{v'^2}{c^2} + \frac{2v'x'}{c^2 t'}} = F(x', t')
\]  \( 12.12 \)

as \( F(x', t') \) is equal to the function \( F \) depending of the variables \( x' \) and \( t' \).

Applying \( 12.8 \) and \( 12.9 \) in \( 1.6 \) we have:
\[
x^2 - c^2 t^2 = \left( \frac{x}{A} - \frac{Bt}{AF} \right)^2 - c^2 \frac{t^2}{F^2}
\]

From where we have:
\[
x^2 - c^2 t^2 = \left( \frac{x^2}{A^2} \right) - c^2 t^2 \left[ \frac{1}{F^2} - \frac{B^2}{A^2 c^2 F^2} + \frac{2Bx}{A^2 c^2 Ft} \right]
\]

where making \( A^2 = 1 \) in the bracket in arc and \( \left[ \frac{1}{F^2} - \frac{B^2}{A^2 c^2 F^2} + \frac{2Bx}{A^2 c^2 Ft} \right] = l \) in the straight bracket we have the equality between both sides of the equal signal of the equation.

Applying \( A = 1 \) and \( 12.10 \) in \( \left[ \frac{1}{F^2} - \frac{B^2}{A^2 c^2 F^2} + \frac{2Bx}{A^2 c^2 Ft} \right] = l \) we have:
\[
F = \sqrt{\frac{l}{l + \frac{v^2}{c^2} - \frac{2vx}{c^2 t}}} = F(x, t)
\]  \( 12.13 \)

as \( F(x, t) \) is equal to the function \( F \) depending on the variables \( x \) and \( t \).

We must make the following naming according to \( 2.5 \) and \( 2.6 \):
\[
K' = l + \frac{v'^2}{c^2} + \frac{2v'x'}{c^2 t'} \Rightarrow F = \sqrt{K'}
\]  \( 12.14 \)

\[
K = l + \frac{v^2}{c^2} - \frac{2vx}{c^2 t} \Rightarrow F = \frac{1}{\sqrt{K}}
\]  \( 12.15 \)

As the equation to \( F(x', t') \) from \( 12.12 \) and \( F(x, t) \) from \( 12.13 \) must be equal, we have:

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\[ F = \sqrt{1 + \frac{v'^2}{c^2} + \frac{2v'x'}{c^2't'}} = \frac{1}{\sqrt{1 + \frac{v'^2 - 2vx}{c^2't'}}} \quad 12.16 \]

Thus:

\[ \frac{1}{\sqrt{1 + \frac{v'^2 - 2vx}{c^2't'}}} \cdot \frac{1}{\sqrt{1 + \frac{v'^2 - 2vx}{c^2't'}}} = 1 \quad \text{or} \quad \sqrt{K} \cdot \sqrt{K'} = 1 \quad 12.17 \]

Exactly equal to 1.10.

Rewriting the equations 12.6, 12.7, 12.8 and 12.9 according to the function of \( v, v' \) and \( F \) we have:

\[ x = x' + v't' \quad 12.6 \]
\[ t = Ft' \quad 12.7 \]

With the respective inverted corrected equations:

\[ x' = x - vt \quad 12.8 \]
\[ t' = \frac{t}{F} \quad 12.9 \]

We have the equations 12.6, 12.7, 12.8 and 12.9 finals replacing \( F \) by the corresponding formulae:

\[ x = x' + v't' \quad 12.6 \]
\[ t = t'\sqrt{1 + \frac{v'^2 + 2v'x'}{c^2't'}} \quad 12.7 \]

With the respective inverted final equations:

\[ x' = x - vt \quad 12.8 \]
\[ t' = \frac{t}{\sqrt{1 + \frac{v'^2 - 2vx}{c^2't}}} \quad 12.9 \]

That are exactly the equations of the table I

As \( v = \frac{B}{F} \) and \( v' = \frac{B'}{F} \) then the relations between \( v \) and \( v' \) are \( v = \frac{v'}{F} \) or \( v' = v.F \) \quad 12.18

We will transform \( F \) (12.12) function of the elements \( v', x', \) and \( t' \) for \( F \) (12.13) function of the elements \( v, x \) and \( t \), replacing in 12.12 the equations 12.8, 12.9 and 12.18:

\[ F = \sqrt{1 + \frac{v'^2}{c^2} + \frac{2v'x'}{c^2't'}} = \frac{1}{\sqrt{1 + \frac{(vF)^2}{c^2} + \frac{2vF(x - vt)}{c^2't/F}}} \]
\[ F = \sqrt{1 + \frac{v'^2 F^2}{c^2} + \frac{2vxF^2}{c^2't}}-\frac{2v'^2 F^2}{c^2} = \sqrt{1 + \frac{2vxF^2}{c^2t} - \frac{v'^2 F^2}{c^2}} \]
\[ F^2 = 1 + \frac{2vxF^2}{c^2 t} - \frac{v^2F^2}{c^2} \quad \Rightarrow \quad F^2 + \frac{v^2F^2}{c^2} - \frac{2vxF^2}{c^2 t} = 0 \quad \Rightarrow \quad F = \frac{l}{\sqrt{1 + \frac{v^2}{c^2} - \frac{2vx}{c^2 t}}} \]

That is exactly the equation 12.13.

We will transform \( F \) (12.13) function of the elements \( v, x, \) and \( t \) for \( F \) (12.12) function of the elements \( v', x' \) and \( t' \), replacing in 12.13 the equations 12.6, 12.7 and 12.18:

\[ F = \frac{l}{\sqrt{1 + \frac{v^2}{c^2} - \frac{2vx}{c^2 t}}} \quad \Rightarrow \quad F^2 \left( 1 - \frac{v^2}{c^2} \frac{F^2}{F} - \frac{2vx}{c^2 t} \frac{F^2}{F} \right) = 0 \quad \Rightarrow \quad F = \sqrt{1 + \frac{v^2}{c^2} + \frac{2vx}{c^2 t}} \]

That is exactly the equation 12.12.

We have to calculate the total differential of \( F(x', t') \) (12.12):

\[ dF = \frac{\partial F}{\partial x'} dx' + \frac{\partial F}{\partial t'} dt' \]

as:

\[ \frac{\partial F}{\partial x'} = \frac{1}{\sqrt{K'}} \frac{v'}{c't'} \text{ and } \frac{\partial F}{\partial t'} = -\frac{1}{\sqrt{K'}} \frac{v'x'}{c't'} \]  

12.19

we have:

\[ dF = \frac{1}{\sqrt{K'}} \frac{v'}{c't'} dx' - \frac{1}{\sqrt{K'}} \frac{v'x'}{c't'} dt' \]

where applying 1.18 we find:

\[ dF = \frac{1}{\sqrt{K'}} \frac{v'}{c't'} dx' - \frac{1}{\sqrt{K'}} \frac{v'}{c't'} dx' dt' = 0 \]  

12.20

From where we conclude that \( F \) function of \( x' \) and \( t' \) is a constant.

We have to calculate the total differential of \( F(x, t) \) (12.13):

\[ dF = \frac{\partial F}{\partial x} dx + \frac{\partial F}{\partial t} dt \]

as:

\[ \frac{\partial F}{\partial x} = \frac{1}{K^2} \frac{v}{c^2 t} \text{ and } \frac{\partial F}{\partial t} = -\frac{1}{K^2} \frac{v}{c^2 t} \frac{x}{t} \]  

12.21

we have:

\[ dF = \frac{1}{K^2} \frac{v}{c^2 t} dx - \frac{1}{K^2} \frac{v}{c^2 t} dx dt \]  

12.22
The equations 1.13 and 1.18 represent to the observers O and O' the principle of constancy of the light speed valid from infinitely small to the infinitely big and mean that in the Undulating Relativity the space and time are measure simultaneously. They shouldn’t be interpreted with a dependency between space and time.

The time has its own interpretation that can be understood if we analyze to a determined observer the emission of two rays of light from the instant t=zero. If we add the times we get, for each ray of light, we will get a result without any use for the physics.

If in the instant t = t' = zero, the observer O' emits two rays of light, one alongside the axis x and the other alongside the axis y, after the interval of time t', the rays hit for the observer O', simultaneously, the points A₁ and A₂, to the distance ct' from the origin, although for the observer O, the points won't be hit simultaneously.

According to § 1, both referentials of the observers O and O' are inertial, thus the light spreads in a straight line according to what is demanded by the fundamental axiom of the Undulating Relativity § 12, because of this, the difference in velocities v and v' is due to only a difference in time between the referentials.

We can also relate na inertial referential for which the light spread in a straight line according to what is demanded by the fundamental axiom of the Undulating Relativity, with an accelerated moving referential for which the light spread in a curve line, considering that in this case the difference v and v' isn’t due to only the difference of time between the referentials.

According to § 1, if the observer O at the instant t = t' = zero, emits a ray of light from the origin of its referential, after an interval of time t₁, the ray of light hits the point A₁ with coordinates (x₁, y₁, z₁, t₁) to the distance ct₁ of the origin of the observer O, then we have:

\[
t'_1 = t_1 \sqrt{1 + \frac{v^2}{c^2} - \frac{2vx_1}{c^2t_1}}
\]

After hitting the point A₁, the ray of light still spread in the same direction and in the same way, after an interval of time t₂, the ray of light hits the point A₂ with coordinates (x₁ + x₂, y₁ + y₂, z₁ + z₂, t₁ + t₂) to the distance ct₂ to the point A₁, then we have:

\[
x = \frac{dx}{dt} = ux \Rightarrow \frac{x_2}{t_1} = \frac{x_2}{t_1} = ux \Rightarrow \sqrt{1 + \frac{v^2}{c^2} - \frac{2vx_2}{c^2t_2}} = \sqrt{1 + \frac{v^2}{c^2} - \frac{2vx_2}{c^2t_2}} = \sqrt{1 + \frac{v^2}{c^2} - \frac{2vx_2}{c^2}}
\]

and with this we get:

\[
t'_2 = t_2 \sqrt{1 + \frac{v^2}{c^2} - \frac{2vx_2}{c^2t_2}} = t_2 \sqrt{1 + \frac{v^2}{c^2} - \frac{2vx_2}{c^2}}
\]

\[
t'_1 + t'_2 = t_1 \sqrt{1 + \frac{v^2}{c^2} - \frac{2vx_2}{c^2t_1}} + t_2 \sqrt{1 + \frac{v^2}{c^2} - \frac{2vx_2}{c^2t_2}} = (t_1 + t_2) \sqrt{1 + \frac{v^2}{c^2} - \frac{2vx_2}{c^2}} = (t_1 + t_2) \sqrt{1 + \frac{v^2}{c^2} - \frac{2vx_2}{c^2(t_1 + t_2)}}
\]

The geometry of space and time in the Undulating Relativity is summarized in the figure below that can be expanded to Aₙ points and several observers.
In the figure the angles have a relation $\psi = \phi' - \phi$ and are equal to the following segments:

- $O_1$ to $O_1$ is equal to $O_1$ to $O_1'$, $(O_i \leftrightarrow O_i' = vt_i = v't_i')$

- $O_2$ to $O_1$ is equal to $O_1'$ to $O_2'$, $(O_2 \leftrightarrow O_2' = v(t_2 + t_2) = v'(t_2' + t_2') \rightarrow vt_2 = v't_2' = O_2 \leftrightarrow O_1 + O_1' \leftrightarrow O_2')$

And are parallel to the following segments:

- $O_2$ to $A_2$ is parallel to $O_1$ to $A_1$

- $O_2'$ to $A_2$ is parallel to $O_1'$ to $A_1$

- $X \equiv X'$ is parallel to $X_1 \equiv X'_1$

The cosine of the angles of inclination $\phi$ and $\phi'$ to the rays for the observers $O$ and $O'$ according to 2.3 and 2.4 are:

$$u'x' = \frac{ux - v}{\sqrt{1 + \frac{v^2}{c^2} - \frac{2vux}{c^2}}} \Rightarrow u'x' = \frac{ux - v}{c} \sqrt{1 + \frac{v^2}{c^2} - \frac{2vux}{c^2}} \Rightarrow \cos \phi' = \frac{\cos \phi - v/c}{\sqrt{K}}$$

$$\cos \phi' = \frac{\cos \phi - v/c}{\sqrt{K}} \quad 12.23$$

And with this we have: $\sin \phi' = \frac{\sin \phi}{\sqrt{K}} \quad 12.24$

$$ux = \frac{u'x' + v'}{\sqrt{1 + \frac{v'^2}{c^2} + \frac{2v'ux'}{c^2}}} \Rightarrow ux = \frac{u'x' + v'}{c} \sqrt{1 + \frac{v'^2}{c^2} + \frac{2v'ux'}{c^2}} \Rightarrow \cos \phi = \frac{\cos \phi' + v'/c}{\sqrt{K'}}$$

$$\cos \phi = \frac{\cos \phi' + v'/c}{\sqrt{K'}} \quad 12.25$$
And with this we have \( \text{sen} \phi = \frac{\text{sen}' \phi}{\sqrt{K'}} \)

The cosine of the angle \( \psi \) with intersection of rays equal to:

\[
\cos \psi = \frac{1 - \frac{v u x}{c^2}}{\sqrt{K}} = \frac{1 + \frac{v' u' x'}{c^2}}{\sqrt{K'}} = \frac{1 - \frac{v \cos \phi}{c}}{\sqrt{K}} = \frac{1 + \frac{v' \cos \phi'}{c}}{\sqrt{K'}}
\]

And with this we have: \( \text{sen} \psi = \frac{v \text{sen} \phi}{c \sqrt{K}} = \frac{v' \text{sen} \phi'}{c \sqrt{K'}} \)

The invariance of the \( \cos \psi \) shows the harmony of all adopted hypotheses for space and time in the Undulating Relativity.

The \( \cos \psi \) is equal to the Jacobians of the transformations for the space and time of the picture I, where the radicals

\[
\sqrt{K} = \sqrt{1 + \frac{v^2}{c^2}} - \frac{2vx}{c^2 \tau} \quad \text{and} \quad \sqrt{K'} = \sqrt{1 + \frac{v'^2}{c^2} + \frac{2v' x'}{c^2 \tau'}}
\]

are considered variables and are derived.

\[
\cos \psi = J = \frac{\partial x_i}{\partial x'^j} = \frac{\partial (x', y', z', \tau')}{\partial (x, y, z, \tau)} = \begin{vmatrix} 1 & 0 & 0 & -v \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{vmatrix} = \frac{1 - \frac{vx}{c^2 \tau}}{\sqrt{K}} = \frac{1 - \frac{v u x}{c^2}}{\sqrt{K}}
\]

\[
\cos \psi = J' = \frac{\partial x^k}{\partial x'^i} = \frac{\partial (x', y', z', \tau')}{\partial (x, y, z, \tau)} = \begin{vmatrix} 1 & 0 & 0 & v' \\ 0 & 1 & 0 & 0 \\ v'/c^2 & 0 & 1 & 0 \end{vmatrix} = \frac{1 + \frac{v' x'}{c^2 \tau'}}{\sqrt{K'}} = \frac{1 + \frac{v' u' x'}{c^2 \tau'}}{\sqrt{K'}}
\]

\section*{§13 Richard C. Tolman}

The §4 Transformations of the Momenta of Undulating Relativity was developed based on the experience conducted by Lewis and Tolman, according to the reference [3]. Where the collision of two spheres preserving the principle of conservation of energy and the principle of conservation of momenta, shows that the mass is a function of the velocity according to:

\[
m = \frac{m_o}{\sqrt{1 - (u^2/c^2)}}
\]

where \( m_o \) is the mass of the sphere when in resting position and \( u = |\vec{u}| = \sqrt{u^2} \) the module of its speed.

Analyzing the collision between two identical spheres when in relative resting position, that for the observer O' are named \( S'_1 \) and \( S'_2 \) are moving along the axis \( x' \) in the contrary way with the following velocities before the collision:

<table>
<thead>
<tr>
<th>Sphere ( S'_1 )</th>
<th>Sphere ( S'_2 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( u' x'_1 = v' )</td>
<td>( u' x'_2 = -v' )</td>
</tr>
<tr>
<td>( u' y'_1 = \text{zero} )</td>
<td>( u' y'_2 = \text{zero} )</td>
</tr>
<tr>
<td>( u' z'_1 = \text{zero} )</td>
<td>( u' z'_2 = \text{zero} )</td>
</tr>
</tbody>
</table>

For the observer O the same spheres are named \( S_1 \) and \( S_2 \) and have the velocities \( (u x_1, u x_2, u y_1 = u z_1 = \text{zero}) \) before the collision calculated according to the table 2 as follows:
The velocity $ux_i$ of the sphere $S_1$ is equal to:

$$ux_i = \frac{u'x'_i + v'}{\sqrt{1 + \frac{v'^2}{c^2} + \frac{2v'x'_i}{c^2}}} = \frac{v' + v'}{\sqrt{1 + \frac{v'^2}{c^2} + \frac{2v'v'}{c^2}}} = \frac{2v'}{\sqrt{1 + \frac{3v'^2}{c^2}}}.$$  

The transformation from $v'$ to $v$ according to 1.20 from Table 2 is:

$$v = \frac{v'}{\sqrt{1 + \frac{v'^2}{c^2} + \frac{2v'u'x'_i}{c^2}}} = \frac{v'}{\sqrt{1 + \frac{v'^2}{c^2} + \frac{2v'v'}{c^2}}} = \frac{v'}{\sqrt{1 + \frac{3v'^2}{c^2}}}.$$  

That applied in $ux_i$ supplies:

$$ux_i = 2 \left( \frac{v'}{\sqrt{1 + \frac{3v'^2}{c^2}}} \right) = 2v$$

The velocity $ux_2$ of the sphere $S_2$ is equal to:

$$ux_2 = \frac{u'x'_2 + v'}{\sqrt{1 + \frac{v'^2}{c^2} + \frac{2v'u'x'_2}{c^2}}} = \frac{-v' + v'}{\sqrt{1 + \frac{v'^2}{c^2} + \frac{2v'(-v')}{c^2}}} = zero$$

<table>
<thead>
<tr>
<th>Sphere $S_1$</th>
<th>Sphere $S_2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$ux_i = \frac{2v'}{\sqrt{1 + \frac{3v'^2}{c^2}}} = 2v$</td>
<td>$ux_2 = zero$</td>
</tr>
<tr>
<td>$uy_i = zero$</td>
<td>$uy_2 = zero$</td>
</tr>
<tr>
<td>$uz_i = zero$</td>
<td>$uz_2 = zero$</td>
</tr>
</tbody>
</table>

For the observers $O$ and $O'$ the two spheres have the same mass when in relative resting position. And for the observer $O'$ the two spheres collide with velocities of equal module and opposite direction because of this the momenta $\left( p'_1 = p'_2 \right)$ null themselves during the collision, forming for a brief time $\left( \Delta t' \right)$ only one body of mass

$$m_0 = m'_1 + m'_2.$$  

According to the principle of conservation of momenta for the observer $O$ we will have to impose that the momenta before the collision are equal to the momenta after the collision, thus:

$$m_1ux_1 + m_2ux_2 = (m_1 + m_2)w$$

Where for the observer $O$, $w$ is the arbitrary velocity that supposedly for a brief time $\left( \Delta t' \right)$ will also see the masses united $\left( m = m'_1 + m'_2 \right)$ moving. As the masses $m_i$ have different velocities and the masses vary according to their own velocities, this equation cannot be simplified algebraically, having this variation of masses:

To the left side of the equal sign in the equation we have:

$$u = ux_i = 2v$$

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\[ m_j = \frac{m_o}{\sqrt{1 - \left( \frac{u}{c} \right)^2}} = \frac{m_o}{\sqrt{1 - \left( \frac{ux_j}{c} \right)^2}} = \frac{m_o}{\sqrt{1 - \left( \frac{2v}{c} \right)^2}} = \frac{m_o}{\sqrt{1 - \frac{4v^2}{c^2}}} \]

\[ u = ux_2 = \text{zero} \]

\[ m_2 = \frac{m_o}{\sqrt{1 - \left( \frac{u}{c} \right)^2}} = \frac{m_o}{\sqrt{1 - \left( \frac{ux_2}{c} \right)^2}} = \frac{m_o}{\sqrt{1 - \left( \text{zero} \right)^2}} = m_o \]

To the right side of the equal sign in the equation we have:

\[ u = w \]

\[ m_jux_j + m_2ux_2 = (m_j + m_2)w = m_jw + m_2w \]

\[ \frac{m_o}{\sqrt{1 - \frac{4v^2}{c^2}}} 2v + m_0w = \frac{m_o}{\sqrt{1 - \frac{w^2}{c^2}}} w + \frac{m_o}{\sqrt{1 - \frac{w'^2}{c^2}}} w \]

From where we have:

\[ \frac{2m_0v}{\sqrt{1 - \frac{4v^2}{c^2}}} = \frac{2m_0w}{\sqrt{1 - \frac{w'^2}{c^2}}} \Rightarrow \frac{v}{\sqrt{1 - \frac{4v^2}{c^2}}} = \frac{w}{\sqrt{1 - \frac{w'^2}{c^2}}} \]

\[ w = \frac{v}{\sqrt{1 - \frac{3v^2}{c^2}}} \]

As \( w \neq v \) for the observer \( O \) the masses united \( m = m_j + m_2 \) wouldn’t move momentarily alongside to the observer \( O' \) which is conceivable if we consider that the instants \( \Delta t \neq \Delta t' \) are different where supposedly the masses would be in a resting position from the point of view of each observer and that the mass acting with velocity \( 2v \) is bigger than the mass in resting position.

If we operate with these variables in line we would have:

\[ m_jux_j + m_2ux_2 = (m_j + m_2)w = m_jw + m_2w \]

\[ \frac{m_o}{\sqrt{1 - \frac{1}{c^2}}} \left( \frac{2v}{1 + \frac{3v^2}{c^2}} \right)^2 + m_o \cdot \theta = \frac{m_o}{\sqrt{1 - \frac{w^2}{c^2}}} w + \frac{m_o}{\sqrt{1 - \frac{w'^2}{c^2}}} w - \frac{2m_0w}{\sqrt{1 - \frac{w'^2}{c^2}}} \]
\[
\left( \frac{1+3v'}{c^2} \right) \left( \frac{1-\frac{1}{c^2} \left( \frac{4v'^2}{1+3v'} \right) }{c^2} \right) = \frac{2m_0 v'}{\sqrt{1-w'^2}}
\]

\[
\frac{2m_0 v'}{\sqrt{1+3v'^2} \frac{4v'^2}{c^2}} = \frac{2m_0 w'}{\sqrt{1-w'^2}}
\]

\[
\frac{2m_0 v'}{\sqrt{1-w'^2}} = \frac{2m_0 w}{\sqrt{1-w^2}}
\]

From where we conclude that \( w = v' \) which must be equal to the previous value of \( w \), that is:

\[
w = v' = \frac{v}{\sqrt{1-\frac{3v^2}{c^2}}}
\]

A relation between \( v \) and \( v' \) that is obtained from Table 2 when \( ux = 2v \) that corresponds for the observer \( O' \) to the velocity acting over the sphere in resting position.

### §14 Velocities composition

Reference – Millennium Relativity

URL: http://www.mrelativity.net/MBriefs/VComp_Sci_Estab_Way.htm

Let’s write the transformations of Hendrik A. Lorentz for space and time in the Special Theory of Relativity:

<table>
<thead>
<tr>
<th>( x' = \frac{x-vt}{\sqrt{1-v^2/c^2}} )</th>
<th>14.1a</th>
<th>( x = \frac{x'+vt'}{\sqrt{1-v^2/c^2}} )</th>
<th>14.3a</th>
</tr>
</thead>
<tbody>
<tr>
<td>( y' = y )</td>
<td>14.1b</td>
<td>( y = y' )</td>
<td>14.3b</td>
</tr>
<tr>
<td>( z' = z )</td>
<td>14.1c</td>
<td>( z = z' )</td>
<td>14.3c</td>
</tr>
<tr>
<td>( t' = \frac{t-\frac{vx}{c^2}}{\sqrt{1-v^2/c^2}} )</td>
<td>14.2</td>
<td>( t = \frac{t'+\frac{vx'}{c^2}}{\sqrt{1-v^2/c^2}} )</td>
<td>14.4</td>
</tr>
</tbody>
</table>

From them we obtain the equations of velocity transformation:

| \( u'x' = \frac{ux-v}{l-\frac{vu'x}{c^2}} \) | 14.5a | \( ux = \frac{u'x'+v}{l+\frac{vu'x'}{c^2}} \) | 14.6a |
| \( u'y' = \frac{uy}{l-\frac{vu'x}{c^2}} \) | 14.5b | \( uy = \frac{u'y'+v}{l+\frac{vu'x'}{c^2}} \) | 14.6b |
| \( u'z' = \frac{uz}{l-\frac{vu'x}{c^2}} \) | 14.5c | \( uz = \frac{u'z'+v}{l+\frac{vu'x'}{c^2}} \) | 14.6c |

Let’s consider that in relation to the observer \( O' \) an object moves with velocity:

\( u'x' = 1.5 \cdot 10^5 \text{ km/s} (= 0.5c) \)
And that the velocity of the observer $O'$ in relation to the observer $O$ is:

$$v=1.5 \times 10^5 \text{ km/s} (=0.50c).$$

The velocity $ux$ of the object in relation to the observer $O$ must be calculated by the formula 14.6a:

$$ux = \frac{u'x' + v}{1 + \frac{vux'}{c^2}} = \frac{1.5 \times 10^5 + 1.5 \times 10^5}{1 + \frac{1.5 \times 10^5 \times 1.5 \times 10^5}{(3 \times 10^5)^2}} = 2.4 \times 10^3 \text{ km/s} (=0.80c).$$

Where we use $c=3 \times 10^5 \text{ km/s} (=1.00c)$.

Considering that the object has moved during one second in relation to the observer $O$ ($t=1.00s$) we can then with 14.2 calculate the time passed to the observer $O'$:

$$t' = \frac{t - \frac{vx}{c^2}}{\sqrt{1 - \frac{v^2}{c^2}}} = \frac{1.00 \left(1 - \frac{1.5 \times 10^5 \times 2.4 \times 10^5}{(3 \times 10^5)^2}\right)}{\sqrt{1 - \left(\frac{1.5 \times 10^5}{3 \times 10^5}\right)^2}} = \frac{0.60}{\sqrt{0.75}} \Rightarrow t'=0.693s.$$

To the observer $O$ the observer $O'$ is away the distance $d$ given by the formula:

$$d=vt = 1.5 \times 10^5 \times 1.00 = 1.5 \times 10^5 \text{ km}.$$

To the observer $O'$ the observer $O$ is away the distance $d'$ given by the formula:

$$d'=vt' = 1.5 \times 10^5 \times \frac{0.60}{\sqrt{0.75}} = 1.03923 \times 10^5 \text{ km}.$$

To the distance of the object ($d_o$, $d'_o$) in relation to the observers $O$ and $O'$ is given by the formulae:

$$d_o = uxt = 2.4 \times 10^5 \times 1.00 = 2.4 \times 10^5 \text{ km}.$$

$$d'_o = u'x't' = 1.5 \times 10^5 \times \frac{0.60}{\sqrt{0.75}} = 1.03923 \times 10^5 \text{ km}.$$

To the observer $O$ the distance between the object and the observer $O'$ is given by the formula:

$$\Delta d = d_o - d = 2.4 \times 10^5 - 1.5 \times 10^5 = 0.90 \times 10^5 \text{ km}.$$

To the observer $O$ the velocity of the object in relation to the observer $O'$ is given by:

$$\frac{\Delta d}{t} = \frac{0.90 \times 10^5 \text{ km}}{1.00s} = 0.90 \times 10^5 \text{ km/s} (=.30c)$$

Relating the times $t$ and $t'$ using the formula $t'=t \frac{1 - \frac{v^2}{c^2}}{\sqrt{1 - \frac{v^2}{c^2}}}$ is only possible and exclusively when $ux=v$ and $u'x'=zero$ what isn’t the case above, to make it possible to understand this we write the equations 14.2 and 14.4 in the formula below:

<table>
<thead>
<tr>
<th>$t' = \frac{\left(1 - \frac{v \cos \phi}{c}\right)}{\sqrt{1 - \frac{v^2}{c^2}}}$</th>
<th>14.2</th>
<th>$t = \frac{\left(1 + \frac{v \cos \phi}{c}\right)}{\sqrt{1 - \frac{v^2}{c^2}}}$</th>
<th>14.4</th>
</tr>
</thead>
<tbody>
<tr>
<td>$t'$</td>
<td>$t$</td>
<td>37/172</td>
<td></td>
</tr>
</tbody>
</table>
Where \( \cos \phi = \frac{x}{ct} \) and \( \cos \phi' = \frac{x'}{ct'} \).

The equations above can be written as:

\[
t' = f(t, \phi) \quad \text{and} \quad t = f'(t', \phi')
\]

14.7

In each referential of the observers O and O' the light propagation creates a sphere with radius \( ct \) and \( ct' \) that intercept each other forming a circumference that propagates with velocity c. The radius \( ct \) and \( ct' \) and the positive way of the axis \( x \) and \( x' \) form the angles \( \phi \) and \( \phi' \) constant between the referentials. If for the same pair of referentials \( \theta \) and \( \phi' \) the angles were variable the time would be aleatory and would become useless for the Physics. In the equation \( t' = f(t, \phi) \) we have \( t' \) identical function of \( t \) and \( \phi \), if we have in it \( \phi \) constant and \( t' \) varies according to \( t \) we get the common relation between the times \( t \) and \( t' \) between two referentials, however if we have \( t \) constant and \( t' \) varies according to \( \phi \) we will have for each value of \( \phi \) one value of \( t' \) and \( t \) between two different referentials, and this analysis is also valid for \( t = f'(t', \phi') \).

Dividing 14.5a by c we have:

\[
\frac{u'x'}{c} = \frac{u}{c} - \frac{v}{c} \quad \frac{\cos \phi' - v}{c} = \frac{\cos \phi - \frac{v}{c}}{c}
\]

14.8

Where \( \cos \phi = \frac{x}{ct} \) and \( \cos \phi' = \frac{x'}{ct'} \).

Isolating the velocity we have:

\[
\frac{v}{c} = \frac{(\cos \phi - \cos \phi')}{(I - \cos \phi \cos \phi')} \quad \text{or} \quad v = \frac{u - u'x'}{c^2}
\]

14.9

From where we conclude that we must have angles \( \phi \) and \( \phi' \) constant so that we have the same velocity between the referentials.

This demand of constant angles between the referentials must solve the controversies of Herbert Dingle.

§15 Invariance

The transformations to the space and time of table I, group 1.2 plus 1.7, in the matrix form is written like this:

\[
\begin{bmatrix}
x' \\
y' \\
z' \\
t'
\end{bmatrix} = \begin{bmatrix}
1 & 0 & 0 & -v \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & \sqrt{K}
\end{bmatrix} \begin{bmatrix}
x \\
y \\
z \\
t
\end{bmatrix}
\]

15.1

That written in the form below represents the same coordinate transformations:

\[
\begin{bmatrix}
x' \\
y' \\
z' \\
t'
\end{bmatrix} = \begin{bmatrix}
1 & 0 & 0 & -v/c \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & \sqrt{K}/ct
\end{bmatrix} \begin{bmatrix}
x \\
y \\
z \\
t
\end{bmatrix}
\]

15.2

We call as:

\[
x' = x^i = \begin{bmatrix}
x' \\
y' \\
z' \\
t'
\end{bmatrix}, \quad \alpha = a_{ij} = \begin{bmatrix}
1 & 0 & 0 & -v/c \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & \sqrt{K}/ct
\end{bmatrix}, \quad x = x^i = \begin{bmatrix}
x \\
y \\
z \\
t
\end{bmatrix}
\]

15.3
That are the functions \( x'^i = x'^i(x^j) = x'^i(x^1, x^2, x^3, c x^4) = x'^i(x, y, z, c t) \) 15.4

That in the symbolic form is written:
\[ x'^i = \alpha \cdot x^i \] or in the indexed form \( x'^i = \sum_{j=1}^{d} \alpha_{ij} x^j \Rightarrow x'^i = \alpha_{ij} x^j \) 15.5

Where we use Einstein’s sum convention.

The transformations to the space and time of table I, group 1.4 plus 1.8, in the matrix form is written:
\[
\begin{bmatrix}
 x' \\
 y' \\
 z' \\
 c t'
\end{bmatrix} =
\begin{bmatrix}
 1 & 0 & 0 & v' / c \\
 0 & 1 & 0 & 0 \\
 0 & 0 & 1 & 0 \\
 0 & 0 & 0 & \sqrt{K'}
\end{bmatrix}
\begin{bmatrix}
 x \\
 y \\
 z \\
 c t
\end{bmatrix}
\] 15.6

That written in the form below represents the same coordinate transformations:
\[
\begin{bmatrix}
 x' \\
 y' \\
 z' \\
 c t'
\end{bmatrix} =
\begin{bmatrix}
 1 & 0 & 0 & v' / c \\
 0 & 1 & 0 & 0 \\
 0 & 0 & 1 & 0 \\
 0 & 0 & 0 & \sqrt{K'}
\end{bmatrix}
\begin{bmatrix}
 x \\
 y \\
 z \\
 c t
\end{bmatrix} =
\begin{bmatrix}
 1 & 0 & 0 & v' / c \\
 0 & 1 & 0 & 0 \\
 0 & 0 & 1 & 0 \\
 0 & 0 & 0 & \sqrt{K'}
\end{bmatrix}
\begin{bmatrix}
 x \\
 y \\
 z \\
 c t
\end{bmatrix}
\] 15.7

That we call as:
\[
x = x^k =
\begin{bmatrix}
 x \\
 y \\
 z \\
 c t
\end{bmatrix} =
\begin{bmatrix}
 x^1 \\
 x^2 \\
 x^3 \\
 c x^4
\end{bmatrix}, \quad \alpha' = \alpha'_{kl} =
\begin{bmatrix}
 1 & 0 & 0 & v' / c \\
 0 & 1 & 0 & 0 \\
 0 & 0 & 1 & 0 \\
 0 & 0 & 0 & \sqrt{K'}
\end{bmatrix},
\quad x' = x'^l =
\begin{bmatrix}
 x' \\
 y' \\
 z' \\
 c t'
\end{bmatrix} =
\begin{bmatrix}
 x' \\
 y' \\
 z' \\
 c x'^4
\end{bmatrix}
\] 15.8

That are the functions \( x^k = x^k(x'^i) = x^k(x'^1, x'^2, x'^3, c x'^4) = x^k(x', y', z', c t') \) 15.9

That in the symbolic form is written:
\[ x = \alpha' \cdot x' \] or in the indexed form \( x^k = \sum_{i=1}^{d} \alpha'_{ki} x'^i \Rightarrow x^k = \alpha'_{ki} x'^i \) 15.10

Being \( \sqrt{K} = \sqrt{1 + \frac{v^2}{c^2}} \) (1.7), \( \sqrt{K'} = \sqrt{1 + \frac{v^2}{c^2} + \frac{2v'^2}{c^2}} \) (1.8) and \( \sqrt{K} \sqrt{K'} = 1 \) (1.10).

The transformation matrices \( \alpha = \alpha_{ij} \) and \( \alpha' = \alpha'_{kl} \) have the properties:
\[
\alpha \cdot \alpha' = \sum_{j=1}^{4} \alpha_{ij} \alpha'_{kl} =
\begin{bmatrix}
 1 & 0 & 0 & 0 \\
 0 & 1 & 0 & 0 \\
 0 & 0 & 1 & 0 \\
 0 & 0 & 0 & 1
\end{bmatrix}
\begin{bmatrix}
 1 & 0 & 0 & 0 \\
 0 & 1 & 0 & 0 \\
 0 & 0 & 1 & 0 \\
 0 & 0 & 0 & 1
\end{bmatrix} =
\begin{bmatrix}
 1 & 0 & 0 & 0 \\
 0 & 1 & 0 & 0 \\
 0 & 0 & 1 & 0 \\
 0 & 0 & 0 & 1
\end{bmatrix} = I = \delta_i^j
\] 15.11

\[
\alpha'^t \cdot \alpha'^t = \sum_{i=1}^{4} \alpha'^t_{ji} \alpha'^t_{ik} =
\begin{bmatrix}
 1 & 0 & 0 & 0 \\
 0 & 1 & 0 & 0 \\
 0 & 0 & 1 & 0 \\
 -v / c \sqrt{K'} & 0 & 0 & 1
\end{bmatrix}
\begin{bmatrix}
 1 & 0 & 0 & 0 \\
 0 & 1 & 0 & 0 \\
 0 & 0 & 1 & 0 \\
 0 & 0 & 0 & 1
\end{bmatrix} =
\begin{bmatrix}
 1 & 0 & 0 & 0 \\
 0 & 1 & 0 & 0 \\
 0 & 0 & 1 & 0 \\
 0 & 0 & 0 & 1
\end{bmatrix} = I = \delta_i^j
\] 15.12

Where \( \alpha'^t = \alpha'^t_{ji} \) is the transposed matrix of \( \alpha = \alpha_{ij} \) and \( \alpha'^t = \alpha'^t_{ik} \) is the transpose matrix of \( \alpha' = \alpha'_{kl} \) and \( \delta \) is the Kronecker’s delta.
\[
\alpha' \cdot \alpha' = \sum_{i=1}^{4} \alpha'_{ki} \alpha_{ij} =
\begin{bmatrix}
 1 & 0 & 0 & v' / c \\
 0 & 1 & 0 & 0 \\
 0 & 0 & 1 & 0 \\
 0 & 0 & 0 & \sqrt{K'}
\end{bmatrix}
\begin{bmatrix}
 1 & 0 & 0 & 0 \\
 0 & 1 & 0 & 0 \\
 0 & 0 & 1 & 0 \\
 0 & 0 & 0 & 1
\end{bmatrix} =
\begin{bmatrix}
 1 & 0 & 0 & 0 \\
 0 & 1 & 0 & 0 \\
 0 & 0 & 1 & 0 \\
 0 & 0 & 0 & 1
\end{bmatrix} = I = \delta_i^j
\] 15.13
\[
\alpha^i = \alpha'_{ik} \alpha_j = \sum_{k=1}^{4} \alpha'_{ik} \alpha_j = \begin{bmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
v/c & 0 & 0 & \sqrt{\gamma}
\end{bmatrix} \begin{bmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
-v/c & 0 & 0 & \sqrt{\gamma}
\end{bmatrix} = I = \delta^i_j
\]

Where \( \alpha'' = \alpha'_{ik} \) is the transposed matrix of \( \alpha' = \alpha'_{kl} \) and \( \alpha' = \alpha'_{ji} \) is the transposed matrix of \( \alpha = \alpha_j \) and \( \delta \) is the Kronecker’s delta.

Observation: the matrices \( \alpha_j \) and \( \alpha'_{kl} \) are inverse of one another but are not orthogonal, that is: \( \alpha_j \neq \alpha'_{kl} \) and \( \alpha_j \neq \alpha'_{ik} \).

The partial derivatives \( \frac{\partial x^i}{\partial x^j} \) of the total differential \( dx^i = \frac{\partial x^i}{\partial x^j} dx^j \) of the coordinate components that correlate according to \( x^i = x^i(x^j) \), where in the transformation matrix \( \alpha = \alpha_j \) the radical \( \sqrt{\gamma} \) is considered constant and equal to:

<table>
<thead>
<tr>
<th>Table 10, partial derivatives of the coordinate components:</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \frac{\partial x^i}{\partial x^j} = \frac{\partial x^i}{\partial x^j} = 1 )</td>
</tr>
<tr>
<td>( \frac{\partial x^i}{\partial x^j} = \frac{\partial x^i}{\partial x^j} = 0 )</td>
</tr>
<tr>
<td>( \frac{\partial x^i}{\partial x^j} = \frac{\partial x^i}{\partial x^j} = -v/c )</td>
</tr>
<tr>
<td>( \frac{\partial x^i}{\partial x^j} = \frac{\partial x^i}{\partial x^j} = 0 )</td>
</tr>
<tr>
<td>( \frac{\partial x^i}{\partial x^j} = \frac{\partial x^i}{\partial x^j} = 1 )</td>
</tr>
<tr>
<td>( \frac{\partial x^i}{\partial x^j} = \frac{\partial x^i}{\partial x^j} = 0 )</td>
</tr>
<tr>
<td>( \frac{\partial x^i}{\partial x^j} = \frac{\partial x^i}{\partial x^j} = \sqrt{\gamma} )</td>
</tr>
</tbody>
</table>

The total differential of the coordinates in the matrix form is equal to:

\[
\begin{bmatrix}
dx^1 \\
dx^2 \\
dx^3 \\
dx^4 \\
dx^4
\end{bmatrix} = \begin{bmatrix}
1 & 0 & 0 & -v/c \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
v/c & 0 & 0 & \sqrt{\gamma}
\end{bmatrix} \begin{bmatrix}
dx^1 \\
dx^2 \\
dx^3 \\
\dx^4
\end{bmatrix}
\]

That we call as:

\[
dx^i = dx^i = \begin{bmatrix}
dx^1 \\
dx^2 \\
dx^3 \\
\cdx^4
\end{bmatrix}, \quad A = A_j^i = \frac{\partial x^i}{\partial x^j} = \begin{bmatrix}
1 & 0 & 0 & -v/c \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & \sqrt{\gamma}
\end{bmatrix}, \quad dx^i = dx^i = \begin{bmatrix}
dx^1 \\
dx^2 \\
dx^3 \\
\cdx^4
\end{bmatrix}
\]

Then we have \( dx^i = Adx \Rightarrow dx^i = \sum_{j=1}^{4} A_j^i dx^j \Rightarrow dx^i = \frac{\partial x^i}{\partial x^j} dx^j \)

The partial derivatives \( \frac{\partial x^k}{\partial x^l} \) of the total differential \( dx^i = \frac{\partial x^k}{\partial x^j} dx^j \) of the coordinate components that correlate according to \( x^i = x^i(x^j) \), where in the transformation matrix \( \alpha' = \alpha'_{kl} \) the radical \( \sqrt{\gamma} \) is considered constant and equal to:

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Table 11 partial derivatives of the coordinate components:

<table>
<thead>
<tr>
<th></th>
<th>(\frac{\partial x^1}{\partial x'^i})</th>
<th>(\frac{\partial x^1}{\partial x'^j})</th>
<th>(\frac{\partial x^2}{\partial x'^i})</th>
<th>(\frac{\partial x^2}{\partial x'^j})</th>
<th>(\frac{\partial x^3}{\partial x'^i})</th>
<th>(\frac{\partial x^3}{\partial x'^j})</th>
<th>(\frac{\partial x^4}{\partial x'^i})</th>
<th>(\frac{\partial x^4}{\partial x'^j})</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>(\frac{\partial x^1}{\partial x'^1})</td>
<td>(\frac{\partial x^1}{\partial x'^2})</td>
<td>(\frac{\partial x^2}{\partial x'^1})</td>
<td>(\frac{\partial x^2}{\partial x'^2})</td>
<td>(\frac{\partial x^3}{\partial x'^1})</td>
<td>(\frac{\partial x^3}{\partial x'^2})</td>
<td>(\frac{\partial x^4}{\partial x'^1})</td>
<td>(\frac{\partial x^4}{\partial x'^2})</td>
</tr>
<tr>
<td>2</td>
<td>(\frac{\partial x^1}{\partial x'^3})</td>
<td>(\frac{\partial x^1}{\partial x'^4})</td>
<td>(\frac{\partial x^2}{\partial x'^3})</td>
<td>(\frac{\partial x^2}{\partial x'^4})</td>
<td>(\frac{\partial x^3}{\partial x'^3})</td>
<td>(\frac{\partial x^3}{\partial x'^4})</td>
<td>(\frac{\partial x^4}{\partial x'^3})</td>
<td>(\frac{\partial x^4}{\partial x'^4})</td>
</tr>
</tbody>
</table>

The total differential of the coordinates in the matrix form is equal to:

\[
\begin{bmatrix}
\frac{dx^1}{cdx^4} & \frac{dx^2}{cdx^4} & \frac{dx^3}{cdx^4} & \frac{dx^4}{cdx^4}
\end{bmatrix}
\begin{bmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & \sqrt{K}
\end{bmatrix}
\begin{bmatrix}
\frac{dx^1}{dx^i} & \frac{dx^2}{dx^i} & \frac{dx^3}{dx^i} & \frac{dx^4}{dx^i}
\end{bmatrix}
\]

That we call as:

\[
dx^4 = dx^i = \begin{bmatrix}
\frac{dx^1}{dx^i} \\
\frac{dx^2}{dx^i} \\
\frac{dx^3}{dx^i} \\
\frac{dx^4}{dx^i}
\end{bmatrix}, \quad A'=A^i_j = \frac{\partial x^i}{\partial x'^j} = \begin{bmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & \sqrt{K}
\end{bmatrix}, \quad dx^i = dx^i = \begin{bmatrix}
\frac{dx^1}{dx^i} \\
\frac{dx^2}{dx^i} \\
\frac{dx^3}{dx^i} \\
\frac{dx^4}{dx^i}
\end{bmatrix}
\]

Then we have: \(dx = A'dx' \Rightarrow dx^4 = \sum_{j=1}^{4} A^i_j dx'^j \Rightarrow dx^4 = \frac{\partial x^4}{\partial x'^j} dx'^j \)

The Jacobians of the transformations 15.15 and 15.18 are:

\[
J = \begin{vmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & \sqrt{K}
\end{vmatrix} = \sqrt{K}
\]

\[
J' = \begin{vmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & \sqrt{K'}
\end{vmatrix} = \sqrt{K'}
\]

Where \(\sqrt{K} = \sqrt{1+\frac{\nu^2}{c^2} - \frac{2\nuux}{c^2}} \) (2.5), \(\sqrt{K'} = \sqrt{1+\frac{\nu'^2}{c'^2} + \frac{2\nu'u'x'}{c'^2}} \) (2.6) and \(\sqrt{K'} = 1 \) (1.23).

The matrices of the transformation \(A\) and \(A'\) also have the properties 15.11, 15.12, 15.13 and 15.14 of the matrices \(\alpha\) and \(\alpha'\).

From the function \(\phi = \phi(x^4) = \phi'[x^4(x'^4)]\) where the coordinates correlate in the form \(x^4 = x^4(x'^4)\) we have \(\frac{\partial \phi}{\partial x'^4} = \frac{\partial \phi}{\partial x^4} \frac{\partial x^4}{\partial x'^4}\) described as:
From the function $\phi$ we get:

\[
\begin{align*}
\frac{\partial \phi}{\partial x^1} &= \frac{\partial \phi}{\partial x^k} \frac{\partial x^k}{\partial x^1} = \frac{\partial \phi}{\partial x^1} \frac{\partial x^1}{\partial x^1} + \frac{\partial \phi}{\partial x^2} \frac{\partial x^2}{\partial x^1} + \frac{\partial \phi}{\partial x^3} \frac{\partial x^3}{\partial x^1} + \frac{\partial \phi}{\partial x^4} \frac{\partial x^4}{\partial x^1} \\
\frac{\partial \phi}{\partial x^2} &= \frac{\partial \phi}{\partial x^k} \frac{\partial x^k}{\partial x^2} = \frac{\partial \phi}{\partial x^1} \frac{\partial x^1}{\partial x^2} + \frac{\partial \phi}{\partial x^2} \frac{\partial x^2}{\partial x^2} + \frac{\partial \phi}{\partial x^3} \frac{\partial x^3}{\partial x^2} + \frac{\partial \phi}{\partial x^4} \frac{\partial x^4}{\partial x^2} \\
\frac{\partial \phi}{\partial x^3} &= \frac{\partial \phi}{\partial x^k} \frac{\partial x^k}{\partial x^3} = \frac{\partial \phi}{\partial x^1} \frac{\partial x^1}{\partial x^3} + \frac{\partial \phi}{\partial x^2} \frac{\partial x^2}{\partial x^3} + \frac{\partial \phi}{\partial x^3} \frac{\partial x^3}{\partial x^3} + \frac{\partial \phi}{\partial x^4} \frac{\partial x^4}{\partial x^3} \\
\frac{\partial \phi}{\partial x^4} &= \frac{\partial \phi}{\partial x^k} \frac{\partial x^k}{\partial x^4} = \frac{\partial \phi}{\partial x^1} \frac{\partial x^1}{\partial x^4} + \frac{\partial \phi}{\partial x^2} \frac{\partial x^2}{\partial x^4} + \frac{\partial \phi}{\partial x^3} \frac{\partial x^3}{\partial x^4} + \frac{\partial \phi}{\partial x^4} \frac{\partial x^4}{\partial x^4}
\end{align*}
\]

That in the matrix form and without presenting the function $\phi$ becomes:

\[
\begin{bmatrix}
\frac{\partial \phi}{\partial x^1} \\
\frac{\partial \phi}{\partial x^2} \\
\frac{\partial \phi}{\partial x^3} \\
\frac{\partial \phi}{\partial x^4}
\end{bmatrix} = 
\begin{bmatrix}
\frac{\partial}{\partial x^1} \frac{\partial}{\partial x^2} \frac{\partial}{\partial x^3} \frac{\partial}{\partial x^4}
\end{bmatrix}
\begin{bmatrix}
\frac{\partial x^1}{\partial x^1} & \frac{\partial x^1}{\partial x^2} & \frac{\partial x^1}{\partial x^3} & \frac{\partial x^1}{\partial x^4} \\
\frac{\partial x^2}{\partial x^1} & \frac{\partial x^2}{\partial x^2} & \frac{\partial x^2}{\partial x^3} & \frac{\partial x^2}{\partial x^4} \\
\frac{\partial x^3}{\partial x^1} & \frac{\partial x^3}{\partial x^2} & \frac{\partial x^3}{\partial x^3} & \frac{\partial x^3}{\partial x^4} \\
\frac{\partial x^4}{\partial x^1} & \frac{\partial x^4}{\partial x^2} & \frac{\partial x^4}{\partial x^3} & \frac{\partial x^4}{\partial x^4}
\end{bmatrix}
\begin{bmatrix}
\frac{\partial \phi}{\partial x^1} \\
\frac{\partial \phi}{\partial x^2} \\
\frac{\partial \phi}{\partial x^3} \\
\frac{\partial \phi}{\partial x^4}
\end{bmatrix}
\]

Where replacing the items below:

\[
\begin{align*}
\frac{\partial x^4}{\partial x^1} &= \frac{v'}{c^2 \sqrt{K}} = \frac{v}{c^2} \\
\frac{\partial x^1}{\partial x^4} &= v = \frac{v}{\sqrt{K}} \\
\frac{\partial x^4}{\partial x^4} &= \frac{1}{\sqrt{K}} \left( 1 + \frac{v^2}{c^2} \right) = \frac{\partial x^4}{\partial x^4} = \frac{1}{\sqrt{K}} \left( 1 + \frac{v^2}{c^2} \right) \\
\frac{\partial x^4}{\partial x^4} &= \frac{1}{\sqrt{K}} \left( 1 + \frac{v^2}{c^2} \right) = \frac{\partial x^4}{\partial x^4} = \frac{1}{\sqrt{K}} \left( 1 + \frac{v^2}{c^2} \right)
\end{align*}
\]

Observation: this last relation shows that the time varies in an equal form between the referentials.

We get:

\[
\begin{bmatrix}
\frac{\partial \phi}{\partial x^1} \\
\frac{\partial \phi}{\partial x^2} \\
\frac{\partial \phi}{\partial x^3} \\
\frac{\partial \phi}{\partial x^4}
\end{bmatrix} = 
\begin{bmatrix}
\frac{\partial}{\partial x^1} \frac{\partial}{\partial x^2} \frac{\partial}{\partial x^3} \frac{\partial}{\partial x^4}
\end{bmatrix}
\begin{bmatrix}
\frac{\partial x^1}{\partial x^1} & \frac{\partial x^1}{\partial x^2} & \frac{\partial x^1}{\partial x^3} & \frac{\partial x^1}{\partial x^4} \\
\frac{\partial x^2}{\partial x^1} & \frac{\partial x^2}{\partial x^2} & \frac{\partial x^2}{\partial x^3} & \frac{\partial x^2}{\partial x^4} \\
\frac{\partial x^3}{\partial x^1} & \frac{\partial x^3}{\partial x^2} & \frac{\partial x^3}{\partial x^3} & \frac{\partial x^3}{\partial x^4} \\
\frac{\partial x^4}{\partial x^1} & \frac{\partial x^4}{\partial x^2} & \frac{\partial x^4}{\partial x^3} & \frac{\partial x^4}{\partial x^4}
\end{bmatrix}
\begin{bmatrix}
\frac{\partial \phi}{\partial x^1} \\
\frac{\partial \phi}{\partial x^2} \\
\frac{\partial \phi}{\partial x^3} \\
\frac{\partial \phi}{\partial x^4}
\end{bmatrix}
\]

That is the group 8.1 plus 8.3 of the table 9, differential operators, in the matrix form.

From the function $\phi' = \phi(x') = \phi(x^i(x'))$ where the coordinates correlate in the form $x'^i = x^i(x')$ we have $\frac{\partial \phi'}{\partial x^i} = \frac{\partial \phi'}{\partial x^i} \frac{\partial x^i}{\partial x^i}$ described as:
\[
\frac{\partial \phi}{\partial x^1} = \frac{\partial \phi}{\partial x^2} = \frac{\partial \phi}{\partial x^3} = \frac{\partial \phi}{\partial x^4} = \frac{\partial \phi}{\partial x^1} + \frac{\partial \phi}{\partial x^2} + \frac{\partial \phi}{\partial x^3} + \frac{\partial \phi}{\partial x^4}
\]

That in the matrix form and without presenting the function \( \phi \) becomes:

\[
\frac{\partial \phi}{\partial x^j} = \begin{bmatrix}
\begin{array}{cccc}
\partial & \partial & \partial & \partial \\
\partial x^1 & \partial x^2 & \partial x^3 & \partial x^4
\end{array}
\end{bmatrix}
= \begin{bmatrix}
\begin{array}{cccc}
\partial & \partial & \partial & \partial \\
\partial x^1 & \partial x^2 & \partial x^3 & \partial x^4
\end{array}
\end{bmatrix}
\]

Where replacing the items below:

\[
\frac{dx^4}{\partial x^4} = -\frac{\sqrt{K}v u x^4}{c^2}
\]

\[
\frac{dx^4}{\partial x^3} = -\frac{\sqrt{K}v u x^4}{c^2}
\]

\[
\frac{dx^4}{\partial x^2} = -\frac{\sqrt{K}v u x^4}{c^2}
\]

Where replacing the items below:

\[
\frac{dx^1}{\partial x^1} = \frac{\sqrt{K}v u x^4}{c^2}
\]

\[
\frac{dx^2}{\partial x^2} = \frac{\sqrt{K}v u x^4}{c^2}
\]

\[
\frac{dx^3}{\partial x^3} = \frac{\sqrt{K}v u x^4}{c^2}
\]

Observation: this last relation shows that the time varies in an equal form between the referentials.

We get:

\[
\frac{\partial \phi}{\partial x^j} = \begin{bmatrix}
\begin{array}{cccc}
\partial & \partial & \partial & \partial \\
\partial x^1 & \partial x^2 & \partial x^3 & \partial x^4
\end{array}
\end{bmatrix}
= \begin{bmatrix}
\begin{array}{cccc}
\partial & \partial & \partial & \partial \\
\partial x^1 & \partial x^2 & \partial x^3 & \partial x^4
\end{array}
\end{bmatrix}
\]

That is the group 8.2 plus 8.4 from the table 9, differential operators in the matrix form.

Applying 8.5 in 8.3 and in 8.4 we simplify these equations in the following way:
Table 9B, differential operators with the equations 8.3 and 8.4 simplified:

<table>
<thead>
<tr>
<th>[\frac{\partial}{\partial x^i} - \frac{\partial}{\partial x^i} + \frac{v}{c^2} \frac{\partial}{\partial x^i}]</th>
<th>8.1</th>
<th>[\frac{\partial}{\partial x^i} - \frac{\partial}{\partial x^i} - \frac{v'}{c^2} \frac{\partial}{\partial x^i}]</th>
<th>8.2</th>
</tr>
</thead>
<tbody>
<tr>
<td>[\frac{\partial}{\partial x^2} = \frac{\partial}{\partial x^2}]</td>
<td>8.1.1</td>
<td>[\frac{\partial}{\partial x^2} = \frac{\partial}{\partial x^2}]</td>
<td>8.2.1</td>
</tr>
<tr>
<td>[\frac{\partial}{\partial x^3} = \frac{\partial}{\partial x^3}]</td>
<td>8.1.2</td>
<td>[\frac{\partial}{\partial x^3} = \frac{\partial}{\partial x^3}]</td>
<td>8.2.2</td>
</tr>
<tr>
<td>[\frac{-\partial}{c\partial x^4} = \sqrt{K} - \frac{\partial}{c\partial x^4}]</td>
<td>8.3B</td>
<td>[\frac{-\partial}{c\partial x^4} = \sqrt{K'} - \frac{\partial}{c\partial x^4}]</td>
<td>8.4B</td>
</tr>
<tr>
<td>[\frac{\partial}{\partial x^i} + \frac{u'x^1}{c^2} \frac{\partial}{\partial x^4} = \text{zero}]</td>
<td>8.5</td>
<td>[\frac{\partial}{\partial x^i} + \frac{u'x^1}{c^2} \frac{\partial}{\partial x^4} = \text{zero}]</td>
<td>8.5</td>
</tr>
</tbody>
</table>

The table 9B, in the matrix form becomes:

\[
\begin{bmatrix}
\frac{\partial}{\partial x^1} & \frac{\partial}{\partial x^2} & \frac{\partial}{\partial x^3} & \frac{\partial}{\partial x^4} \\
\frac{\partial}{\partial x^1} & \frac{\partial}{\partial x^2} & \frac{\partial}{\partial x^3} & \frac{\partial}{\partial x^4} \\
\frac{\partial}{\partial x^1} & \frac{\partial}{\partial x^2} & \frac{\partial}{\partial x^3} & \frac{\partial}{\partial x^4} \\
\frac{\partial}{\partial x^1} & \frac{\partial}{\partial x^2} & \frac{\partial}{\partial x^3} & \frac{\partial}{\partial x^4}
\end{bmatrix} =
\begin{bmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
-\nu/c & 0 & 0 & \sqrt{K}
\end{bmatrix} \quad 15.23
\]

\[
\begin{bmatrix}
\frac{\partial}{\partial x^1} & \frac{\partial}{\partial x^2} & \frac{\partial}{\partial x^3} & \frac{\partial}{\partial x^4} \\
\frac{\partial}{\partial x^1} & \frac{\partial}{\partial x^2} & \frac{\partial}{\partial x^3} & \frac{\partial}{\partial x^4} \\
\frac{\partial}{\partial x^1} & \frac{\partial}{\partial x^2} & \frac{\partial}{\partial x^3} & \frac{\partial}{\partial x^4} \\
\frac{\partial}{\partial x^1} & \frac{\partial}{\partial x^2} & \frac{\partial}{\partial x^3} & \frac{\partial}{\partial x^4}
\end{bmatrix} =
\begin{bmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
\nu/c & 0 & 0 & \sqrt{K'}
\end{bmatrix} \quad 15.24
\]

The squared matrices of the transformations above are transposed of the matrices A and A'.

**Invariance of the Total Differential**

In the observer O referential the total differential of a function \(\phi(x^i)\) is equal to:

\[
d\phi(x^i) = \frac{\partial \phi}{\partial x^i} dx^i = \frac{\partial \phi}{\partial x^i} dx^i + \frac{\partial \phi}{\partial x^i} dx^2 + \frac{\partial \phi}{\partial x^i} dx^3 + \frac{\partial \phi}{\partial x^i} dx^4 = \left[\begin{array}{cccc}
\frac{dx^1}{dx^1} & \frac{dx^1}{dx^2} & \frac{dx^1}{dx^3} & \frac{dx^4}{dx^1} \\
\frac{dx^2}{dx^1} & \frac{dx^2}{dx^2} & \frac{dx^2}{dx^3} & \frac{dx^4}{dx^2} \\
\frac{dx^3}{dx^1} & \frac{dx^3}{dx^2} & \frac{dx^3}{dx^3} & \frac{dx^4}{dx^3} \\
\frac{dx^4}{dx^1} & \frac{dx^4}{dx^2} & \frac{dx^4}{dx^3} & \frac{dx^4}{dx^4}
\end{array}\right] \quad 15.25
\]

Where the coordinates correlate with the ones from the observer O' according to \(x^i = x^i(x'^i)\), replacing the transformations 15.24 and 15.18 and without presenting the function \(\phi\) we have:

\[
d\phi = \frac{\partial \phi}{\partial x^i} dx^i \left[\begin{array}{cccc}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
-\nu/c & 0 & 0 & \sqrt{K'}
\end{array}\right] = \left[\begin{array}{cccc}
1 & 0 & 0 & \nu/c' \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
-\nu/c & 0 & 0 & \sqrt{K'}
\end{array}\right] \quad 15.26
\]

The multiplication of the middle matrices supplies:

\[
\left[\begin{array}{cccc}
1 & 0 & 0 & \nu/c \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
-\nu/c & 0 & 1 & 0 + 2\nu' dx^4
\end{array}\right] = \left[\begin{array}{cccc}
1 & 0 & 0 & \nu/c \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
-\nu/c & 0 & 0 & \sqrt{K'}
\end{array}\right] \quad 15.27
\]

Result that can be divided in two matrices:

\[
\left[\begin{array}{cccc}
1 & 0 & 0 & \nu/c \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
-\nu/c & 0 & 0 & \sqrt{K'}
\end{array}\right] = \left[\begin{array}{cccc}
0 & 0 & \nu/c \\
0 & 0 & 0 \\
0 & 0 & 0 \\
-\nu/c & 0 & 0 & \sqrt{K'}
\end{array}\right] \quad 15.28
\]
That applied to the total differential supplies:

\[ d\phi = \frac{\partial \phi}{\partial x^i} dx^i = \begin{bmatrix} \frac{\partial}{\partial x^1} & \frac{\partial}{\partial x^2} & \frac{\partial}{\partial x^3} & \frac{\partial}{\partial x^4} \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} + \begin{bmatrix} 0 & 0 & v'/c \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ -v'/c & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} dx^1 \\ dx^2 \\ dx^3 \\ dx^4 \end{bmatrix} \]

15.29

Executing the operations of the second term we have:

\[ \begin{bmatrix} \frac{\partial}{\partial x^1} & \frac{\partial}{\partial x^2} & \frac{\partial}{\partial x^3} & \frac{\partial}{\partial x^4} \end{bmatrix} \begin{bmatrix} 0 & 0 & v'/c \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ -v'/c & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} dx^1 \\ dx^2 \\ dx^3 \\ dx^4 \end{bmatrix} = \frac{-v'}{c^2} \begin{bmatrix} dx^2 \\ dx^3 \\ dx^4 \end{bmatrix} + \frac{2v}{c^2} \begin{bmatrix} dx^1 \\ dx^4 \end{bmatrix} \]

Where applying 8.5 we have:

\[ \frac{v'}{c^2} \frac{\partial}{\partial x^4} dx^1 + \frac{1}{c^2} \frac{dx^1}{dx^4} \frac{\partial}{\partial x^4} dx^4 + \frac{2v}{c^2} \frac{dx^1}{dx^4} \frac{\partial}{\partial x^4} dx^4 = 0 \]

Then we have:

\[ \frac{v'}{c^2} \frac{\partial}{\partial x^4} dx^1 + \frac{1}{c^2} \frac{dx^1}{dx^4} \frac{\partial}{\partial x^4} dx^4 + \frac{2v}{c^2} \frac{dx^1}{dx^4} \frac{\partial}{\partial x^4} dx^4 = 0 \]

With this result we have in 15.29 the invariance of the total differential:

\[ d\phi = \frac{\partial \phi}{\partial x^i} dx^i = \begin{bmatrix} \frac{\partial}{\partial x^1} & \frac{\partial}{\partial x^2} & \frac{\partial}{\partial x^3} & \frac{\partial}{\partial x^4} \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} + \begin{bmatrix} 0 & 0 & v'/c \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ -v'/c & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} dx^1 \\ dx^2 \\ dx^3 \\ dx^4 \end{bmatrix} = \frac{\partial \phi}{\partial x^i} dx^i = d\phi \]

15.31

In the observer O’ referential the total differential of a function \( \phi(x^i) \) is equal to:

\[ d\phi(x^i) \frac{\partial \phi}{\partial x^i} dx^i = \begin{bmatrix} \frac{\partial}{\partial x^1} & \frac{\partial}{\partial x^2} & \frac{\partial}{\partial x^3} & \frac{\partial}{\partial x^4} \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} + \begin{bmatrix} 0 & 0 & v'/c \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ -v'/c & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} dx^1 \\ dx^2 \\ dx^3 \\ dx^4 \end{bmatrix} \]

15.32

Where the coordinates correlate with the ones from the observer O referential according to \( x^i = x^i(x^j) \), replacing the transformations 15.23 and 15.15 and without presenting the function \( \phi \) we have:

\[ d\phi = \frac{\partial \phi}{\partial x^i} dx^i = \begin{bmatrix} \frac{\partial}{\partial x^1} & \frac{\partial}{\partial x^2} & \frac{\partial}{\partial x^3} & \frac{\partial}{\partial x^4} \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} + \begin{bmatrix} 0 & 0 & v'/c \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ -v'/c & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} dx^1 \\ dx^2 \\ dx^3 \\ dx^4 \end{bmatrix} \]

15.33

The multiplication of the middle matrices supplies:

\[ \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ v/c & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -v'/c & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} dx^1 \\ dx^2 \\ dx^3 \\ dx^4 \end{bmatrix} \]

15.34

Result that can be divided in two matrices:
The wave equation to the observer O is equal to:

\[ \nabla^2 \phi = \frac{1}{c^2} \frac{\partial^2 \phi}{\partial t^2} = \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} + \frac{\partial^2 \phi}{\partial z^2} \]

Where applying 8.5 we have:

\[ \frac{v}{c^2} \frac{\partial}{\partial x} dx^4 \left( \frac{1}{c^2} \frac{dx^4}{dx^4} \frac{\partial}{\partial x^4} \right) = \frac{2v}{c^2} dx^4 \frac{\partial}{\partial x^4} dx^4 = 0 \]

Then we have:

\[ \frac{\partial}{\partial x} dx^4 = \frac{\partial^2 \phi}{\partial x^2} dx^4 \]

With this result we have in 15.36 the invariance of the total differential:

\[ d\phi = \frac{\partial \phi}{\partial x^4} dx^4 = \frac{\partial \phi}{\partial x} dx^4 = d\phi \]

**Invariance of the Wave Equation**

The wave equation to the observer O is equal to:

\[ \nabla^2 \phi = \frac{1}{c^2} \frac{\partial^2 \phi}{\partial t^2} = \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} + \frac{\partial^2 \phi}{\partial z^2} \]
Where applying 15.24 and the transposed from 15.24 we have:

\[
\nabla^2 \phi \left( \frac{1}{c^2} \frac{\partial \phi}{\partial (x^4)} \right) = \left[ \frac{\partial}{\partial x^1} \frac{\partial}{\partial x^2} \frac{\partial}{\partial x^3} \frac{\partial}{\partial c \partial x^4} \right] \left[ \begin{array}{cccc}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
-\frac{\nu'}{c} & 0 & 0 & 0
\end{array} \right]^{-1} \left[ \begin{array}{cccc}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
-\frac{\nu'}{c} & 0 & 0 & 0
\end{array} \right]
\]

The multiplication of the three middle matrices supplies:

\[
\left[ \begin{array}{cccc}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
-\frac{\nu'}{c} & 0 & 0 & -1
\end{array} \right] \left[ \begin{array}{cccc}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
-\frac{\nu'}{c} & 0 & 0 & -1
\end{array} \right] = \left[ \begin{array}{cccc}
1 & 0 & 0 & -\frac{\nu'}{c} \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
-\frac{\nu'}{c} & 0 & 0 & -1
\end{array} \right]
\]

Result that can be divided in two matrices:

\[
\left[ \begin{array}{cccc}
1 & 0 & 0 & -\frac{\nu'}{c} \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
-\frac{\nu'}{c} & 0 & 0 & -1
\end{array} \right] = \left[ \begin{array}{cccc}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
-\frac{\nu'}{c} & 0 & 0 & 0
\end{array} \right] + \left[ \begin{array}{cccc}
0 & 0 & 0 & -\frac{\nu'}{c} \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
-\frac{\nu'}{c} & 0 & 0 & -1
\end{array} \right]
\]

That applied in the wave equation supplies:

\[
\nabla^2 \phi \left( \frac{1}{c^2} \frac{\partial \phi}{\partial (x^4)} \right) = \left[ \frac{\partial}{\partial x^1} \frac{\partial}{\partial x^2} \frac{\partial}{\partial x^3} \frac{\partial}{\partial c \partial x^4} \right] \left[ \begin{array}{cccc}
0 & 0 & 0 & -\frac{\nu'}{c} \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
-\frac{\nu'}{c} & 0 & 0 & -1
\end{array} \right] + \left[ \begin{array}{cccc}
0 & 0 & 0 & -\frac{\nu'}{c} \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
-\frac{\nu'}{c} & 0 & 0 & -1
\end{array} \right]
\]

Executing the operations of the second term we have:

\[
\left[ \frac{\partial}{\partial x^1} \frac{\partial}{\partial x^2} \frac{\partial}{\partial x^3} \frac{\partial}{\partial c \partial x^4} \right] \left[ \begin{array}{cccc}
0 & 0 & 0 & -\frac{\nu'}{c} \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
-\frac{\nu'}{c} & 0 & 0 & -1
\end{array} \right] \left[ \frac{\partial}{\partial x^1} \frac{\partial}{\partial x^2} \frac{\partial}{\partial x^3} \frac{\partial}{\partial c \partial x^4} \right] \left[ \begin{array}{cccc}
0 & 0 & 0 & -\frac{\nu'}{c} \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
-\frac{\nu'}{c} & 0 & 0 & -1
\end{array} \right]
\]

Executing the operations we have:

\[
\frac{2\nu'}{c^2} \frac{\partial}{\partial x^1} \frac{\partial}{\partial x^4} - \frac{2\nu'}{c^2} \frac{\partial}{\partial (x^4)^2}
\]

Where applying 8.5 we have:

\[
\frac{2\nu'}{c^2} \left( \frac{\partial}{\partial x^1} \frac{\partial}{\partial x^4} - \frac{2\nu'}{c^2} \frac{\partial}{\partial (x^4)^2} \right) = 0
\]
Then we have:

\[
\begin{bmatrix}
\frac{\partial}{\partial x^1} & \frac{\partial}{\partial x^2} & \frac{\partial}{\partial x^3} & \frac{\partial}{\partial x^4}
\end{bmatrix}
\begin{bmatrix}
0 & 0 & -\frac{v}{c} & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
-\frac{v}{c} & 0 & 0 & -2v'x^4/c^2 \\
\end{bmatrix}
\begin{bmatrix}
\frac{\partial}{\partial x^1} & \frac{\partial}{\partial x^2} & \frac{\partial}{\partial x^3} & \frac{\partial}{\partial x^4}
\end{bmatrix}
= \text{zero}
\]

15.44

With this result we have in 15.43 the invariance of the wave equation:

\[
\nabla^2 \phi - \frac{1}{c^2} \frac{\partial^2 \phi^2}{\partial (x^4)^2} = \left[ \frac{\partial}{\partial x^1} \frac{\partial}{\partial x^2} \frac{\partial}{\partial x^3} \frac{\partial}{\partial x^4} \right] \begin{bmatrix}
1000 & 0 & 0100 & 0000-1
\end{bmatrix} = \nabla^2 \phi - \frac{1}{c^2} \frac{\partial^2 \phi^2}{\partial (x^4)^2}
\]

15.45

The wave equation to the observer O’ is equal to:

\[
\nabla^2 \phi^* - \frac{1}{c^2} \frac{\partial^2 \phi^2}{\partial (x^4)^2} = \left[ \frac{\partial}{\partial x^1} \frac{\partial}{\partial x^2} \frac{\partial}{\partial x^3} \frac{\partial}{\partial x^4} \right] \begin{bmatrix}
1000 & 0 & 0100 & 0000-1
\end{bmatrix} = 0
\]

15.46

Where applying 15.23 and the transposed from 15.23 we have:

\[
\nabla^2 \phi^* - \frac{1}{c^2} \frac{\partial^2 \phi^2}{\partial (x^4)^2} = \left[ \frac{\partial}{\partial x^1} \frac{\partial}{\partial x^2} \frac{\partial}{\partial x^3} \frac{\partial}{\partial x^4} \right] \begin{bmatrix}
1000 & 0 & 0100 & 0000-1
\end{bmatrix} = \nabla^2 \phi^* - \frac{1}{c^2} \frac{\partial^2 \phi^2}{\partial (x^4)^2}
\]

15.47

The multiplication of the three middle matrices supplies:

\[
\begin{bmatrix}
1000 & 0 & 0100 & 0000-1
\end{bmatrix} \begin{bmatrix}
1000 & 0 & 0100 & 0000-1
\end{bmatrix} = \begin{bmatrix}
100 & 0 & 010 & 0010
\end{bmatrix}
\]

15.48

Result that can be divided in two matrices:

\[
\begin{bmatrix}
100 & 0 & 010 & 0010
\end{bmatrix} = \begin{bmatrix}
1000 & 0 & 0100 & 0000-1
\end{bmatrix} \begin{bmatrix}
000 & 0 & 000 & 0
\end{bmatrix}
\]
Replacing 2.4, 8.2, 8.4B in 8.5 we have:

\[
\frac{\partial u' x' + v'}{c^2 \partial x' \partial^4} = 0
\]
The continuity equation to the observer O is equal to:
\[ \vec{\nabla} \cdot \mathbf{J} + \frac{\partial J_\rho}{\partial x^\tau} = \frac{\partial J_\rho}{\partial x^1} + \frac{\partial J_\rho}{\partial x^2} + \frac{\partial J_\rho}{\partial x^3} + \frac{\partial J_\rho}{\partial c \rho} = \frac{\partial J_1}{\partial x^1} + \frac{\partial J_2}{\partial x^2} + \frac{\partial J_3}{\partial x^3} + \frac{\partial J_\rho}{\partial c \rho} = \text{zero} \]

Invariance of the Continuity Equation

The table 4 in a matrix form becomes:

\[
\begin{bmatrix}
px^1 \\
px^2 \\
px^3 \\
E'/c
\end{bmatrix} = \begin{bmatrix}
1 & 0 & 0 & -v/c \\
0 & 1 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & \sqrt{K} & 0
\end{bmatrix} \begin{bmatrix}
px^1 \\
px^2 \\
px^3 \\
E/c
\end{bmatrix}
\]

The table 6 in a matrix form becomes:

\[
\begin{bmatrix}
J'x^1 \\
J'x^2 \\
J'x^3 \\
c\rho'
\end{bmatrix} = \begin{bmatrix}
1 & 0 & 0 & -v/c \\
0 & 1 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & \sqrt{K} & 0
\end{bmatrix} \begin{bmatrix}
Jx^1 \\
Jx^2 \\
Jx^3 \\
c\rho
\end{bmatrix}
\]

\[
\begin{bmatrix}
Jx^1 \\
Jx^2 \\
Jx^3 \\
c\rho
\end{bmatrix} = \begin{bmatrix}
1 & 0 & 0 & -v/c \\
0 & 1 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & \sqrt{K} & 0
\end{bmatrix} \begin{bmatrix}
J'x^1 \\
J'x^2 \\
J'x^3 \\
c\rho'
\end{bmatrix}
\]

Replacing 2.3, 8.1, 8.3B in 8.5 we have:
\[ \frac{\partial}{\partial x^\tau} + \frac{u' x^\prime}{c^2} \frac{\partial}{\partial x^\tau} = \frac{\partial}{\partial x^\tau} + \frac{v}{c^2} \frac{\partial}{\partial x^\tau} + \frac{1}{c} \left( \frac{u x^1}{c} - v \right) \frac{1}{\sqrt{K}} \frac{\partial}{\partial x^4} = \text{zero} \]

Executing the operations we have:
\[ \frac{\partial}{\partial x^\tau} + \frac{u' x^\prime}{c^2} \frac{\partial}{\partial x^\tau} = \frac{\partial}{\partial x^\tau} + \frac{v}{c^2} \frac{\partial}{\partial x^\tau} + \frac{u x^1}{c^2} \frac{\partial}{\partial x^4} - \frac{v}{c^2} \frac{\partial}{\partial x^4} = \text{zero} \]

That simplified supplies the invariance of the equation 8.5:

\[ \frac{\partial}{\partial x^\tau} + \frac{u' x^\prime}{c^2} \frac{\partial}{\partial x^\tau} = \frac{\partial}{\partial x^\tau} + \frac{u x^1}{c^2} \frac{\partial}{\partial x^4} = \text{zero} \]
Where replacing 15.24 and 15.56 we have:

\[
\nabla J + \frac{\partial \rho}{\partial x^4} \left[ \frac{\partial}{\partial x^1} \frac{\partial}{\partial x^2} \frac{\partial}{\partial x^3} \frac{\partial}{\partial x^4} \right] \begin{bmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
-\nu/c & 0 & 0 & 0
\end{bmatrix} \begin{bmatrix}
J^1 x^1 \\
J^2 x^2 \\
J^3 x^3 \\
J \cdot' \cdot'
\end{bmatrix} = 0
\]

15.58

The product of the transformation matrices is given in 15.27 and 15.28 with this:

\[
\nabla J + \frac{\partial \rho}{\partial x^4} \left[ \frac{\partial}{\partial x^1} \frac{\partial}{\partial x^2} \frac{\partial}{\partial x^3} \frac{\partial}{\partial x^4} \right] \begin{bmatrix}
1000 \\
0100 \\
0010 \\
-\nu/c \cdot 0001
\end{bmatrix} \begin{bmatrix}
J^1 x^1 \\
J^2 x^2 \\
J^3 x^3 \\
J \cdot' \cdot'
\end{bmatrix} = 0
\]

15.59

Executing the operations of the second term we have:

\[
\nabla J \cdot' + \frac{\partial \rho}{\partial x^4} \left[ \frac{\partial}{\partial x^1} \frac{\partial}{\partial x^2} \frac{\partial}{\partial x^3} \frac{\partial}{\partial x^4} \right] \begin{bmatrix}
0 & 0 & 0 & \frac{\nu}{c} \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
-\nu/c \cdot 0001
\end{bmatrix} \begin{bmatrix}
J^1 x^1 \\
J^2 x^2 \\
J^3 x^3 \\
J \cdot' \cdot'
\end{bmatrix} = \frac{\nu}{c} \frac{\partial J}{\partial x^4} + \frac{\partial J}{\partial x^4} \frac{\partial \rho}{\partial x^4} + \frac{2 \nu u x^1}{c^2} \frac{\partial \rho}{\partial x^4} = 0
\]

15.60

Then we have:

\[
\nabla J + \frac{\partial \rho}{\partial x^4} \left[ \frac{\partial}{\partial x^1} \frac{\partial}{\partial x^2} \frac{\partial}{\partial x^3} \frac{\partial}{\partial x^4} \right] \begin{bmatrix}
1000 \\
0100 \\
0010 \\
-\nu/c \cdot 0001
\end{bmatrix} \begin{bmatrix}
J^1 x^1 \\
J^2 x^2 \\
J^3 x^3 \\
J \cdot' \cdot'
\end{bmatrix} = 0
\]

15.61

With this result we have in 15.59 the invariance of the continuity equation:

The continuity equation to the observer O' is equal to:

\[
\nabla J + \frac{\partial \rho}{\partial x^4} \left[ \frac{\partial}{\partial x^1} \frac{\partial}{\partial x^2} \frac{\partial}{\partial x^3} \frac{\partial}{\partial x^4} \right] \begin{bmatrix}
1000 \\
0100 \\
0010 \\
-\nu/c \cdot 0001
\end{bmatrix} \begin{bmatrix}
J^1 x^1 \\
J^2 x^2 \\
J^3 x^3 \\
J \cdot' \cdot'
\end{bmatrix} = 0
\]

15.62

Where replacing 15.23 and 15.55 we have:

\[
\nabla J + \frac{\partial \rho}{\partial x^4} \left[ \frac{\partial}{\partial x^1} \frac{\partial}{\partial x^2} \frac{\partial}{\partial x^3} \frac{\partial}{\partial x^4} \right] \begin{bmatrix}
1000 -\nu/c \\
0100 \\
0010 \\
0001 \cdot 0001
\end{bmatrix} \begin{bmatrix}
J^1 x^1 \\
J^2 x^2 \\
J^3 x^3 \\
J \cdot' \cdot'
\end{bmatrix} = 0
\]

15.63
The product of the transformation matrices is given in 15.34 and 15.35 then we have:

\[
\nabla \cdot J^4 \frac{\partial \rho'}{\partial x^4} = \begin{bmatrix} \frac{\partial \rho}{\partial x^1} & \frac{\partial \rho}{\partial x^2} & \frac{\partial \rho}{\partial x^3} & \frac{\partial \rho}{\partial x^4} \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} J_{x1}^4 \\ J_{x2}^4 \\ J_{x3}^4 \\ J_{x4}^4 \end{bmatrix}
\]

15.64

Executing the operations of the second term we have:

\[
\begin{bmatrix} \frac{\partial \rho}{\partial x^1} & \frac{\partial \rho}{\partial x^2} & \frac{\partial \rho}{\partial x^3} & \frac{\partial \rho}{\partial x^4} \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 & -v/c \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ v/c & 0 & 0 & -2vux^4/c^2 \end{bmatrix} = \begin{bmatrix} v \frac{\partial J_{x1}^4}{\partial x^4} - \frac{\partial \rho}{\partial x^1} \cdot \frac{2vux^4}{c^2} \\ \rho \frac{\partial J_{x2}^4}{\partial x^4} - \frac{\partial \rho}{\partial x^2} \cdot \frac{2vux^4}{c^2} \\ \rho \frac{\partial J_{x3}^4}{\partial x^4} - \frac{\partial \rho}{\partial x^3} \cdot \frac{2vux^4}{c^2} \\ \rho \frac{\partial J_{x4}^4}{\partial x^4} - \frac{\partial \rho}{\partial x^4} \cdot \frac{2vux^4}{c^2} \end{bmatrix}
\]

Where replacing \( J_{x1}^4 = \rho u x^4 \) and 8.5 we have:

\[
\frac{vux^4}{c^2} \frac{\partial \rho}{\partial x^4} \cdot \frac{u x^4}{c^2} \cdot \frac{\partial}{\partial x^4} - \frac{2vux^4}{c^2} \frac{\partial \rho}{\partial x^4} = 0
\]

Then we have:

\[
\begin{bmatrix} \frac{\partial \rho}{\partial x^1} & \frac{\partial \rho}{\partial x^2} & \frac{\partial \rho}{\partial x^3} & \frac{\partial \rho}{\partial x^4} \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 & -v/c \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ v/c & 0 & 0 & -2vux^4/c^2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}
\]

15.65

With this result we have in 15.64 the invariance of the continuity equation:

\[
\nabla \cdot J^4 \frac{\partial \rho'}{\partial x^4} = \begin{bmatrix} \frac{\partial \rho}{\partial x^1} & \frac{\partial \rho}{\partial x^2} & \frac{\partial \rho}{\partial x^3} & \frac{\partial \rho}{\partial x^4} \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} J_{x1}^4 \\ J_{x2}^4 \\ J_{x3}^4 \\ J_{x4}^4 \end{bmatrix} = \nabla \cdot \frac{\partial \rho}{\partial x^4}
\]

15.66

Invariance of the line differential element:

That to the observer O is written this way:

\[
(dx)^2 + (dx)^2 + (cdx)^2 - (cdx^4)^2 = \begin{bmatrix} dx^1 & dx^2 & dx^3 & dx^4 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} dx^1 \\ dx^2 \\ dx^3 \\ dx^4 \end{bmatrix}
\]

15.67

Where replacing 15.18 and the transposed from 15.18 we have:

\[
(dx)^2 = \begin{bmatrix} dx^1 & dx^2 & dx^3 & dx^4 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \frac{c'}{c} & 0 & 0 \sqrt{K'} & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} dx^1 \\ dx^2 \\ dx^3 \\ dx^4 \end{bmatrix}
\]

15.68

The multiplication of the three central matrices supplies:

\[
\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \frac{c'}{c} & 0 & 0 \sqrt{K'} & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \frac{c'}{c} & 0 & 0 \sqrt{K'} & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \frac{c'}{c} & 0 & 0 \sqrt{K'} & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \frac{c'}{c} & 0 & 0 \sqrt{K'} & 0 \end{bmatrix}
\]

15.69
Result that can be divided in two matrices:

$$\begin{bmatrix}
1 & 0 & v' \\
0 & 1 & 0 \\
0 & 0 & 1 \\
v'' & 0 & 0 - \frac{2v' dv''}{c^2 dx^4}
\end{bmatrix} = \begin{bmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1 \\
v'' & 0 & 0 - \frac{2v' dv''}{c^2 dx^4}
\end{bmatrix} + \begin{bmatrix}
0 & 0 & v' \\
0 & 0 & 0 \\
0 & 0 & 0 \\
v'' & 0 & 0 - \frac{2v' dv''}{c^2 dx^4}
\end{bmatrix}$$

That applied in the line differential element supplies:

$$(dx) = \left[ dx^1 dx^2 dx^3 cdx^4 \right] \begin{bmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 \\
v'' & 0 & 0 & - \frac{2v' dv''}{c^2 dx^4}
\end{bmatrix} \begin{bmatrix}
1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
v'' & 0 & 0 & - \frac{2v' dv''}{c^2 dx^4}
\end{bmatrix} = zero$$

Executing the operations of the second term we have:

$$\begin{bmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 \\
v'' & 0 & 0 & - \frac{2v' dv''}{c^2 dx^4}
\end{bmatrix} \begin{bmatrix}
0 & 0 & v' \\
0 & 0 & 0 \\
0 & 0 & 0 \\
v'' & 0 & 0 - \frac{2v' dv''}{c^2 dx^4}
\end{bmatrix} \begin{bmatrix}
\frac{dv^1}{dx} \\
\frac{dv^2}{dx} \\
\frac{dv^3}{dx} \\
\frac{dv^4}{dx}
\end{bmatrix} = \frac{\frac{dv^1}{dx}}{c} + \frac{\frac{dv^2}{dx}}{c} + \frac{\frac{dv^3}{dx}}{c} + \frac{\frac{dv^4}{dx}}{c} = zero$$

Then we have:

$$\begin{bmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 \\
v'' & 0 & 0 & - \frac{2v' dv''}{c^2 dx^4}
\end{bmatrix} \begin{bmatrix}
0 & 0 & v' \\
0 & 0 & 0 \\
0 & 0 & 0 \\
v'' & 0 & 0 - \frac{2v' dv''}{c^2 dx^4}
\end{bmatrix} = zero$$

With this result we have in 15.71 the invariance of the line differential element:

$$(dx^2) = \left[ dx^1 dx^2 dx^3 cdx^4 \right] \begin{bmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 \\
v'' & 0 & 0 & - \frac{2v' dv''}{c^2 dx^4}
\end{bmatrix} \begin{bmatrix}
1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
v'' & 0 & 0 & - \frac{2v' dv''}{c^2 dx^4}
\end{bmatrix} = (dx^1)^2 + (dx^2)^2 + (dx^3)^2 - (dx^4)^2 = (dx)^2$$

To the observer O’ the line differential element is written this way:

$$(dx')^2 = (dx^1)^2 + (dx^2)^2 + (dx^3)^2 - (dx^4)^4 = \left[ dx^1 dx^2 dx^3 cdx^4 \right] \begin{bmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 \\
v'' & 0 & 0 & - \frac{2v' dv''}{c^2 dx^4}
\end{bmatrix} \begin{bmatrix}
1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
v'' & 0 & 0 & - \frac{2v' dv''}{c^2 dx^4}
\end{bmatrix}$$

Where replacing 15.15 and the transposed from 15.15 we have:

$$(dx')^2 = \left[ dx^1 dx^2 dx^3 cdx^4 \right] \begin{bmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 \\
v'' & 0 & 0 & - \frac{2v' dv''}{c^2 dx^4}
\end{bmatrix} \begin{bmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 \\
v'' & 0 & 0 & - \frac{2v' dv''}{c^2 dx^4}
\end{bmatrix} \begin{bmatrix}
1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
v'' & 0 & 0 & - \frac{2v' dv''}{c^2 dx^4}
\end{bmatrix} \begin{bmatrix}
1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
v'' & 0 & 0 & - \frac{2v' dv''}{c^2 dx^4}
\end{bmatrix}$$
The multiplication of the three central matrices supplies:

\[
\begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
-\frac{\nu}{c} & 0 & 0 & 1 \\
0 & 0 & \sqrt{K} & 0
\end{pmatrix}
\begin{pmatrix}
1 & 0 & 0 & -\nu \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{pmatrix}
\begin{pmatrix}
1 & 0 & 0 & \frac{\nu}{c} \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
-\frac{\nu}{c} & 0 & 0 & -1 + \frac{2\nu dx^1}{c^2 dx^2}
\end{pmatrix}
\]

\[
= \begin{pmatrix}
1 & 0 & 0 & \frac{\nu}{c} \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
-\frac{\nu}{c} & 0 & 0 & -1 + \frac{2\nu dx^1}{c^2 dx^2}
\end{pmatrix}
\]

Result that can be divided in two matrices:

\[
\begin{pmatrix}
1 & 0 & 0 & \frac{\nu}{c} \\
0 & 1 & 0 & 0 \\
-\frac{\nu}{c} & 0 & 0 & -1 + \frac{2\nu dx^1}{c^2 dx^2}
\end{pmatrix}
\begin{pmatrix}
0 & 0 & \frac{\nu}{c} \\
0 & 0 & 0 \\
0 & 0 & 0 \\
\frac{2\nu dx^1}{c^2 dx^2}
\end{pmatrix}
\]

That applied in the line differential element supplies:

\[
(dx')^2 = \left[dx^1 dx^2 dx^3 cdx^4 \right] \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & -1
\end{pmatrix} + \left[dx^1 dx^2 dx^3 dx^4 \right] \begin{pmatrix}
0 & 0 & \frac{\nu}{c} \\
0 & 0 & 0 \\
0 & 0 & 0 \\
\frac{2\nu dx^1}{c^2 dx^2}
\end{pmatrix} = \frac{dx^1 dx^2}{c^2}
\]

Executing the operations of the second term we have:

\[
\begin{pmatrix}
0 & 0 & \frac{\nu}{c} \\
0 & 0 & 0 \\
0 & 0 & 0 \\
\frac{2\nu dx^1}{c^2 dx^2}
\end{pmatrix}
\begin{pmatrix}
0 & 0 & \frac{\nu}{c} \\
0 & 0 & 0 \\
0 & 0 & 0 \\
\frac{2\nu dx^1}{c^2 dx^2}
\end{pmatrix} = \frac{dx^1 dx^2}{c^2}
\]

Then we have:

\[
\begin{pmatrix}
0 & 0 & \frac{\nu}{c} \\
0 & 0 & 0 \\
0 & 0 & 0 \\
\frac{2\nu dx^1}{c^2 dx^2}
\end{pmatrix}
\begin{pmatrix}
0 & 0 & \frac{\nu}{c} \\
0 & 0 & 0 \\
0 & 0 & 0 \\
\frac{2\nu dx^1}{c^2 dx^2}
\end{pmatrix} = zero
\]

With this result we have in 15.78 the invariance of the line differential element:

\[
(dx')^2 = \left[dx^1 dx^2 dx^3 cdx^4 \right] + \left(dx^2 dx^3 \right) + \left(dx^3 \right) = (dx)^2
\]

In §7 as a consequence of 5.3 we had the invariance of \( E \ddot{u} = \ddot{E}' \ddot{u}' \) where now applying 7.3.1, 7.3.2, 7.4.1, 7.4.2 and the velocity transformation formulae from table 2 we have new relations between \( E\ddot{x} \) and \( E'\ddot{x}' \) distinct from 7.3 and 7.4 and with them we rewrite the table 7 in the form below:
Table 7B

<table>
<thead>
<tr>
<th>Expression</th>
<th>Formula</th>
<th>Notes</th>
</tr>
</thead>
<tbody>
<tr>
<td>$E' = \frac{E}{1 - \frac{v}{ux}}$</td>
<td>7.3B</td>
<td></td>
</tr>
<tr>
<td>$Ex' = \frac{Ex}{1 + \frac{v}{ux'}}$</td>
<td>7.4B</td>
<td></td>
</tr>
<tr>
<td>$E' = Ey' \sqrt{K}$</td>
<td>7.3.1</td>
<td>$Ey = E' y' \sqrt{K'}$</td>
</tr>
<tr>
<td>$E' = Ez' \sqrt{K}$</td>
<td>7.3.2</td>
<td>$Ez = E' z' \sqrt{K'}$</td>
</tr>
<tr>
<td>$B' = Bx$</td>
<td>7.5</td>
<td>$Bx = B' x'$</td>
</tr>
<tr>
<td>$B' = Bx \pm \frac{v}{c^2} = Ez$</td>
<td>7.5.1</td>
<td>$By = B' y' \pm \frac{v'}{c^2} = E' z'$</td>
</tr>
<tr>
<td>$B' = Bz - \frac{v}{c^2} = Ey$</td>
<td>7.5.2</td>
<td>$Bz = B' z' \pm \frac{v'}{c^2} = E' y'$</td>
</tr>
<tr>
<td>$By = - \frac{ux}{c^2} = Ez$</td>
<td>7.9</td>
<td>$B' y' = - \frac{ux'}{c^2} E' z'$</td>
</tr>
<tr>
<td>$Bz = \frac{ux}{c^2} = Ey$</td>
<td>7.9.1</td>
<td>$B' z' = \frac{ux'}{c^2} E' y'$</td>
</tr>
</tbody>
</table>

With the tables 7B and 9B we can have the invariance of all Maxwell’s equations.

Invariance of the Gauss’ Law for the electrical field:

$$\frac{\partial E' x'}{\partial x'} + \frac{\partial E' y'}{\partial y'} + \frac{\partial E' z'}{\partial z'} = \frac{\rho'}{\varepsilon_0}$$

Where applying the tables 6, 7B and 9B we have:

$$\left( \frac{\partial}{\partial x} + \frac{v}{c^2} \frac{\partial}{\partial t} \right) \frac{Ex}{1 - \frac{v}{ux}} \frac{Ex}{1 - \frac{v}{ux'}} \frac{Ey}{1 - \frac{v}{ux'}} + \frac{Ez}{1 - \frac{v}{ux'}} \frac{Ez}{1 - \frac{v}{ux'}} = \frac{\rho}{\varepsilon_0}$$

Where simplifying and replacing 8.5 we have:

$$\left[ \frac{\partial}{\partial x} + \left( \frac{-1}{ux} \frac{\partial}{\partial x} \right) \frac{Ex}{1 - \frac{v}{ux}} \right] \frac{Ex}{1 - \frac{v}{ux}} + \frac{Ez}{1 - \frac{v}{ux'}} \frac{Ez}{1 - \frac{v}{ux'}} = \frac{\rho}{\varepsilon_0}$$

That reordered supplies:

$$\left[ \frac{\partial}{\partial x} \left( \frac{v}{ux} \right) \right] \frac{Ex}{1 - \frac{v}{ux}} + \frac{Ez}{1 - \frac{v}{ux'}} \frac{Ez}{1 - \frac{v}{ux'}} = \frac{\rho}{\varepsilon_0}$$

That simplified supplies the invariance of the Gauss’ Law for the electrical field.

Invariance of the Gauss’ Law for the magnetic field:

$$\frac{\partial B' x'}{\partial x'} + \frac{\partial B' y'}{\partial y'} + \frac{\partial B' z'}{\partial z'} = \text{zero}$$

Where applying the tables 7B and 9B we have:

$$\left( \frac{\partial}{\partial x} + \frac{v}{c^2} \frac{\partial}{\partial t} \right) Bx + \frac{\partial}{\partial y} \left( By + \frac{v}{c^2} Ez \right) + \frac{\partial}{\partial z} \left( Bz - \frac{v}{c^2} Ey \right) = 0$$
That reordered supplies:

\[
\frac{\partial B_x}{\partial x} + \frac{\partial B_y}{\partial y} + \frac{\partial B_z}{\partial z} + \frac{v}{c^2} \left( \frac{\partial E_z}{\partial y} - \frac{\partial E_y}{\partial z} + \frac{\partial B_x}{\partial t} \right) = 0
\]

Where the term in parenthesis is the Faraday-Henry’s Law (8.19) that is equal to zero from where we have the invariance of the Gauss’ Law for the magnetic field.

**Invariance of the Faraday-Henry’s Law:**

\[
\frac{\partial E'_{y'}}{\partial x'} \frac{\partial E'_{x'}}{\partial y'} = \frac{\partial B'_{z'}}{\partial t'}
\]

Where applying the tables 7B and 9B we have:

\[
\left( \frac{\partial}{\partial x} + \frac{v}{c^2} \frac{\partial}{\partial t} \right) E_y \sqrt{K} \frac{\partial}{\partial y} \left( B_z - \frac{v}{c^2} E_y \right) = -\sqrt{K} \frac{\partial}{\partial t} \left( B_z - \frac{v}{c^2} E_y \right)
\]

That simplified and multiplied by \(1-v/ux\) we have:

\[
\frac{\partial B_z}{\partial t} = \frac{\partial B_z}{\partial t} \left( 1 - \frac{v}{ux} \right)
\]

Where executing the products and replacing 7.9.1 we have:

\[
\frac{\partial E_y}{\partial x} \left( 1 - \frac{v}{ux} \right) + \frac{\partial E_x}{\partial y} \left( \frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y} \right) + \frac{\partial E_y}{\partial z} \left( \frac{\partial E_y}{\partial z} \right)
\]

As the term in parenthesis is the equation 8.5 that is equal to zero then we have the invariance of the Faraday-Henry’s Law.

**Invariance of the Faraday-Henry’s Law:**

\[
\frac{\partial E'_{z'}}{\partial y'} \frac{\partial E'_{y'}}{\partial z'} = \frac{\partial B'_{x'}}{\partial t'}
\]

Where applying the tables 7B and 9B we have:

\[
\frac{\partial E_z}{\partial y} \sqrt{K} - \frac{\partial E_y}{\partial z} \sqrt{K} = -\sqrt{K} \frac{\partial B_x}{\partial t}
\]

That simplified supplies the invariance of the Faraday-Henry’s Law.

**Invariance of the Faraday-Henry’s Law:**

\[
\frac{\partial E'_{x'}}{\partial z'} \frac{\partial E'_{z'}}{\partial x'} = \frac{\partial B'_{y'}}{\partial t'}
\]

Where applying the tables 7B and 9B we have:

\[
\frac{\partial}{\partial z} \left( \frac{\partial E_x}{\partial z} \right) \left( \frac{\partial E_y}{\partial z} \right) = -\sqrt{K} \frac{\partial B_y}{\partial t} \left( \frac{\partial E_y}{\partial z} \right)
\]

That simplified and multiplied by \(1-v/ux\) we have:

\[
\frac{\partial E_x}{\partial z} \left( 1 - \frac{v}{ux} \right) - \frac{\partial E_z}{\partial x} \left( 1 - \frac{v}{ux} \right) = \frac{\partial B_y}{\partial t} \left( 1 - \frac{v}{ux} \right) - \frac{\partial E_z}{\partial x} \left( 1 - \frac{v}{ux} \right)
\]
That simplifying and making the operations we have:

\[
\frac{\partial E_x}{\partial z} \frac{\partial E_z}{\partial x} = \frac{\partial B_y}{\partial t} \frac{v}{ux} \left( \frac{\partial E_z}{\partial x} \frac{\partial B_y}{\partial t} \right)
\]

Where applying 7.9 we have:

\[
\frac{\partial E_x}{\partial z} \frac{\partial E_z}{\partial x} = \frac{\partial B_y}{\partial t} \frac{v}{ux} \left( \frac{\partial E_z}{\partial x} \frac{ux \partial E_z}{c^2 \partial t} \right)
\]

As the term in parenthesis is the equation 8.5 that is equal to zero then we have the invariance of the Faraday-Henry’s Law.

**Invariance of the Ampere-Maxwell’s Law:**

\[
\frac{\partial B'_{x'}}{\partial x'} \frac{\partial B'_{y'}}{\partial y'} = \mu_o J'z' + \varepsilon_o \mu_o \frac{\partial E'z'}{\partial t'}
\]

Where applying the tables 6, 7B and 9B we have:

\[
\left( \frac{\partial}{\partial x} + \frac{v}{c^2} \frac{\partial}{\partial t} \right) \left( B_y + \frac{v}{c^2} E_z \right) \frac{\partial B_x}{\partial y} = \mu_o J_z + \varepsilon_o \mu_o \sqrt{K} \frac{\partial}{\partial t} E_z \sqrt{K}
\]

That simplifying and making the operations we have:

\[
\frac{\partial B_y}{\partial x} \frac{\partial B_x}{\partial y} = \mu_o J_z + \varepsilon_o \mu_o \frac{\partial E_z}{\partial t} \frac{1}{c^2} \left[ \frac{1}{c^2} \frac{\partial E_z}{\partial t} \frac{2}{c^2} \frac{\partial E_z}{\partial \check{x}} \frac{v}{c^2} \frac{\partial E_z}{\partial \check{y}} \frac{2}{c^2} \frac{\partial E_z}{\partial \check{t}} \right]
\]

Where simplifying and applying 7.9 we have:

\[
\frac{\partial B_y}{\partial x} \frac{\partial B_x}{\partial y} = \mu_o J_z + \varepsilon_o \mu_o \frac{\partial E_z}{\partial t} \frac{1}{c^2} \left[ \frac{1}{c^2} \frac{\partial E_z}{\partial t} \frac{2}{c^2} \frac{\partial E_z}{\partial \check{x}} \frac{v}{c^2} \frac{\partial E_z}{\partial \check{y}} \frac{2}{c^2} \frac{\partial E_z}{\partial \check{t}} \right]
\]

That reorganized supplies

\[
\frac{\partial B_y}{\partial x} \frac{\partial B_x}{\partial y} = \mu_o J_z + \varepsilon_o \mu_o \frac{\partial E_z}{\partial t} \frac{1}{c^2} \left[ \frac{1}{c^2} \frac{\partial E_z}{\partial t} \frac{2}{c^2} \frac{\partial E_z}{\partial \check{x}} \frac{v}{c^2} \frac{\partial E_z}{\partial \check{y}} \frac{2}{c^2} \frac{\partial E_z}{\partial \check{t}} \right]
\]

As the term in parenthesis is the equation 8.5 that is equal to zero then we have the invariance of the Ampere-Maxwell’s Law:

**Invariance of the Ampere-Maxwell’s Law:**

\[
\frac{\partial B'_{z'}}{\partial z'} \frac{\partial B'_{y'}}{\partial y'} = \mu_o J'x' + \varepsilon_o \mu_o \frac{\partial E'x'}{\partial t'}
\]

Where applying the tables 6, 7B and 9B we have:

\[
\frac{\partial}{\partial y} \left( B_z - \frac{v}{c^2} E_y \right) - \frac{\partial}{\partial z} \left( B_y + \frac{v}{c^2} E_z \right) = \mu_o (J_x - \rho v) + \varepsilon_o \mu_o \sqrt{K} \frac{\partial}{\partial t} (1 - \frac{v}{c^2})
\]

Making the operations we have:

\[
\frac{\partial B_z}{\partial y} + \frac{\partial B_y}{\partial z} = \mu_o J_x + \varepsilon_o \mu_o \left( \frac{c^2}{2} \rho \right) + \varepsilon_o \mu_o \left( \frac{1}{c^2} \frac{\partial E}{\partial t} \frac{1}{(1 - \frac{v}{c^2})} \right)
\]
Replacing in the first parenthesis the Gauss’ Law and multiplying by \( 1 - \frac{v}{ux} \) we have:

\[
\frac{\partial B_z}{\partial y} - \frac{\partial B_y}{\partial z} = \mu_0 J_x + \varepsilon_0 \mu_0 \frac{\partial E_x}{\partial t} - \frac{\partial E_x}{\partial y} - \frac{\partial E_y}{\partial z} \left( \mu_0 J_x \right) = v \left( \frac{\partial B_z}{\partial y} - \frac{\partial B_y}{\partial z} \right) + \frac{v^2}{c^2} \left( \frac{1}{ux} \frac{\partial E_x}{\partial x} + \frac{1}{c^2} \frac{\partial E_y}{\partial x} \right) + \frac{1}{c^2} \frac{\partial E_x}{\partial t} - \frac{1}{c^2} \frac{\partial E_y}{\partial t}
\]

Where replacing \( J_x = \rho u_x \), 7.9.1, 7.9 and 8.5 we have:

\[
\frac{\partial B_z}{\partial y} - \frac{\partial B_y}{\partial z} = \mu_0 J_x + \varepsilon_0 \mu_0 \frac{\partial E_x}{\partial t} + v \left( \frac{\partial E_x}{\partial y} + \frac{\partial E_y}{\partial z} \right) \left( \mu_0 \rho u_x \right) = v \frac{\partial E_x}{\partial y} + \frac{v^2}{c^2} \left( \frac{1}{ux} \frac{\partial E_x}{\partial x} + \frac{1}{c^2} \frac{\partial E_y}{\partial x} \right) + \frac{1}{c^2} \frac{\partial E_x}{\partial t} - \frac{1}{c^2} \frac{\partial E_y}{\partial t}
\]

That simplified supplies:

\[
\frac{\partial B_z}{\partial y} - \frac{\partial B_y}{\partial z} = \mu_0 J_x + \varepsilon_0 \mu_0 \frac{\partial E_x}{\partial t} + v \frac{\partial E_x}{\partial y} + \frac{\partial E_y}{\partial z} \left( \mu_0 \rho u_x \right) = v \frac{\partial E_x}{\partial y} + \frac{v^2}{c^2} \frac{\partial E_x}{\partial x} + \frac{1}{c^2} \frac{\partial E_x}{\partial t}
\]

Replacing in the first parenthesis the Gauss’ Law we have:

\[
\frac{\partial B_z}{\partial y} - \frac{\partial B_y}{\partial z} = \mu_0 J_x + \varepsilon_0 \mu_0 \frac{\partial E_x}{\partial t} + v \frac{\partial E_x}{\partial y} + \frac{\partial E_y}{\partial z} \left( \mu_0 \rho u_x \right) = v \frac{\partial E_x}{\partial y} + \frac{v^2}{c^2} \frac{\partial E_x}{\partial x} + \frac{1}{c^2} \frac{\partial E_x}{\partial t}
\]

That reorganized makes:

\[
\frac{\partial B_z}{\partial y} - \frac{\partial B_y}{\partial z} = \mu_0 J_x + \varepsilon_0 \mu_0 \frac{\partial E_x}{\partial t} + \frac{v^2}{c^2} \frac{\partial E_x}{\partial y} + \frac{v^2}{c^2} \frac{\partial E_y}{\partial z} = v \frac{\partial E_x}{\partial y} + \frac{v^2}{c^2} \frac{\partial E_x}{\partial x} + \frac{1}{c^2} \frac{\partial E_x}{\partial t}
\]

As the term in parenthesis is the equation 8.5 that is equal to zero then we have the invariance of the Ampere-Maxwell’s Law:

**Invariance of the Ampere-Maxwell’s Law:**

\[
\frac{\partial B_x'}{\partial z'} = \frac{\partial B_y'}{\partial x'} \frac{\partial B_z'}{\partial x'} = \mu_0 J_x + \varepsilon_0 \mu_0 \frac{\partial E_y'}{\partial t'} = \mu_0 J' + \varepsilon_0 \mu_0 \frac{\partial E_y'}{\partial t'}
\]

Where applying the tables 6, 7B and 9B we have:

\[
\frac{\partial B_x}{\partial z} \left( \frac{\partial}{\partial x} + \frac{v}{c^2} \frac{\partial}{\partial t} \right) \left( B_z - \frac{v}{c^2} E_y \right) = \mu_0 J_y + \varepsilon_0 \mu_0 \sqrt{K} \frac{\partial}{\partial t} E_y \sqrt{K}
\]

Making the operations we have:

\[
\frac{\partial B_x}{\partial z} \frac{\partial B_z}{\partial x} = \mu_0 J_y + \varepsilon_0 \mu_0 \frac{\partial E_y}{\partial t} + \frac{1}{c^2} \frac{\partial E_y}{\partial t} + \frac{1}{c^2} \frac{\partial E_y}{\partial t} + \frac{v}{c^2} \frac{\partial B_z}{\partial t} = \frac{1}{c^2} \frac{\partial E_y}{\partial t} + \frac{1}{c^2} \frac{\partial E_y}{\partial t}
\]

Where simplifying and applying 7.9.1 we have:

\[
\frac{\partial B_x}{\partial z} \frac{\partial B_z}{\partial x} = \mu_0 J_y + \varepsilon_0 \mu_0 \frac{\partial E_y}{\partial t} + \frac{1}{c^2} \frac{\partial E_y}{\partial t} + \frac{v}{c^2} \frac{\partial E_y}{\partial t} + \frac{v}{c^2} \left( \frac{\partial E_y}{\partial t} \right)
\]

That reorganized makes:

\[
\frac{\partial B_x}{\partial z} \frac{\partial B_z}{\partial x} = \mu_0 J_y + \varepsilon_0 \mu_0 \frac{\partial E_y}{\partial t} + \frac{1}{c^2} \frac{\partial E_y}{\partial t} + \frac{v}{c^2} \left( \frac{\partial E_y}{\partial t} \right)
\]

As the term in parenthesis is the equation 8.5 that is equal to zero then we have the invariance of the Ampere-Maxwell’s Law:
Invariance of the Gauss' Law for the electrical field without electrical charge:

\[ \frac{\partial E_x'}{\partial x'} + \frac{\partial E_y'}{\partial y'} + \frac{\partial E_z'}{\partial z'} = 0 \tag{8.30} \]

Where applying the tables 7B and 9B we have:

\[ \left( \frac{\partial}{\partial x} + \frac{v}{c^2} \frac{\partial}{\partial t} \right) \frac{E_x}{1-\frac{v}{c u}} \frac{\partial E_y}{\partial y} + \frac{\partial E_y}{\partial y} + \frac{\partial E_z}{\partial z} = 0 \]

Where simplifying and replacing 8.5 we have:

\[ \left[ \frac{\partial}{\partial x} + \frac{1}{c u} \left( \frac{\partial}{\partial x} \right) \right] \frac{E_x}{1-\frac{v}{c u}} \frac{\partial E_y}{\partial y} + \frac{\partial E_y}{\partial y} + \frac{\partial E_z}{\partial z} = 0 \]

That reorganized makes:

\[ \left[ \frac{\partial}{\partial x} \right] \frac{E_x}{1-\frac{v}{c u}} \frac{\partial E_y}{\partial y} + \frac{\partial E_y}{\partial y} + \frac{\partial E_z}{\partial z} = 0 \]

That simplified supplies the Gauss' Law for the electrical field without electrical charge.

**Invariance of the Ampere-Maxwell's Law without electrical charge:**

\[ \frac{\partial B_x'}{\partial x'} + \frac{\partial B_y'}{\partial y'} + \frac{\partial B_z'}{\partial z'} = \varepsilon_0 \mu_0 \frac{\partial E_x'}{\partial t'} \tag{8.40} \]

Where applying the tables 7B and 9B we have:

\[ \left( \frac{\partial}{\partial x} + \frac{v}{c^2} \frac{\partial}{\partial t} \right) \left( \frac{B_y}{c^2} + \frac{v}{c^2} \frac{E_z}{c^2} \right) = \varepsilon_0 \mu_0 \sqrt{\varepsilon_0 \mu_0} \frac{\partial E_z}{\partial t} \]

Making the operations we have:

\[ \frac{\partial B_y}{\partial x} \frac{\partial B_x}{\partial y} = \varepsilon_0 \mu_0 \frac{\partial E_z}{\partial t} + \frac{1}{c^2} \frac{\partial E_z}{c^2} + \frac{2}{c^2} \frac{v u x}{c^2} \frac{\partial E_z}{c^2} - \frac{v}{c^2} \frac{\partial E_z}{c^2} + \frac{v}{c^2} \frac{\partial E_z}{c^2} + \frac{1}{c^2} \frac{\partial E_z}{c^2} + \frac{1}{c^2} \frac{\partial E_z}{c^2} \]

Where simplifying and applying 7.9 we have:

\[ \frac{\partial B_y}{\partial x} \frac{\partial B_x}{\partial y} = \varepsilon_0 \mu_0 \frac{\partial E_z}{\partial t} + \frac{1}{c^2} \frac{\partial E_z}{c^2} + \frac{2}{c^2} \frac{v u x}{c^2} \frac{\partial E_z}{c^2} - \frac{v}{c^2} \frac{\partial E_z}{c^2} + \frac{v}{c^2} \frac{\partial E_z}{c^2} + \frac{1}{c^2} \frac{\partial E_z}{c^2} + \frac{1}{c^2} \frac{\partial E_z}{c^2} \]

That reorganized makes:

\[ \frac{\partial B_y}{\partial x} \frac{\partial B_x}{\partial y} = \varepsilon_0 \mu_0 \frac{\partial E_z}{\partial t} + \frac{v}{c^2} \frac{\partial E_z}{c^2} + \frac{v}{c^2} \frac{\partial E_z}{c^2} + \frac{v}{c^2} \frac{\partial E_z}{c^2} \]

As the term in parenthesis is the equation 8.5 that is equal to zero then we have the invariance of the Ampere-Maxwell's Law without electrical charge:

**Invariance of the Ampere-Maxwell's Law without electrical charge:**

\[ \frac{\partial B_x'}{\partial y'} + \frac{\partial B_y'}{\partial z'} = \varepsilon_0 \mu_0 \frac{\partial E_x'}{\partial t'} \tag{8.42} \]

Where applying the tables 7B and 9B we have:
\[
\frac{\partial}{\partial y} \left( B_z \frac{v}{c^2} E_y \right) - \frac{\partial}{\partial z} \left( B_y + \frac{v}{c^2} E_z \right) = \varepsilon_0 \mu_0 \sqrt{K} \frac{\partial}{\partial t} E_x \sqrt{K}
\]

Making the operations we have:

\[
\frac{\partial B_z}{\partial y} \frac{\partial B_y}{\partial z} = \frac{v}{c^2} \left( \frac{\partial E_y}{\partial y} + \frac{\partial E_z}{\partial z} \right) + \varepsilon_0 \mu_0 \left( 1 + \frac{v^2}{c^2} - \frac{2vux}{c^2} \right) \frac{\partial E_x}{\partial t} \frac{1}{1 - v/ux}
\]

Replacing in the first parenthesis the Gauss’ Law without electrical charge and multiplying by \((1 - v/ux)\) we have:

\[
\frac{\partial B_z}{\partial y} \frac{\partial B_y}{\partial z} = \varepsilon_0 \mu_0 \frac{\partial E_x}{\partial t} + v \left( \frac{\partial B_z}{\partial y} \frac{\partial B_y}{\partial z} \right) - \frac{v}{c^2} \left( \frac{\partial E_x}{\partial y} \frac{\partial E_z}{\partial z} \right) \frac{v}{c^2} \left( \frac{\partial E_x}{\partial z} \frac{\partial E_z}{\partial y} \right) + \frac{v^2}{c^2} \left( \frac{1}{\partial y} \frac{\partial E_x}{\partial z} + \frac{1}{\partial z} \frac{\partial E_x}{\partial y} \right) \frac{1}{c^2} \frac{2vux}{c^2} \frac{\partial E_x}{\partial t} \frac{1}{c^2}
\]

Where replacing 7.9, 7.9.1 and 8.5 we have:

\[
\frac{\partial B_z}{\partial y} \frac{\partial B_y}{\partial z} = \varepsilon_0 \mu_0 \frac{\partial E_x}{\partial t} + \frac{v}{c^2} \left( \frac{\partial E_y}{\partial y} \frac{\partial E_z}{\partial z} \right) \frac{v}{c^2} \left( \frac{\partial E_x}{\partial z} \frac{\partial E_z}{\partial y} \right) + \frac{v^2}{c^2} \left( \frac{1}{\partial y} \frac{\partial E_x}{\partial z} + \frac{1}{\partial z} \frac{\partial E_x}{\partial y} \right) \frac{1}{c^2} \frac{2vux}{c^2} \frac{\partial E_x}{\partial t} \frac{1}{c^2}
\]

That simplified supplies:

\[
\frac{\partial B_z}{\partial y} \frac{\partial B_y}{\partial z} = \varepsilon_0 \mu_0 \frac{\partial E_x}{\partial t} + \frac{v}{c^2} \left( \frac{\partial E_y}{\partial y} \frac{\partial E_z}{\partial z} \right) \frac{v}{c^2} \left( \frac{\partial E_x}{\partial z} \frac{\partial E_z}{\partial y} \right) + \frac{v^2}{c^2} \left( \frac{1}{\partial y} \frac{\partial E_x}{\partial z} + \frac{1}{\partial z} \frac{\partial E_x}{\partial y} \right) \frac{1}{c^2} \frac{2vux}{c^2} \frac{\partial E_x}{\partial t} \frac{1}{c^2}
\]

Replacing in the first parenthesis the Gauss’ Law without electrical charge we have:

\[
\frac{\partial B_z}{\partial y} \frac{\partial B_y}{\partial z} = \varepsilon_0 \mu_0 \frac{\partial E_x}{\partial t} + \frac{v}{c^2} \left( \frac{\partial E_y}{\partial y} \frac{\partial E_z}{\partial z} \right) \frac{v}{c^2} \left( \frac{\partial E_x}{\partial z} \frac{\partial E_z}{\partial y} \right) + \frac{v^2}{c^2} \left( \frac{1}{\partial y} \frac{\partial E_x}{\partial z} + \frac{1}{\partial z} \frac{\partial E_x}{\partial y} \right) \frac{1}{c^2} \frac{2vux}{c^2} \frac{\partial E_x}{\partial t} \frac{1}{c^2}
\]

That reorganized makes:

\[
\frac{\partial B_z}{\partial y} \frac{\partial B_y}{\partial z} = \mu_0 \frac{\partial E_x}{\partial t} + \varepsilon_0 \mu_0 \frac{\partial E_x}{\partial t} \left( \frac{2v}{c^2} \left( \frac{\partial E_x}{\partial z} + \frac{\partial E_y}{\partial y} \frac{\partial E_z}{\partial z} \right) \frac{1}{c^2} \frac{2vux}{c^2} \frac{\partial E_x}{\partial t} \frac{1}{c^2}
\]

As the term in parenthesis is the equation 8.5 that is equal to zero then we have the invariance of the Ampere-Maxwell’s Law without electrical charge:

**Invariance of the Ampere-Maxwell’s Law without electrical charge:**

\[
\frac{\partial B_x^x'}{\partial x^x'} - \frac{\partial B_z^x'}{\partial x^y'} = \varepsilon_0 \mu_0 \frac{\partial E_y^{y'}}{\partial t} \tag{8.44}
\]

Where applying the tables 6, 7B and 9B we have:

\[
\frac{\partial B_x}{\partial y} - \frac{\partial B_z}{\partial z} = \varepsilon_0 \mu_0 \sqrt{K} \frac{\partial E_y}{\partial t} \sqrt{K}
\]

Making the operations we have:

\[
\frac{\partial B_x}{\partial y} \frac{\partial B_z}{\partial z} = \varepsilon_0 \mu_0 \frac{\partial E_y}{\partial t} + \frac{v}{c^2} \frac{\partial E_y}{\partial t} + \frac{1}{c^2} \frac{2vux}{c^2} \frac{\partial E_y}{\partial t} + \frac{v}{c^2} \frac{\partial E_y}{\partial t} + \frac{1}{c^2} \frac{2vux}{c^2} \frac{\partial E_y}{\partial t}
\]

Where simplifying and applying 7.9.1 we have:

\[
\frac{\partial B_x}{\partial y} \frac{\partial B_z}{\partial z} = \varepsilon_0 \mu_0 \frac{\partial E_y}{\partial t} \left( \frac{1}{c^2} \frac{2vux}{c^2} \frac{\partial E_y}{\partial t} + \frac{v}{c^2} \frac{\partial E_y}{\partial t} + \frac{1}{c^2} \frac{2vux}{c^2} \frac{\partial E_y}{\partial t} \right)
\]

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That reorganized makes:

\[ \frac{\partial B_x}{\partial z} - \frac{\partial B_z}{\partial x} = \varepsilon_0 \mu_0 \frac{\partial E_y}{\partial t} - \frac{v}{c^2} \left( \frac{\partial u x}{\partial t} + \frac{\partial E_y}{\partial x} \right) \]

As the term in parenthesis is the equation 8.5 that is equal to zero then we have the invariance of the Ampere-Maxwell's Law without electrical charge:

\[ \frac{\partial B_x}{\partial z} - \frac{\partial B_z}{\partial x} = 0 \]

§15 Invariance (continuation)

A function \( f(\theta) = f(kr - wt) \)

Where the phase is equal to \( \theta = (kr - wt) \)

In order to represent an undulating movement that goes on in one arbitrary direction must comply with the wave equation and because of this we have:

\[ \frac{k^2}{r^2} \left[ 3r - \frac{(x^2 + y^2 + z^2)}{r} \right] \frac{\partial f(\theta)}{\partial \theta} + \frac{k^2}{r^2} \left( x^2 + y^2 + z^2 \right) \frac{\partial^2 f(\theta)}{\partial \theta^2} - k^2 \frac{\partial^2 f(\theta)}{\partial \theta^2} = 0 \]

That doesn't meet with the wave equation because the two last elements get nule but the first one doesn't.

In order to overcome this problem we reformulate the phase \( \theta \) of the function in the following way.

A unitary vector such as

\[ \vec{n} = \cos \phi \vec{i} + \cos \alpha \vec{j} + \cos \beta \vec{k} \]

where \( \cos \phi = \frac{x}{r}, \cos \alpha = \frac{y}{r}, \cos \beta = \frac{z}{r} \)

has the module equal to \( n = |\vec{n}| = \sqrt{\vec{n} \cdot \vec{n}} = \sqrt{\cos^2 \phi + \cos^2 \alpha + \cos^2 \beta} = 1. \)

Making the product

\[ \vec{n} \cdot \vec{R} = (\cos \phi \vec{i} + \cos \alpha \vec{j} + \cos \beta \vec{k}) \cdot (x \vec{i} + y \vec{j} + z \vec{k}) = \cos \phi x + \cos \alpha y + \cos \beta z = \frac{x^2 + y^2 + z^2}{r} = \frac{r^2}{r} = r \]

we have \( r = \vec{n} \cdot \vec{R} = \cos \phi x + \cos \alpha y + \cos \beta z \) that applied to the phase \( \theta \) supplies a new phase

\[ \Phi = (kr - wt) = (k \vec{n} \cdot \vec{R} - wt) = (k \cos \phi x + k \cos \alpha y + k \cos \beta z - wt) \]

with the same meaning of the previous phase \( \theta = \Phi. \)

Replacing \( r = \vec{n} \cdot \vec{R} = \cos \phi x + \cos \alpha y + \cos \beta z \) \( e \ k = \frac{w}{c} \) in the phase \( \theta \) multiplied by \( -1 \) we also get another phase in the form

\[ \Phi = (-1)(kr - wt) = (wt - kr) = \left[ \frac{w}{c} \left( t - \frac{r}{c} \right) \right] = \left[ \frac{w}{c} \left( t - \frac{\cos \phi x + \cos \alpha y + \cos \beta z}{c} \right) \right] \]

with the same meaning of the previous phase \( (-1)\theta = \Phi. \)

Thus we can write a new function as:

\[ f(\Phi) = f \left[ \frac{w}{c} \left( t - \frac{\cos \phi x + \cos \alpha y + \cos \beta z}{c} \right) \right] \]
That replaced in the wave equation with the director cosine considered constant supplies:

\[
\frac{\partial^2 f(\Phi)}{\partial \Phi^2} \frac{w^2}{c^2} \cos^2 \phi + \frac{\partial^2 f(\Phi)}{\partial \Phi^2} \frac{w^2}{c^2} \cos^2 \alpha + \frac{\partial^2 f(\Phi)}{\partial \Phi^2} \frac{w^2}{c^2} \cos^2 \beta - \frac{\partial^2 f(\Phi)}{\partial \Phi^2} \frac{w^2}{c^2} = \text{zero}
\]

that simplified meets the wave equation.

The positive result of the phase $\Phi$ in the wave equation is an exclusive consequence of the director cosines being constant in the partial derivatives showing that the wave equation demands the propagation to have one steady direction in the space (plane wave).

For the observer $O$ a source located in the origin of its referential produces in a random point located at the distance $r = ct = \sqrt{x^2 + y^2 + z^2}$ of the origin, an electrical field $\vec{E}$ described by:

\[
\vec{E} = E_x \hat{i} + E_y \hat{j} + E_z \hat{k}
\]

Where the components are described as:

\[
E_x = E_{xo} \cdot f(\Phi) \\
E_y = E_{yo} \cdot f(\Phi) \\
E_z = E_{zo} \cdot f(\Phi)
\]

That applied in $\vec{E}$ supplies:

\[
\vec{E} = E_{xo} f(\Phi) \hat{i} + E_{yo} f(\Phi) \hat{j} + E_{zo} f(\Phi) \hat{k} = [E_{xo} \hat{i} + E_{yo} \hat{j} + E_{zo} \hat{k}] f(\Phi) = \vec{E}_o f(\Phi).
\]

with module equal to $E = \sqrt{(E_{xo})^2 + (E_{yo})^2 + (E_{zo})^2} \cdot f(\Phi) \Rightarrow E = E_o \cdot f(\Phi)$

Being $\vec{E}_o = E_{xo} \hat{i} + E_{yo} \hat{j} + E_{zo} \hat{k}$

The maximum amplitude vector Constant with the components $E_{xo}$, $E_{yo}$, $E_{zo}$

And module $E_o = \sqrt{(E_{xo})^2 + (E_{yo})^2 + (E_{zo})^2}$

Being $f(\Phi)$ a function with the phase $\Phi$ equal to 15.87 or 15.88.

Deriving the component $E_x$ in relation to $x$ and $t$ we have:

\[
\frac{\partial E_x}{\partial x} = E_{xo} \frac{\partial f(\Phi)}{\partial \Phi} \frac{\partial \Phi}{\partial x} = E_{xo} \frac{\partial f(\Phi)}{\partial \Phi} \frac{kr - wt}{r} = E_{xo} \frac{\partial f(\Phi)}{\partial \Phi} \frac{kr - wt}{ct}
\]

\[
\frac{\partial E_x}{\partial t} = E_{xo} \frac{\partial f(\Phi)}{\partial \Phi} \frac{\partial \Phi}{\partial t} = E_{xo} \frac{\partial f(\Phi)}{\partial \Phi} (-w)
\]

that applied in 8.5 supplies

\[
\frac{\partial E_x}{\partial x} + \frac{x/t \cdot \partial E_x}{\partial t} = 0 \Rightarrow E_{xo} \frac{\partial f(\Phi)}{\partial \Phi} \frac{\partial \Phi}{\partial x} + \frac{x/t \cdot E_{xo} \frac{\partial f(\Phi)}{\partial \Phi} \frac{\partial \Phi}{\partial t}}{c^2} = 0 \Rightarrow E_{xo} \frac{\partial f(\Phi)}{\partial \Phi} \left( \frac{\partial \Phi}{\partial x} + \frac{x/t \cdot \partial \Phi}{c^2} \right) = 0
\]

\[
E_{xo} \frac{\partial f(\Phi)}{\partial \Phi} \left( \frac{\partial \Phi}{\partial x} + \frac{x/t \cdot \partial \Phi}{c^2} \right) = 0 \Rightarrow \frac{\partial \Phi}{\partial x} + \frac{x/t \cdot \partial \Phi}{c^2} = 0
\]

demonstrating that it is the phase $\Phi$ that must comply with 8.5.
\[ \frac{\partial \Phi}{\partial x} + \frac{x}{c^2} \frac{\partial \Phi}{\partial t} = \text{zero} \Rightarrow \frac{\partial (kr-wt)}{\partial x} + \frac{x}{c^2} \frac{\partial (kr-wt)}{\partial t} = \text{zero} \Rightarrow \frac{kx}{ct} + \frac{x}{c^2}(-w) = \text{zero} \Rightarrow \frac{x}{ct} \left( k - \frac{w}{c} \right) = \text{zero} \]

as \( k = \frac{w}{c} \) then \( E_z \) complies with 8.5.

As the phase is the same for the components \( E_y \) and \( E_z \) then they also comply with 8.5.

As the phases for the observers \( O \) and \( O' \) are equal \((kr-wt) = (k'r'-w't')\) then the components of the observer \( O' \) also comply with 8.5.

\[ \frac{\partial (kr-wt)}{\partial x} + \frac{x}{c^2} \frac{\partial (kr-wt)}{\partial t} = \frac{\partial (k'r'-w't')}{\partial x'} + \frac{x'}{c^2} \frac{\partial (k'r'-w't')}{\partial t'} = \text{zero} \]

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The components relatively to the observer \( O \) of the electrical field are transformed for the referential of the observer \( O' \) according to the tables 7, 7B and 8.

Applying in 8.5 a wave function written in the form:

\[ \Psi = e^{i(kx-wt)} = e^{i\Phi} = \cos(kx-wt) + i\sin(kx-wt) = \cos \Phi + i\sin \Phi \]

where \( i = \sqrt{-1} \).

Deriving we have:

\[ \frac{\partial \Psi}{\partial x} = -k \sin \Phi + ki \cos \Phi \quad \text{and} \quad \frac{\partial \Psi}{\partial t} = w \sin \Phi - wi \cos \Phi \]

15.103

or \( \frac{\partial \Psi}{\partial x} = ke^{\Phi} \) and \( \frac{\partial \Psi}{\partial t} = -we^{\Phi} \)

15.104

That applied in 8.5 supplies:

\[ \frac{\partial \Psi}{\partial x} + \frac{x}{c^2} \frac{\partial \Psi}{\partial t} = \text{zero} \Rightarrow \left( -k \sin \Phi + ki \cos \Phi \right) + \frac{x}{c^2} \left( w \sin \Phi - wi \cos \Phi \right) = \text{zero} \]

that is equal to:

\[ \left( -k + \frac{xw}{c^2 t} \right) \sin \Phi + \left( ki - \frac{xwi}{c^2 t} \right) \cos \Phi = \text{zero} \]

or \( \frac{\partial \Psi}{\partial x} + \frac{x}{c^2} \frac{\partial \Psi}{\partial t} = \text{zero} \Rightarrow \left( ke^{\Phi} \right) + \frac{x}{c^2} \left( -we^{\Phi} \right) = \text{zero} \]

where we must have the coefficients equal to zero so that we get an identity, then:

\[ -k + \frac{xw}{c^2 t} = \text{zero} \Rightarrow k = \frac{xw}{c^2 t} \]

\[ ki - \frac{xwi}{c^2 t} = \text{zero} \Rightarrow k = \frac{xw}{c^2 t} \]

\[ \left( ke^{\Phi} \right) + \frac{x}{c^2} \left( -we^{\Phi} \right) = \text{zero} \Rightarrow k = \frac{xw}{c^2 t} \]

Where applying \( w = ck \) we have:
\[ k = \frac{xw}{c^2t} = \frac{xck}{c^2t} \Rightarrow \frac{x}{t} = c \]

Then to meet with the equation 8.5 we must have a wave propagation along the axis x with the speed c.

If we apply \( w = uk \) and \( v = \frac{x}{t} \) we have:

\[ k = \frac{xw}{c^2t} = \frac{vak}{c^2t} \Rightarrow u = c^2 \cdot \frac{v}{v}. \]

A result also gotten from the Louis de Broglie's wave equation.

### §16 Time and Frequency

Considering the Doppler effect as a law of physics.

We can define a clock as any device that produces a frequency of identical events in a series possible to be enlisted and added in such a way that a random event \( n \) of a device will be identical to any event in the series of events produced by a replica of this device when the events are compared in a relative resting position.

The cyclical movement of a clock in a resting position according to the observer O referential sets the time in this referential and the cyclical movement of the arms of a clock in a resting position according to the observer O' sets the time in this referential. The formulas of time transformation 1.7 and 1.8 relate the times between the referentials in relative movement thus, relate movements in relative movement.

The relative movement between the inertial referentials produces the Doppler effect that proves that the frequency varies with velocity and as the frequency can be interpreted as being the frequency of the cyclical movement of the arms of a clock then the time varies in the same proportion that varies the frequency with the relative movement that is, it is enough to replace the time \( t \) and \( t' \) in the formulas 1.7 and 1.8 by the frequencies \( y \) and \( y' \) to get the formulas of frequency transformation, then:

\[ t' = t\sqrt{K} \Rightarrow y' = y\sqrt{K} \quad 1.7 \text{ becomes } 2.22 \]
\[ t = t'\sqrt{K'} \Rightarrow y = y'\sqrt{K'} \quad 1.8 \text{ becomes } 2.22 \]

The Galileo's transformation of velocities \( u'' = u - v \) between two inertial referentials presents intrinsically three defects that can be described this way:

a) The Galileo's transformation of velocity to the axis x is \( u'x' = ux - v \). In that one if we have \( ux = c \) then \( u'x' = c - v \) and if we have \( u'x' = c \) then \( ux = c + v \). As both results are not simultaneously possible or else we have \( ux = c \) or \( u'x' = c \) then the transformation doesn't allow that a ray of light be simultaneously observed by the observers O and O' what shows the privilege of an observer in relation to the other because each observer can only see the ray of light running in its own referential (intrinsic defect to the classic analysis of the Sagnac's effect).

b) It cannot also comply to Newton's first law of inertia because a ray of light emitted parallel to the axis x from the origin of the respective inertial referentials at the moment that the origins are coincident and at the moment in which \( t = t' = 0 \) will have by the Galileo's transformation the velocity c of light altered by \( \pm v \) to the referentials, on the contrary of the inertial law that wouldn't allow the existence of a variation in velocity because there is no external action acting on the ray of light and because of this both observers should see the ray of light with velocity c.

c) As it considers the time as a constant between the referentials it doesn't produce the temporal variation between the referentials in movement as it is required by the Doppler effect.

The principle of constancy of light velocity is nothing but a requirement of the Newton's first law, the inertia law.

Newton's first law, the inertia law, is introduced in Galileo's transformation when the principle of constancy of light velocity is applied in Galileo's transformation providing the equation of tables 1 and 2 of the Undulating Relativity that doesn't have the three defects described.
The time and velocity equations of tables 1 and 2 can be written as:

\[ t' = t \sqrt{1 + \frac{v^2}{c^2} - \frac{2v}{c} \cos \phi} \]  

\[ v' = \frac{v}{\sqrt{1 + \frac{v^2}{c^2} - \frac{2v}{c} \cos \phi}} \]  

\[ t = t' \sqrt{1 + \frac{v'^2}{c^2} + \frac{2v'\cos \phi'}{c}} \]  

\[ v = \frac{v'}{\sqrt{1 + \frac{v'^2}{c^2} + \frac{2v'\cos \phi'}{c}}} \]

The distance \( d \) between the referentials is equal to the product of velocity by time this way:

\[ d = vt = v' t' \]

It doesn’t depend on the propagation angle of the ray of light, being exclusively a function of velocity and time, that is, the propagation angle of the ray of light, only alters between the inertial referential the proportion between time and velocity, keeping the distance constant in each moment, to any propagation angle.

The equations above in a function form are written as:

\[ d = e(v, t) = e'(v', t') \]  

\[ t' = f(v, t, \phi) \]  

\[ v' = g(v, \phi) \]  

\[ t = f'(v', t', \phi') \]  

\[ v = g'(v', \phi') \]  

Then we have that the distance is a function of two variables, the time a function of three variables and the velocity a function of two variables.

From the definition of moment 4.1 and energy 4.6 we have:

\[ \vec{p} = \frac{E}{c^2} \vec{u} \]  

The elevated to the power of two supplies:

\[ \frac{u^2}{c^2} = \frac{c^2}{E^2} p^2 \]  

Elevating to the power of two the energy formula we have:

\[ E^2 = \left( \sqrt{\frac{m_0 c^2}{\sqrt{1 - \frac{u^2}{c^2}}}} \right)^2 \implies E^2 - E^2 \frac{u^2}{c^2} = m_0^2 c^4 \]

Where applying 16.2 we have:
\[
E^2 - E^2 \frac{u^2}{c^2} = m_0 c^4 \Rightarrow E^2 - E^2 \frac{c^2}{E^2} p^2 = m_0^2 c^4 \Rightarrow E = c \sqrt{p^2 + m_0^2 c^2}
\]

From where we conclude that if the mass in resting position of a particle is null \( m_0 = 0 \) the particle energy is equal to \( E = c p \).

That applied in 16.2 supplies:

\[
u^2 = \frac{c^2}{E^2} p^2 \Rightarrow \frac{u^2}{c^2} = \frac{c^2}{(cp)^2} p^2 \Rightarrow u = c
\]

From where we conclude that the movement of a particle with a null mass in resting position \( m_0 = 0 \) will always be at the velocity of light \( u = c \).

Applying in \( E = c p \) the relations \( E = yh \) and \( c = \lambda \). we have:

\[y h = \lambda, p \Rightarrow p = \frac{h}{\lambda} \text{ and in the same way } p' = \frac{h}{\lambda'}\]

Equation that relates the moment of a particle with a null mass in resting position with its own way length.

Elevating to the power of two the formula of moment transformation (4.9) we have:

\[\bar{p}' = \bar{p} - \frac{E}{c^2} \bar{v} \Rightarrow p'^2 = p^2 + \frac{E^2}{c^2} v^2 - 2 \frac{E}{c^2} v p x\]

Where applying \( E = c p \) and \( p x = p \cos \phi = p \frac{u x}{c} \) we find:

\[p'^2 = p^2 + \frac{(cp)^2}{c^4} v^2 - 2 \frac{cp}{c^2} v p \frac{u x}{c} \Rightarrow p' = p \sqrt{1 + \frac{v^2}{c^2} - \frac{2v u x}{c^3}} \Rightarrow p' = p \sqrt{\lambda} \]

Where applying 16.5 results in:

\[p' = p \sqrt{\lambda} \Rightarrow \frac{h}{\lambda'} = \frac{h}{\sqrt{\lambda}} \Rightarrow \lambda' = \frac{\lambda}{\sqrt{\lambda}} \text{ or inverted } \lambda = \frac{\lambda'}{\sqrt{\lambda'}}\]

Where applying \( c = \lambda \) and \( c = \lambda' \) we have:

\[y' = y \sqrt{\lambda} \text{ or inverted } y = y' \sqrt{\lambda'}\]

In § 2 we have the equations 2.21 and 2.22 applying the principle of relativity to the wave phase.

### 17 Transformation of H. Lorentz

For two observers in a relative movement, the equation that represents the principle of constancy of light speed for a random point A is:

\[x'^2 + y'^2 + z'^2 - c^2 t'^2 = x^2 + y^2 + z^2 - c^2 t^2\]

In this equation canceling the symmetric terms we have:

Nesta cancelando os termos simétricos obtemos:

\[x'^2 - c^2 t'^2 = x^2 - c^2 t^2\]
That we can write as:

\[(x'-ct')(x'+ct')=(x-ct)(x+ct)\]  \hspace{1cm} (17.03)

If in this equation we define the proportion factors \(\eta\) and \(\mu\) as:

\[
\begin{align*}
(x'-ct') &= \eta(x-ct) & A \\
(x'+ct') &= \mu(x+ct) & B
\end{align*}
\]

where we must have \(\eta, \mu = 1\) to comply 17.03.

The equations 17.04 where first gotten by Albert Einstein.

When a ray of light moves in the plane \(y'z'\) to the observer \(O'\) we have \(x' = 0\) and \(x = vt\) and such conditions applied to the equation 17.02 supplies:

\[0 - c^2t^2 = (vt)^2 - c^2t^2 \Rightarrow t' = t\sqrt{1 - \frac{v^2}{c^2}}\]  \hspace{1cm} (17.05)

This result will also be supplied by the equations \(A\) and \(B\) of the group 17.04 under the same conditions:

\[
\left\{
\begin{align*}
0 - ct\sqrt{1 - \frac{v^2}{c^2}} &= \eta(vt - ct) & A \\
0 + ct\sqrt{1 - \frac{v^2}{c^2}} &= \mu(vt + ct) & B
\end{align*}
\]

From those we have:

\[
\eta = \sqrt{\frac{1 + \frac{v}{c}}{1 - \frac{v}{c}}} \quad \text{and} \quad \mu = \sqrt{\frac{1 - \frac{v}{c}}{1 + \frac{v}{c}}} \]

Where we have proven that \(\eta, \mu = 1\).

From the group 17.04 we have the Transformations of H. Lorentz:

\[
x' = \frac{(\eta + \mu)}{2} x + \frac{(\mu - \eta)}{2} ct
\]

\[
ct' = \frac{(\mu - \eta)}{2} x + \frac{(\eta + \mu)}{2} ct
\]

\[
x = \frac{(\eta + \mu)}{2} x' + \frac{(\eta - \mu)}{2} ct'
\]

\[
ct = \frac{(\eta - \mu)}{2} x' + \frac{(\eta + \mu)}{2} ct'
\]

Indexes equations \(\frac{\eta + \mu}{2}\), \(\frac{\mu - \eta}{2}\) and \(\frac{\eta - \mu}{2}:

\[
\eta + \mu = \sqrt{\frac{1 + \frac{v}{c}}{1 - \frac{v}{c}}} + \sqrt{\frac{1 - \frac{v}{c}}{1 + \frac{v}{c}}} = \frac{2}{\sqrt{\frac{1 - \frac{v^2}{c^2}}{1 + \frac{v^2}{c^2}}} \sqrt{\frac{1 + \frac{v^2}{c^2}}{1 - \frac{v^2}{c^2}}} \Rightarrow \eta + \mu = \frac{1}{\sqrt{\frac{1 - \frac{v^2}{c^2}}{1 + \frac{v^2}{c^2}}}}
\]

\[
\mu - \eta = \sqrt{\frac{1 - \frac{v}{c}}{1 + \frac{v}{c}}} - \sqrt{\frac{1 + \frac{v}{c}}{1 - \frac{v}{c}}} = \frac{2}{\sqrt{\frac{1 - \frac{v^2}{c^2}}{1 + \frac{v^2}{c^2}}} \sqrt{\frac{1 + \frac{v^2}{c^2}}{1 - \frac{v^2}{c^2}}} \Rightarrow \mu - \eta = \frac{-\frac{v}{c}}{\sqrt{\frac{1 - \frac{v^2}{c^2}}{1 + \frac{v^2}{c^2}}}}
\]
\[
\eta - \mu = \frac{l + \frac{V}{c}}{\sqrt{1 - \frac{V^2}{c^2}}} - \frac{l - \frac{V}{c}}{\sqrt{1 + \frac{V^2}{c^2}}} = \frac{l + \frac{V}{c} - l + \frac{V}{c}}{\sqrt{1 - \frac{V^2}{c^2}}} = \frac{2\frac{V}{c}}{\sqrt{1 - \frac{V^2}{c^2}}} \Rightarrow \eta - \mu = \frac{\frac{V}{c}}{\sqrt{1 - \frac{V^2}{c^2}}}
\]

17.14

Sagnac effect

When both observers’ origins are equal the time is zeroed (t = t’ = zero) in both referentials and two rays of light are emitted from the common origin, one in the positive direction (clockwise index c) of the axis x and x’ with a wave front \(A_c\) and another in the negative direction (counter-clockwise index u) of the axis x and x’ with a wave front \(A_u\).

The propagation conditions above applied to the Lorentz equations supply the tables A and B below:

**Table A**

<table>
<thead>
<tr>
<th>Equation</th>
<th>Clockwise ray (c)</th>
<th>Equation Counter-clockwise ray (u)</th>
<th>Sum of rays</th>
</tr>
</thead>
<tbody>
<tr>
<td>Result</td>
<td>(x_c = ct_c)</td>
<td>Condition (x_u = -ct_u)</td>
<td></td>
</tr>
<tr>
<td>17.08</td>
<td>(x'_c = \mu ct_c)</td>
<td>17.08 (x'_u = -\eta ct_u)</td>
<td>(x'_c + x'_u = \mu x_c + \eta x_u)</td>
</tr>
<tr>
<td></td>
<td>(x'_c = \mu x_c)</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>(ct'_c = \mu ct_c)</td>
<td>17.09 (ct'_u = \eta ct_u)</td>
<td>(ct'_c + ct'_u = \mu ct_c + \eta ct_u)</td>
</tr>
<tr>
<td></td>
<td>(x'_c = ct'_c)</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

**Table B**

<table>
<thead>
<tr>
<th>Equation</th>
<th>Clockwise ray (c)</th>
<th>Equation Counter-clockwise ray (u)</th>
<th>Sum of rays</th>
</tr>
</thead>
<tbody>
<tr>
<td>Result</td>
<td>(x'_c = ct'_c)</td>
<td>Condition (x'_u = -ct'_u)</td>
<td></td>
</tr>
<tr>
<td>17.10</td>
<td>(x_c = \eta ct'_c)</td>
<td>17.10 (x_u = -\mu ct'_u)</td>
<td></td>
</tr>
<tr>
<td></td>
<td>(x_c = \eta x'_c)</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>(ct_c = \eta ct'_c)</td>
<td>17.11 (ct_u = \mu ct'_u)</td>
<td>(ct_c + ct_u = \eta ct'_c + \mu ct'_u)</td>
</tr>
<tr>
<td></td>
<td>(x_c = ct_c)</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

We observe that the tables A and B are inverse one to another.

When we form the group of the sum equations of the two rays from tables A and B:

\[
\begin{align*}
D' &= ct'_c + ct'_u = \mu ct_c + \eta ct_u \quad A \\
D &= ct_c + ct_u = \eta ct'_c + \mu ct'_u \quad B
\end{align*}
\]

17.15

Where to the observer \(O'\) \(D' = A_{u} \leftrightarrow A_{c}\) is the distance between the front waves \(A_u\) and \(A_c\) and where to the observer \(O\) \(D = A_{u} \leftrightarrow A_{c}\) is the distance between the front waves \(A_u\) and \(A_c\).

In the equations 17.15 above, due to the isotropy of space and time and the front waves \(A_{u} \leftrightarrow A_{c}\) of the two rays of light being the same for both observers, the sum of rays of light e times must be invariable between the observers, which we can express by:

\[
D' = D \Rightarrow ct'_c + ct'_u = ct_c + ct_u \Rightarrow \sum t' = \sum t
\]

17.16

This result that generates an equation of isotropy of space and time can be called as the conservation of space and time principle.
The three hypothesis of propagation defined as follows will be applied in 17.15 and tested to prove the conservation of space and time principle given by 17.16:

Hypothesis A:

If the space and time are isotropic and there is no movement with no privilege of one observer considered over the other in an empty space then the propagation geometry of rays of light can be given by:

\[ ct_c = ct'_u \] and \[ ct_u = ct'_c \] \hspace{1cm} 17.17

This hypothesis applied to the equation A or B of the group 17.15 complies to the space and time conservation principle given by 17.16.

The hypothesis 17.17 applied to the tables A and B results in:

\[
\begin{align*}
\text{Quadro A} & \quad \begin{cases} ct'_c = \mu ct'_u & A \\ ct'_u = \eta ct'_c & B \end{cases} \\
\text{Quadro B} & \quad \begin{cases} ct = \eta ct'_c & C \\ ct_u = \mu ct'_c & D \end{cases}
\end{align*}
\]

Hypothesis B:

If the space and time are isotropic but the observer O is in an absolute resting position in an empty space then the geometry of propagation of the rays of light is given by:

\[ ct_c = ct'_u \] \hspace{1cm} 17.19

That applied to the table A and B results in:

\[
\begin{align*}
\text{Quadro A} & \quad \begin{cases} ct'_c = \mu ct & A \\ ct'_u = \eta ct & B \end{cases} \\
\text{Quadro B} & \quad \begin{cases} ct = \eta ct'_c & C \\ ct = \mu ct'_c & D \end{cases}
\end{align*}
\]

Hypothesis C:

If the space and time are isotropic but the observer O' is in an absolute resting position in an empty space then the propagation geometry of the rays of light is given:

\[ ct'_c = ct'_u \] \hspace{1cm} 17.23

That applied to the tables A and B results in:

\[
\begin{align*}
\text{Quadro A} & \quad \begin{cases} ct'_c = \mu ct_c & A \\ ct'_u = \eta ct_u & B \end{cases} \\
\text{Quadro B} & \quad \begin{cases} ct_c = \eta ct' & C \\ ct_u = \mu ct' & D \end{cases}
\end{align*}
\]

This result doesn’t comply with the conservation of space and time principle given by 17.16 and as \( D' \neq D \) it results in a situation of four rays of light, two to each observer, and each ray of light with its respective independent front wave from the others.

Hypothesis C:

If the space and time are isotropic but the observer O' is in an absolute resting position in an empty space then the propagation geometry of the rays of light is given:

\[ |ct'_c| = |ct'_u| = |ct'| \] \hspace{1cm} 17.25

That applied to the tables A and B results in:

\[
\begin{align*}
\text{Quadro A} & \quad \begin{cases} ct'_c = \mu ct_c & A \\ ct'_u = \eta ct_u & B \end{cases} \\
\text{Quadro B} & \quad \begin{cases} ct_c = \eta ct' & C \\ ct_u = \mu ct' & D \end{cases}
\end{align*}
\]
\[
\begin{align*}
\text{ct}_c &= \eta^2 \text{ct}_u \quad \text{A} \\
\text{ct}_u &= \mu^2 \text{ct}_c \quad \text{B}
\end{align*}
\]

Summing C and D in 17.24 we have:
\[
\text{ct}_c + \text{ct}_u = 2\text{t}'\left(\frac{\eta + \mu}{2}\right) \Rightarrow D' = D'\left(\frac{\eta + \mu}{2}\right) \Rightarrow D = \frac{D'}{\sqrt{1 - \frac{v'^2}{c^2}}} \Rightarrow t = \frac{t'}{\sqrt{1 - \frac{v'^2}{c^2}}}
\]

This result doesn’t comply with the conservation of space and time principle exactly the same way as hypothesis B given by 17.16 and as \(D' \neq D\) \(D' \neq D\) it results in a situation of four rays of light, two to each observer and each ray of light with its respective independent front wave from the others.

Conclusion

The hypothesis A, B and C are completely compatible with the demand of isotropy of space and time as we can conclude with the geometry of propagations.

The result of hypothesis A is contrary to the result of hypothesis B and C despite of the relative movement of the observers not changing the front wave \(A_c\) relatively to the front wave \(A_c\) because the front waves have independent movement one from the other and from the observers.

The hypothesis A applied in the transformations of H. Lorentz complies with the conservation of space and time principle given by 17.16 showing the compatibility with the transformations of H. Lorentz with the hypothesis A. The application of hypothesis B and C in the transformations of H. Lorentz supplies the space and time deformations given by 17.22 and 17.26 because the transformations of H. Lorentz are not compatible with the hypothesis B and C.

For us to obtain the Sagnac effect we must consider that the observer \(O'\) is in an absolute resting position, hypothesis C above and that the path of the rays of light be of \(2\pi R\):

\[
ct'_c = ct'_u = ct' = 2\pi R
\]

For the observer \(O\) the Sagnac effect is given by the time difference between the clockwise ray of light and the counter-clock ray of light \(\Delta t = t_c - t_u\) that can be obtained using 17.24 (C-D), 17.27 and 17.14:

\[
\Delta t = t_c - t_u = t'(\eta - \mu) = \frac{2\pi R}{c} \left(\frac{\frac{2v}{c}}{\sqrt{1 - \frac{v'^2}{c^2}}}\right) = \frac{4\pi Rv}{c\sqrt{c^2 - v'^2}}
\]

§9 The Sagnac Effect (continuation)

The moment the origins are the same the time is zeroed \((t = t' = 0)\) at both sides of the referential and the rays of light are emitted from the common origin, one in the positive way (clockwise index c) of the axis \(x\) and \(x'\) with a wave front \(A_c\) and the other one in the negative way (counter clockwise index u) of the axis \(x\) and \(x'\) with wave front \(A_u\).

The projected ray of light in the positive way (clockwise index c) of the axis \(x\) and \(x'\) is equationed by \(x'_c = ct_c\) and \(x'_c = ct'_c\) that applied to the Table I supplies:

\[
ct'_c = ct_c \left(1 - \frac{v_c}{c}\right) \Rightarrow ct'_c = ct_c K_c \quad \text{(1.7)}
\]

\[
ct_c = ct'_c \left(1 + \frac{v_c}{c}\right) \Rightarrow ct_c = ct_c K'_c \quad \text{(1.8)}
\]

\[
v'_c = \frac{v_c}{\left(1 - \frac{v_c}{c}\right)} \Rightarrow v'_c = \frac{v_c}{K_c} \quad \text{(1.15)}
\]

\[
v_c = \frac{v'_c}{\left(1 + \frac{v_c}{c}\right)} \Rightarrow v_c = \frac{v'_c}{K'_c} \quad \text{(1.20)}
\]
From those we deduct that the distance between the observers is given by:

\[ d_c = v_c t_c = v'_c t'_c \]

Where we have:

\[
\left( 1 - \frac{v_c}{c} \right) \left( 1 + \frac{v'_c}{c} \right) = K_c K'_c = I
\]

The ray of light project in the negative way (counter clockwise index u) of the axis x and x' is equationed by \( x_u = -c t_u \) and \( x'_u = -c t'_u \); that applied to the Table I gives:

\[
ct'_u = ct_u \left( 1 + \frac{v_u}{c} \right) \Rightarrow ct'_u = ct_u K'_u \quad (1.7) \quad \text{and} \quad ct'_u = ct_u \left( 1 - \frac{v'_u}{c} \right) \Rightarrow ct'_u = ct_u K'_u \quad (1.8)
\]

\[
v'_u = -\frac{v_u}{c} \quad \Rightarrow v'_u = \frac{v_u}{K'_u} \quad (1.15) \quad \text{and} \quad v_u = \frac{v'_u}{K'_u} \quad (1.20)
\]

From those we deduct that the distance between the observers is given by:

\[ d_u = v_u t_u = v'_u t'_u \]

Where we have:

\[
\left( 1 + \frac{v_u}{c} \right) \left( 1 - \frac{v'_u}{c} \right) = K_u K'_u = I
\]

We must observe that at first there is no relationship between the equations 9.11 to 9.14 with the equations 9.15 to 9.18.

With the propagation conditions described we form the following Tables A and B:

**Table A**

<table>
<thead>
<tr>
<th>Equation</th>
<th>Clockwise ray of light (c)</th>
<th>Counter clockwise ray of light (u)</th>
<th>Sum of the rays of light</th>
</tr>
</thead>
<tbody>
<tr>
<td>Result</td>
<td>Result</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Condition</td>
<td>( x_c = c t_c )</td>
<td>Condition ( x_u = -c t_u )</td>
<td></td>
</tr>
<tr>
<td>1.2</td>
<td>( x'_c = c t'_c K'_c )</td>
<td>1.2 ( x'_u = -c t'_u K'_u )</td>
<td></td>
</tr>
<tr>
<td>( x'_c = x'_u K'_u )</td>
<td>( x'_u = x'_u K'_u )</td>
<td>( x'_c + x'_u = x'_c K'_c + x'_u K'_u )</td>
<td></td>
</tr>
<tr>
<td>1.7</td>
<td>( ct'_c = c t_c K'_c )</td>
<td>1.7 ( ct'_u = c t_u K'_u )</td>
<td>( ct'_c + ct'_u = ct'_c K'_c + ct'_u K'_u )</td>
</tr>
<tr>
<td>( x'_c = ct'_c )</td>
<td>( x'_u = -ct'_u )</td>
<td>( x'_c + x'_u = x'_c K'_c + x'_u K'_u )</td>
<td></td>
</tr>
</tbody>
</table>

**Table B**

<table>
<thead>
<tr>
<th>Equation</th>
<th>Clockwise ray of light (c)</th>
<th>Counter clockwise ray of light (u)</th>
<th>Sum of the rays of light</th>
</tr>
</thead>
<tbody>
<tr>
<td>Result</td>
<td>Result</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Condition</td>
<td>( x'_c = c t'_c )</td>
<td>Condition ( x'_u = -c t'_u )</td>
<td></td>
</tr>
<tr>
<td>1.4</td>
<td>( x_c = c t'_c K'_c )</td>
<td>1.4 ( x_u = -c t'_u K'_u )</td>
<td></td>
</tr>
<tr>
<td>( x_c = x'_u K'_u )</td>
<td>( x_u = x'_u K'_u )</td>
<td>( x_c + x_u = x'_c K'_c + x'_u K'_u )</td>
<td></td>
</tr>
<tr>
<td>1.8</td>
<td>( ct_c = c t'_c K'_c )</td>
<td>1.8 ( ct_u = c t'_u K'_u )</td>
<td>( ct_c + ct_u = ct'_c K'_c + ct'_u K'_u )</td>
</tr>
<tr>
<td>( x_c = c t_c )</td>
<td>( x_u = -c t_u )</td>
<td>( x_c + x_u = x'_c K'_c + x'_u K'_u )</td>
<td></td>
</tr>
</tbody>
</table>
We observe that for the rays of light with the same direction the Tables A and B are inverse from each other.

Forming the equations group of the sum of the rays of light of the Tables A and B:

\[
\begin{aligned}
D' &= ct'_c + ct'_u = ct_c K_c + ct_u K_u & \text{A} \\
D &= ct_c + ct_u = ct'_c K'_c + ct'_u K'_u & \text{B}
\end{aligned}
\]

9.19

Where for the observer O' \(D' = A_u \leftrightarrow A_c\) is the distance between the wave fronts \(A_u\) and \(A_c\) and where for the observer O \(D = A_u \leftrightarrow A_c\) is the distance between the wave fronts \(A_u\) and \(A_c\).

In the equations above 9.19 due to the isotropy of the space and time and the wave fronts \(A_u \leftrightarrow A_c\) of the rays of light being the same for both observers, the sumo of the rays of light and of times must be invariable between the observers, which is expressed by:

\[
D' = D \Rightarrow ct'_c + ct'_u = ct_c + ct_u = \sum t' = \sum t
\]

9.20

This result that equations the isotropy of space and time can be called as the space and time conservation principle.

The three hypothesis of propagations defined next will be applied in 9.19 and tested to prove the compliance of the conservation of space and time principle given by 9.20. With these hypotheses we create a bond between the equations 9.11 to 9.14 with the equations 9.15 to 9.18.

**Hypothesis A:**

If the space and time are isotropic and there is movement with any privilege of any observer over each other in the empty space then the propagation geometry of the rays of light is equationed by:

\[
\begin{aligned}
ct_c = ct'_c \Rightarrow t_c = t'_c \Rightarrow v_c = v'_c \Rightarrow K_c = K'_c & \text{A} \\
ct_u = ct'_u \Rightarrow t_u = t'_u \Rightarrow v_u = v'_u \Rightarrow K_u = K'_u & \text{B}
\end{aligned}
\]

9.21

With those we deduct that the distance between the observers is given by:

\[
d_c = d_u = v_c t_c = v'_c t'_c = v_u t_u = v'_u t'_u
\]

9.22

Results that applied in the equations A or B of the group 9.19 complies with the conservation of space and time principle given by 9.20, showing that the Doppler effect in the clockwise and counter clockwise rays of light are compensated in the referentials.

**Hypothesis B:**

If the space and time are isotropic but the observer O is in an absolute resting position in the empty space then the propagation geometry of the rays of light is equationed by:

\[
\begin{aligned}
ct_c = ct_u = ct & \text{A} \\
v_c = v_u = v & \text{B} \\
v_c t_c = v_u t_u = vt & \text{C}
\end{aligned}
\]

9.23

With those we deduct that the distance between the observers is given by:

\[
d_c = d_u = vt = v'_c t'_c = v'_u t'_u
\]

9.24

Results that applied in the equations A or B of the group 9.19 complies with the conservation of space and time principle given by 9.20, showing that the Doppler effect in the clockwise and counter clockwise rays of light are compensated in the referentials.
Hypothesis C:

If the space and time are isotropic but the observer O is in an absolute resting position in the empty space then the propagation geometry of the rays of light is equationed by:

\[
\begin{align*}
ct'_c &= ct'_u = ct' \quad A \\
v'_c &= v'_u = v' \quad B \\
v'_c t'_c &= v'_u t'_u = v't' \quad C
\end{align*}
\]

With those we deduce that the distance between the observers is given by:

\[
d_c = d_u = v't' = v_c t_c = v_u t_u
\]

Results that applied in the equations A or B of the group 9.19 complies with the conservation of space and time principle given by 9.20, showing that the Doppler effect in the clockwise and counter clockwise rays of light are compensated in the referentials.

In order to obtain the Sagnac effect we consider that the observer O’ is in an absolute resting position, hypothesis C above and that the rays of light course must be of \(R\frac{\pi}{2}\):

\[
c t'_c = c t'_u = c t' = 2\pi R
\]

Applying the hypothesis C in 9.11 and 9.15 we have:

\[
t_c = t'_c K'_c \implies t_c = t' \left( 1 + \frac{v'}{c} \right)
\]

\[
t_u = t'_u K'_u \implies t_u = t' \left( 1 - \frac{v'}{c} \right)
\]

For the observer O the Sagnac effect is given by the time difference between course of the clockwise ray of light and the counter clock ray of \(\Delta t = t_c - t_u\) that can be obtained making \((9.28 - 9.29)\) and applying 9.27 making:

\[
\Delta t = t_c - t_u = t' \left( 1 + \frac{v'}{c} \right) - t' \left( 1 - \frac{v'}{c} \right) = 2v't' \left( \frac{1}{c} \right) = \frac{4\pi R v'}{c^2}
\]

The equation \(\Delta t = \frac{2v't'}{c} = \frac{2v_c t_c}{c} = \frac{2v_u t_u}{c}\) is exactly the result obtained from the geometry analysis of the propagation of the clockwise and counter clockwise rays of light in a circumference showing the coherence of the hypothesis adopted by the Undulating Relativity.

In 9.30 applying 9.12 and 9.16 we have the final result due to \(v_c\) and \(v_u\):

\[
\Delta t = t_c - t_u = -\frac{2v't'}{c} = \frac{4\pi R v'}{c^2} = \frac{4\pi R v_c}{\sqrt{c^2 - cv_c}} = \frac{4\pi R v_u}{\sqrt{c^2 + cv_u}}
\]

The classic formula of the Sagnac effect is given as:

\[
\Delta t = t_c - t_u = \frac{4\pi R v}{c^2 - v^2}
\]

From the propagation geometry we have:

\[
\Delta t = \frac{2vt}{c}
\]
The classic times would be given by:

\[ t = \frac{2\pi R}{c} \]  
\[ t_c = \frac{2\pi R}{c-v} \]  
\[ t_u = \frac{2\pi R}{c+v} \]  

Applying 9.34, 9.35 and 9.36 in 9.33 we have:

\[ \Delta t = \frac{2v}{c} \left( \frac{2\pi R}{c} \right) = \frac{4\pi RV}{c^2} \]  
\[ \Delta t_c = \frac{2v}{c} \left( \frac{2\pi R}{c-v} \right) = \frac{4\pi RV}{c^2 - cV} \]  
\[ \Delta t_u = \frac{2v}{c} \left( \frac{2\pi R}{c+v} \right) = \frac{4\pi RV}{c^2 + cV} \]  

The results 9.37, 9.38 and 9.39 are completely different from 9.32.

§18 The Michelson & Morley experience

The traditional analysis that supplies the solution for the null result of this experience considers a device in a resting position at the referential of the observer O' that emits two rays of light, one horizontal in the x' direction (clockwise index c) and another vertical in the direction y'. The horizontal ray of light (clockwise index c) runs until a mirror placed in x' = L at this point the ray of light reflects (counter clockwise index u) and returns to the origin of the referential where x' = zero. The vertical ray of light runs until a mirror placed in y' = L reflects and returns to the origin of the referential where y' = zero.

In the traditional analysis according to the speed of light constancy principle for the observer O' the rays of light track is given by:

\[ ct'_c = ct'_u = L \]  

For the observer O' the sum of times of the track of both rays of light along the x' axis is:

\[ \sum t'_{x'} = t'_c + t'_u = \frac{L}{c} + \frac{L}{c} = \frac{2L}{c} \]  

In the traditional analysis for the observer O' the sum of times of the track of both rays of light along the y' axis is:

\[ \sum t'_{y'} = t'_+ + t'_- = \frac{L}{c} + \frac{L}{c} = \frac{2L}{c} \]  

As we have \( \sum t'_{x'} = \sum t'_{y'} = \frac{2L}{c} \) there is no interference fringe and it is applied the null result of the Michelson & Morley experience.

In this traditional analysis the identical track of the clockwise and counter clockwise rays of light in the equation 18.01 that originates the null result of the Michelson & Morley experience contradicts the Sagnac effect that is exactly the time difference existing between the track of the clockwise and counter clockwise rays of light.

Based on the Undulating Relativity we make a deeper analysis of the Michelson & Morley experience obtaining a result that complies completely with the Sagnac effect.

Observing that the equation 18.01 corresponds to the hypothesis C of the paragraph §9.

Observing that the equation 18.01 corresponds to the hypothesis C of the paragraph §9.
Applying 18.01 in 9.19 we have:

\[
\begin{align*}
D' &= c t'_c + c t'_u = c t_c K_c + c t_u K_u \Rightarrow D' = L + L = c t_c K_c + c t_u K_u \quad \text{A} \\
D &= c t_c + c t_u = c t'_c K'_c + c t'_u K'_u \Rightarrow D = c t_c + c t_u = L K'_c + L K'_u = L (K'_c + K'_u) \quad \text{B}
\end{align*}
\]

From 18.04 A we have:

\[
D' = 2 L = c t_c \left( I - \frac{V_c}{c} \right) + c t_u \left( I + \frac{V_u}{c} \right) \Rightarrow D' = 2 L = c t_c - V_c t_c + c t_u + \frac{V_u}{c} t_u
\]

Where applying 9.26 we have:

\[
D' = 2 L = c t_c + c t_u \Rightarrow \sum t_x = t_c + t_u = \frac{2 L}{c}
\]

In 18.04 B we have:

\[
D = c t_c + c t_u = L \left[ \left( I + \frac{V'_c}{c} \right) + \left( I - \frac{V'_u}{c} \right) \right]
\]

Where applying 9.25 B we have:

\[
D = c t_c + c t_u = 2 L \Rightarrow \sum t_x = t_c + t_u = \frac{2 L}{c}
\]

The equations 18.06 and 18.08 demonstrate that the Doppler effect in the clockwise and counter clockwise rays of light compensate itself in the referential of the observer O resulting in:

\[
\sum t'_y = \sum t'_x = \sum t_x = \frac{2 L}{c}
\]

Because of this, according to the Undulating Relativity in the Michelson & Morley experience we can predict that the clockwise ray of light has a different track from the counter clockwise ray of light according to the formula 18.08 obtaining also the null result for the experience and matching then with the Sagnac effect. This supposition cannot be made based on the Einstein’s Special Relativity because according to 17.26 we have:

\[
\sum t'_x = \sum t_x
\]
§19 Regression of the perihelion of Mercury of 7,13"

Let us imagine the Sun located in the focus of an ellipse that coincides with the origin of a system of coordinates \((x,y,z)\) with no movement in relation to denominated fixed stars and that the planet Mercury is in a movement governed by the force of gravitational attraction with the Sun describing an elliptic orbit in the plan \((x,y)\) according to the laws of Kepler and the formula of the Newton's gravitational attraction law:

\[
\ddot{F} = \frac{GM_o m_o}{r^2} \left( \frac{1.9810^{30}}{1.6710^{-11}} \right) \left( \frac{3.2810^{23}}{r^2} \right) = -k \frac{\dot{r}}{r^2} \tag{19.01}
\]

The sub index "o" indicating mass in relative rest to the observer.

To describe the movement we will use the known formulas:

\[
\ddot{r} = r \ddot{\phi} - \dot{r} \frac{\dot{\phi}}{r}
\]

\[
\dddot{r} = \ddot{r} \frac{\dot{r}}{r} + r \dddot{\phi} - \dot{r} \dddot{\phi} + \dot{\phi} \frac{r}{c^2} \dddot{u}
\]

The formula of the relativity force is given by:

\[
\dddot{F} = \frac{d}{dt} \left( \frac{m_o \ddot{u}}{\sqrt{1 - \frac{u^2}{c^2}}} \right) = m_o \dddot{u} = \frac{m_o u \dddot{u}}{c^2} \left[ \left( 1 - \frac{u^2}{c^2} \right) \dddot{u} + \left( \frac{u \dddot{u}}{c^2} \right) \dddot{u} \right] \tag{19.06}
\]

In this the first term corresponds to the variation of the mass with the speed and the second as we will see later in 19.22 corresponds to the variation of the energy with the time.

With this and the previous formulas we obtain:

\[
\dddot{F} = \frac{m_o}{\left( 1 - \frac{u^2}{c^2} \right)^{3/2}} \left[ \left( 1 - \frac{u^2}{c^2} \right) \dddot{u} + \left( \frac{u \dddot{u}}{c^2} \right) \dddot{u} \right] \tag{19.07}
\]

\[
\dddot{F} = \frac{m_o}{\left( 1 - \frac{u^2}{c^2} \right)^{3/2}} \left[ \left( 1 - \frac{u^2}{c^2} \right) \dddot{u} + \left( \frac{u \dddot{u}}{c^2} \right) \dddot{u} \right] \tag{19.08}
\]
In this we have the transverse and radial component given by:

\[ \vec{F}_r = \frac{m_o}{(1-u^2/c^2)^{3/2}} \left( \begin{array}{c} \frac{d^2r}{dt^2} - r \frac{d\phi}{dt} \\ \frac{d^2r}{dt^2} - r \frac{d\phi}{dt} \end{array} \right) \left( \begin{array}{c} \frac{d\phi}{dt} \\ \frac{d\phi}{dt} \end{array} \right) + \frac{d\phi}{dt} \left( \begin{array}{c} \frac{d^2r}{dt^2} - r \frac{d\phi}{dt} \\ \frac{d^2r}{dt^2} - r \frac{d\phi}{dt} \end{array} \right) + \frac{d\phi}{dt} \left( \begin{array}{c} \frac{d^2r}{dt^2} - r \frac{d\phi}{dt} \\ \frac{d^2r}{dt^2} - r \frac{d\phi}{dt} \end{array} \right) c^2 dt \right) \hat{\mathbf{r}} \]

\[ \vec{F}_\phi = \frac{m_o}{(1-u^2/c^2)^{3/2}} \left( \begin{array}{c} \frac{d^2r}{dt^2} - r \frac{d\phi}{dt} \\ \frac{d^2r}{dt^2} - r \frac{d\phi}{dt} \end{array} \right) \left( \begin{array}{c} \frac{d\phi}{dt} \\ \frac{d\phi}{dt} \end{array} \right) + \frac{d\phi}{dt} \left( \begin{array}{c} \frac{d^2r}{dt^2} - r \frac{d\phi}{dt} \\ \frac{d^2r}{dt^2} - r \frac{d\phi}{dt} \end{array} \right) \frac{r d\phi}{dt} \hat{\mathbf{\phi}} \]

As the gravitational force is central we should have to null the traverse component \( \vec{F}_r = \text{zero} \) so we have:

\[ \vec{F}_\phi = \frac{m_o}{(1-u^2/c^2)^{3/2}} \left( \begin{array}{c} \frac{d^2r}{dt^2} - r \frac{d\phi}{dt} \\ \frac{d^2r}{dt^2} - r \frac{d\phi}{dt} \end{array} \right) \left( \begin{array}{c} \frac{d\phi}{dt} \\ \frac{d\phi}{dt} \end{array} \right) + \frac{d\phi}{dt} \left( \begin{array}{c} \frac{d^2r}{dt^2} - r \frac{d\phi}{dt} \\ \frac{d^2r}{dt^2} - r \frac{d\phi}{dt} \end{array} \right) \frac{r d\phi}{dt} \hat{\mathbf{\phi}} = \text{zero} \]

From where we have:

\[
\begin{align*}
\left( \frac{2 \frac{dr}{dt} + r \frac{d^2\phi}{dt^2}}{c^2} \right) & = -r \frac{dr}{dt} \\
\left( \frac{d^2r}{dt^2} - r \frac{d\phi}{dt} \right) & = \left[ \frac{1}{c^2} \frac{dr}{dt} \right]
\end{align*}
\]

From the radial component \( \vec{F}_r \) we have:

\[ \vec{F}_r = \frac{m_o}{(1-u^2/c^2)^{3/2}} \left( \begin{array}{c} \frac{d^2r}{dt^2} - r \frac{d\phi}{dt} \\ \frac{d^2r}{dt^2} - r \frac{d\phi}{dt} \end{array} \right) \left( \begin{array}{c} \frac{d\phi}{dt} \\ \frac{d\phi}{dt} \end{array} \right) + \frac{d\phi}{dt} \left( \begin{array}{c} \frac{d^2r}{dt^2} - r \frac{d\phi}{dt} \\ \frac{d^2r}{dt^2} - r \frac{d\phi}{dt} \end{array} \right) \frac{r d\phi}{dt} \hat{\mathbf{r}} \]

That applying 19.12 we have:

\[ \vec{F}_r = \frac{m_o}{(1-u^2/c^2)^{3/2}} \left( \begin{array}{c} \frac{d^2r}{dt^2} - r \frac{d\phi}{dt} \\ \frac{d^2r}{dt^2} - r \frac{d\phi}{dt} \end{array} \right) \left( \begin{array}{c} \frac{d\phi}{dt} \\ \frac{d\phi}{dt} \end{array} \right) + \frac{d\phi}{dt} \left( \begin{array}{c} \frac{d^2r}{dt^2} - r \frac{d\phi}{dt} \\ \frac{d^2r}{dt^2} - r \frac{d\phi}{dt} \end{array} \right) \frac{r d\phi}{dt} \hat{\mathbf{r}} \]

That simplifying results in:

\[ \vec{F}_r = \frac{m_o}{\sqrt{\left( \frac{d^2r}{dt^2} - r \frac{d\phi}{dt} \right)^2} \left( \begin{array}{c} \frac{d\phi}{dt} \\ \frac{d\phi}{dt} \end{array} \right) + \frac{d\phi}{dt} \left( \begin{array}{c} \frac{d^2r}{dt^2} - r \frac{d\phi}{dt} \\ \frac{d^2r}{dt^2} - r \frac{d\phi}{dt} \end{array} \right) \frac{r d\phi}{dt} \hat{\mathbf{r}} \]

This equaled to Newton's gravitational force results in the relativistic gravitational force:

\[ \vec{F}_r = \frac{m_o}{\sqrt{\left( \frac{d^2r}{dt^2} - r \frac{d\phi}{dt} \right)^2} \left( \begin{array}{c} \frac{d\phi}{dt} \\ \frac{d\phi}{dt} \end{array} \right) + \frac{d\phi}{dt} \left( \begin{array}{c} \frac{d^2r}{dt^2} - r \frac{d\phi}{dt} \\ \frac{d^2r}{dt^2} - r \frac{d\phi}{dt} \end{array} \right) \frac{r d\phi}{dt} \hat{\mathbf{r}} = -GM_o m_o \frac{\mathbf{r}}{r^3} = -k \frac{\mathbf{r}}{r^2} \hat{\mathbf{r}} \]

\[ \hat{\mathbf{r}} = \frac{GM_o m_o}{r^3} \hat{\mathbf{r}} = -k \frac{\mathbf{r}}{r^2} \hat{\mathbf{r}} \]
As the gravitational force is central it should assist the theory of conservation of the energy (E) that is written as:

\[ E = E_k + E_p = \text{constant.} \quad 19.17 \]

Where the kinetic energy \( E_k \) is given by:

\[ E_k = mc^2 - m_v c^2 = m_v c^2 \left( \frac{1}{\sqrt{1 - \frac{u^2}{c^2}}} - 1 \right) \quad 19.18 \]

And the potential energy \( E_p \) gravitational by:

\[ E_p = \frac{-GMm_v}{r} = \frac{-k}{r} \quad 19.19 \]

Resulting in:

\[ E = m_v c^2 \left[ \frac{1}{\sqrt{1 - \frac{u^2}{c^2}}} - 1 \right] \cdot \frac{k}{r} = \text{Constant.} \quad 19.20 \]

As the total energy (E) it is constant we should have:

\[ \frac{dE}{dt} = \frac{dE_k}{dt} + \frac{dE_p}{dt} = \text{zero.} \quad 19.21 \]

Then we have:

\[ \frac{dE_k}{dt} = \frac{m_v u}{\left( 1 - \frac{u^2}{c^2} \right)^{\frac{3}{2}}} \frac{du}{dt} \quad 19.22 \]

\[ \frac{dE_p}{dt} = \frac{k}{r^2} \frac{dr}{dt} \quad 19.23 \]

Resulting in:

\[ \frac{dE}{dt} = \frac{dE_k}{dt} + \frac{dE_p}{dt} = \text{zero} \Rightarrow \frac{m_v u}{\left( 1 - \frac{u^2}{c^2} \right)^{\frac{3}{2}}} \frac{du}{dt} + \frac{k}{r^2} \frac{dr}{dt} = \text{zero} \Rightarrow \frac{m_v u}{\left( 1 - \frac{u^2}{c^2} \right)^{\frac{3}{2}}} \frac{du}{dt} = -\frac{k}{r^2} \frac{dr}{dt} \]

\[ \frac{dE}{dt} = \frac{dE_k}{dt} + \frac{dE_p}{dt} = \text{zero} \Rightarrow \frac{m_v u}{\left( 1 - \frac{u^2}{c^2} \right)^{\frac{3}{2}}} \frac{du}{dt} = -\frac{k}{r^2} \frac{dr}{dt} \]

This applied in the relativistic force 19.06 and equaled to the gravitational force 19.01 results in:

\[ \vec{F} = \frac{m_v}{\left( 1 - \frac{u^2}{c^2} \right)^{\frac{3}{2}}} \frac{\ddot{u}}{r^2} = \frac{-k}{r^2} \frac{\ddot{r}}{c^2} \quad 19.25 \]

In this substituting the previous variables we get:
\[
\vec{F} = \frac{m_0}{\sqrt{1-u^2/c^2}} \left[ \left( \frac{d^2r}{dt^2} - \frac{r}{c^2} \left( \frac{d\phi}{dt} \right)^2 \right) + \left( 2 \frac{d\rho}{dt} \frac{dr}{dt} + r \frac{d^2\phi}{dt^2} \right) - \frac{l}{c^2 r^2} \frac{dr}{dt} \left( \frac{dr}{dt} + \frac{d\phi}{dt} \right) \right] = -\frac{k}{r^2} \hat{r}
\]

From this we obtain the radial component \( \vec{F}_r \) equals to:

\[
\vec{F}_r = \frac{m_0}{\sqrt{1-u^2/c^2}} \left( \frac{d^2r}{dt^2} - \frac{r}{c^2} \left( \frac{d\phi}{dt} \right)^2 \right) - \frac{l}{c^2 r^2} \frac{dr}{dt} = -\frac{k}{r^2}
\]

That easily becomes the relativistic gravitational force 19.16.

From 19.26 we obtain the traverse component \( \vec{F}_\phi \) equals to:

\[
\vec{F}_\phi = \frac{m_0}{\sqrt{1-u^2/c^2}} \left( 2 \frac{d\rho}{dt} \frac{dr}{dt} + r \frac{d^2\phi}{dt^2} \right) - \frac{l}{c^2 r^2} \frac{dr}{dt} = 0
\]

From this last one we have:

\[
\frac{2r \frac{dr}{dt} + r \frac{d^2\phi}{dt^2}}{r^2 \frac{d\phi}{dt}} = \frac{1}{m_0 c^2 r^2} \frac{dr}{dt} \sqrt{1 - u^2/c^2}
\]

As the gravitational force is central it should also assist the theory of conservation of the angular moment that is written as:

\[
\vec{L} = \vec{r} \times \vec{p} = \text{constant.}
\]

\[
\vec{L} = \vec{r} \times \vec{p} = \vec{r} \times \frac{m_0 \vec{u}}{1 - u^2/c^2} = \vec{r} \times \frac{m_0 \left( \frac{dr}{dt} + \frac{d\phi}{dt} \right)}{1 - u^2/c^2} = \frac{m_0}{1 - u^2/c^2} \frac{r^2}{dt} \frac{d\phi}{dt} \hat{\epsilon}
\]

\[
\vec{L} = \frac{m_0}{1 - u^2/c^2} \frac{r^2}{dt} \frac{d\phi}{dt} \hat{k} = L \hat{k} = \text{constant.}
\]

\[
\frac{d\vec{L}}{dt} = \frac{d\left( L \hat{k} \right)}{dt} = \frac{dL}{dt} \hat{k} + \vec{L} \frac{d\hat{k}}{dt} = \frac{dL}{dt} \hat{k} = 0 \Rightarrow \frac{d\left( L \hat{k} \right)}{dt} = 0
\]

Resulting in \( \vec{L} \) that is constant.

In 19.33 we had \( \frac{d\hat{k}}{dt} = 0 \) because the movement is in the plane \( (x,y) \).
Deriving L we find:
\[
\frac{dL}{dt} = \frac{d}{dt} \left( \frac{m_v}{\sqrt{1-u^2/c^2}} \right) = \frac{1}{c^2} \left( \int \frac{m_u du}{t} + \frac{m_v}{2} \left( \frac{2r}{dt} \frac{dr}{dt} + r^2 \frac{d^2\phi}{dt^2} \right) \right) = 0
\]

From that we have:
\[
\left( \frac{2r}{dt} \frac{dr}{dt} + r^2 \frac{d^2\phi}{dt^2} \right) = -u \frac{du}{dt} \frac{l}{c^2}
\]

Equaling 19.12 originating from the theory of the central force with 19.29 originating from the theory of conservation of the energy and 19.35 originating from the theory of conservation of the angular moment we have:
\[
\left( \frac{2r}{dt} \frac{dr}{dt} + r^2 \frac{d^2\phi}{dt^2} \right) = -u \frac{du}{dt} \frac{l}{c^2}
\]

From the last two equality we obtain 19.24 and from the two of the middle we obtain 19.16.

For solution of the differential equations we will use the same method used in the Newton's theory.

Let us assume \( w = \frac{l}{r} \)

The differential total of this is \( dw = \frac{\partial w}{\partial r} dr \Rightarrow dw = -\frac{l}{r^2} dr \)

From where we have \( dw = -\frac{l}{r^2} \frac{dr}{dt} \frac{d\phi}{dt} \Rightarrow dw = -\frac{l}{r^2} \frac{dr}{dt} \frac{d\phi}{dt} \)

From the module of the angular moment we have \( \frac{d\phi}{dt} = \frac{L}{m_r^2} \frac{1}{c^2} \frac{l}{u^2} \)

From where we have \( \frac{dr}{dt} = \frac{L}{m_r^2} \frac{dr}{d\phi} \frac{1}{c^2} \frac{l}{u^2} \)

Where applying 19.39 we have \( \frac{dr}{dt} = \frac{L}{m_r^2} \frac{dr}{d\phi} \frac{1}{c^2} \frac{l}{u^2} \)

That derived supplies \( \frac{d^2r}{dt^2} = \frac{d^2\phi}{dt^2} \frac{1}{c^2} \frac{l}{u^2} \)
Where applying 19.40 and deriving we have:

\[
\frac{d^2 r}{dt^2} = \frac{L}{m_r r^3} \sqrt{1 - \frac{u^2}{c^2}} \left[ \frac{d}{dt} \left( \frac{L \, dw}{ \phi} \right) \sqrt{1 - \frac{u^2}{c^2}} + \frac{d}{dt} \left( \frac{dw}{d\phi} \right) \sqrt{1 - \frac{u^2}{c^2}} \right]
\]

19.44

In this with 19.36 the radical derived is obtained this way:

\[
\frac{d}{dt} \left( \frac{L \, dw}{ \phi} \right) = \frac{-L \, u \, du}{\sqrt{1 - \frac{u^2}{c^2}}} \frac{dr}{dt} \left( 1 - \frac{u^2}{c^2} \right) = -k \frac{dw}{d\phi} \left( 1 - \frac{u^2}{c^2} \right)
\]

19.45

\[
\frac{d}{d\phi} \left( \frac{L \, dw}{ \phi} \right) = -L \frac{u \, du}{\sqrt{1 - \frac{u^2}{c^2}}} \frac{d\phi}{r^2} = -k \frac{dw}{d\phi} \frac{dr}{dt} \left( 1 - \frac{u^2}{c^2} \right)
\]

19.46

That applied in 19.44 supplies:

\[
\frac{d^2 r}{dt^2} = -\frac{L^2}{m_r r^3} \sqrt{1 - \frac{u^2}{c^2}} \left[ \frac{d^2 w}{d\phi^2} \sqrt{1 - \frac{u^2}{c^2}} + \frac{d}{d\phi} \left( \frac{dw}{d\phi} \right) \sqrt{1 - \frac{u^2}{c^2}} \right]
\]

19.47

Simplified results:

\[
\frac{d^2 r}{dt^2} = \frac{L^2 k}{m_r c^2 r^3} \left( 1 - \frac{u^2}{c^2} \right)^{3/2} \left( \frac{dw}{d\phi} \right)^2 - \frac{L^2}{m_r c^2 r^3} \left( 1 - \frac{u^2}{c^2} \right) \frac{d^2 w}{d\phi^2}
\]

19.48

Let us find the second derived of the angle deriving 19.40:

\[
\frac{d^2 \phi}{dt^2} = \frac{d}{dt} \left( \frac{L}{m_r r^3} \sqrt{1 - \frac{u^2}{c^2}} \right) = -\frac{2L \, dr}{m_r r^3} \frac{d}{dt} \left( 1 - \frac{u^2}{c^2} \right) + \frac{L \, d}{dt} \left( 1 - \frac{u^2}{c^2} \right)
\]

19.49

In this applying 19.42 and 19.45 and simplifying we have:

\[
\frac{d^2 \phi}{dt^2} = \frac{2L^2}{m_r r^3} \frac{dw}{d\phi} \left( 1 - \frac{u^2}{c^2} \right) \frac{L^2 k}{m_r c^2 r^3} \frac{dw}{d\phi} \left( 1 - \frac{u^2}{c^2} \right)^{3/2}
\]

19.50

Applying in 19.04 the equations 19.40 and 19.42 and simplifying we have:

\[
u^2 = \frac{L^2}{m_r} \left( 1 - \frac{u^2}{c^2} \right) \left( \frac{dw}{d\phi} \right)^2 + \frac{1}{r^2}
\]

19.51

The equation of the relativistic gravitational force 19.16 remodeled is:

\[
\frac{d^2 r}{dt^2} \left( \frac{d\phi}{dt} \right)^2 = \sqrt{1 - \frac{u^2}{c^2}} \left[ \frac{d}{dt} \left( \frac{1}{c^2} \right) \right] \left[ \frac{d}{dt} \left( \frac{dr}{dt} \right) \right] = -\frac{k}{m_r r^2}
\]

19.52

In this applying the formulas above we have:

\[
\frac{L^2 k}{m_r c^2 r^3} \left( 1 - \frac{u^2}{c^2} \right)^{3/2} \left( \frac{dw}{d\phi} \right)^2 - \frac{L^2}{m_r c^2 r^3} \frac{d^2 w}{d\phi^2} \left( 1 - \frac{u^2}{c^2} \right) + \frac{L^2}{m_r c^2 r^3} \frac{d}{d\phi} \left( \frac{L}{m_r} \sqrt{1 - \frac{u^2}{c^2}} \right)^2 \left( 1 - \frac{u^2}{c^2} \right) \left( \frac{dw}{d\phi} \right)^2 - \frac{k}{m_r r^2}
\]
\[
\frac{L^2}{m^2 r^2} \left( L \frac{u^2}{c^2} \right) \left( \frac{d w}{d \phi} \right)^2 - L \frac{L^2}{m^2 r^2} \frac{1-u^2}{c^2} d \phi - \frac{L^2}{m^2 r^2} \frac{1-u^2}{c^2} = \left[ \frac{1}{c^2} \frac{L d w}{m \phi} \frac{1-u^2}{c^2} \right]^2 - \frac{k}{m r^2}
\]

\[
\frac{L^2}{m^2 r^2} \left( L \frac{u^2}{c^2} \right) \left( \frac{d w}{d \phi} \right)^2 - L \frac{L^2}{m^2 r^2} \frac{1-u^2}{c^2} d \phi - \frac{L^2}{m^2 r^2} \frac{1-u^2}{c^2} = -\frac{k}{m r^2} + \frac{L^2 k}{m^2 r^2} \left( L \frac{u^2}{c^2} \right) \left( \frac{d w}{d \phi} \right)^2
\]

\[
\frac{L^2}{m^2 r^2} \frac{1-u^2}{c^2} d \phi = \frac{1}{c^2} \frac{L d w}{m \phi} \frac{1-u^2}{c^2}
\]

\[
\frac{d^2 w}{d \phi^2} + \frac{1}{r} = \frac{m k}{L^2} \left( \frac{m_o}{r^2} \frac{d \phi}{d t} \right) \frac{1}{c^2} \frac{d \phi}{d t}
\]

\[
\frac{d^2 w}{d \phi^2} + \frac{1}{r} = \frac{m k}{L^2} \left( \frac{m_o}{r^2} \frac{d \phi}{d t} \right) \left( \frac{d \phi}{d t} \right)^2
\]

\[
\frac{d^2 w}{d \phi^2} + \frac{1}{r} = \frac{m k}{L^2} \frac{1}{r} \frac{1-u^2}{c^2}
\]

\[
\left( \frac{d^2 w}{d \phi^2} + \frac{1}{r} \right)^2 = \left[ \frac{k \sqrt{1-u^2}}{m r^2} \left( \frac{d \phi}{d t} \right)^2 \right]^2
\]

\[
\left( \frac{d^2 w}{d \phi^2} + \frac{1}{r} \right)^2 = \frac{k^2 \left( 1-u^2 \right)}{m r^2} \left( \frac{d \phi}{d t} \right)^4
\]

\[
\left( \frac{d^2 w}{d \phi^2} + \frac{1}{r} \right)^2 = \frac{k^2}{r} \frac{u^2}{c^2} \left( r \frac{d \phi}{d t} \right)^4 \left( \frac{d \phi}{d t} \right)^4
\]

\[
\left( \frac{d^2 w}{d \phi^2} + \frac{1}{r} \right)^2 = \frac{k^2}{r} \left( \frac{dr}{d t} \right)^2 \left( \frac{d \phi}{d t} \right)^4 + \frac{k^2}{r} \left( \frac{d \phi}{d t} \right)^2
\]
Where we have:

\[ \left( \frac{d^2 w}{d \phi^2} \right)^2 + 2 \frac{d^2 w}{d \phi^2} \frac{1}{r} \frac{d}{dt} \left( \frac{r}{r^2} \right) \frac{d^2}{dt^2} \frac{1}{r^2} \] 
\[ m_o^2 r^6 \left( \frac{d \phi}{dt} \right)^2 - \frac{k^2}{c^2} \left( \frac{d^2}{dt^2} \frac{1}{r^2} \right) \] 
\[ m_o^2 r^4 \left( \frac{d \phi}{dt} \right)^2 \] 
\[ m_o^2 r^2 \left( \frac{d \phi}{dt} \right)^2 \] 

In this we will consider constant the Newton's angular moment in the form:

\[ L = r^2 \frac{d \phi}{dt} \]

That it is really the known theoretical angular moment.

\[ \left( \frac{d^2 w}{d \phi^2} \right)^2 + 2 \frac{d^2 w}{d \phi^2} \frac{1}{r} \frac{d}{dt} \left( \frac{r}{r^2} \right) \frac{d^2}{dt^2} \frac{1}{r^2} \] 
\[ m_o^2 c^6 \frac{L^2}{L^2} - \frac{k^2}{m_o^2 c^6} \] 
\[ \frac{k^2}{m_o^2 c^6} \] 
\[ \frac{k^2}{m_o^2 c^6} \] 

\[ \left( \frac{d^2 w}{d \phi^2} \right)^2 + 2 \frac{d^2 w}{d \phi^2} \frac{1}{r} \frac{d}{dt} \left( \frac{r}{r^2} \right) \frac{d^2}{dt^2} \frac{1}{r^2} = B - A \left( \frac{d w}{d \phi} \right)^2 - A w^2 \]

\[ \left( \frac{d^2 w}{d \phi^2} \right)^2 + 2 \frac{d^2 w}{d \phi^2} \frac{1}{r} \frac{d}{dt} \left( \frac{r}{r^2} \right) \frac{d^2}{dt^2} \frac{1}{r^2} + A \left( \frac{d w}{d \phi} \right)^2 + (A + 1) w^2 = B = zero \]

Where we have:

\[ A = \frac{k^2}{m_o^2 c^6} \]
\[ B = \frac{k^2}{m_o^2 L^2} \]
The equation 19.54 has as solution:
\[
w=\frac{1}{\xi D}[I-\epsilon \cos(\phi \sqrt{I+A}+\phi_0)] \Rightarrow w=\frac{1}{\xi D}[I-\epsilon \cos(\phi Q)]
\]
19.57

Where we consider \(\phi_0=\text{zero}\).

It is denominated in 19.57 \(Q^2=I+A\).

The equation 19.58 is function only of \(A\) demonstrating the intrinsic union between the variation of the mass with the variation of the energy in the time, because both as already described, participate in the relativistic force 19.06 in this relies the essential difference between the mass and the electric charge that is invariable and indivisible in the electromagnetic theory.

From 19.57 we obtain the ray of a conical:
\[
r=\frac{\xi D}{w} \frac{I-\epsilon \cos(\phi \sqrt{I+A})}{I-\epsilon \cos(\phi Q)} \Rightarrow r=\frac{\xi D}{I-\epsilon \cos(\phi Q)}
\]
19.59

Where \(\epsilon\) is the eccentricity and \(D\) the director distance of the focus.

Deriving 19.57 we have
\[
\frac{dw}{d\phi}=\frac{Q \sin(\phi Q)}{D}
\]
19.60

That derived results in
\[
\frac{d^2 w}{d\phi^2}=\frac{Q^2 \cos(\phi Q)}{D}
\]
19.61

Applying in 19.54 the variables we have:
\[
\left(\frac{d^2 w}{d\phi^2}\right)^2 + 2 \frac{d^2 w}{d\phi^2} w + A \left(\frac{dw}{d\phi}\right)^2 + (A+1)w^2 - B = \text{zero}.
\]
19.62

\[
\frac{Q^4 \cos^2(\phi Q)}{D^2} + 2 \frac{Q^2 \cos(\phi Q)}{D} \left(1-\epsilon \cos(\phi Q)\right) + A \frac{Q^2 \cos^2(\phi Q)}{D^2} + \frac{(A+1) \left(1-\epsilon \cos(\phi Q)\right)^2}{\xi D} - B = \text{zero}
\]
19.63

In this applying in the first parenthesis \(Q^2=I+A\) we have:
\[
\left(Q^4 - 2Q^2 - AQ^2 + A + I\right) - \left[(I+A)^2 - 2(I+A) - A(I+A) + A + I\right] = (I+2A + A^2 - 2A - A^2 + A + I) = \text{zero}
\]
In 19.63 applying in the second parenthesis $Q^2=1+A$ we have:

$$\left(\frac{2Q^2}{\varepsilon D^2} - \frac{2A}{\varepsilon D} - \frac{2}{2} \right) \left[ \frac{2(1+A)}{\varepsilon D} - \frac{2A}{\varepsilon D} - \frac{2}{2} \right] = 0$$

The rest of the equation 19.63 is therefore:

$$\frac{AQ^2}{D^2} + \frac{(A+1)}{\varepsilon^2 D^2} - B = 0$$  \hspace{1cm} 19.64

The data of the elliptic orbit of the planet Mercury is [1]:

Eccentricity of the orbit $\varepsilon=0.206$.

Larger semi-axis $a = 5.79 \cdot 10^{10}$ m.

Smaller semi-axis $b = a\sqrt{1-\varepsilon^2} = 5.79 \cdot 10^{10} \sqrt{1-0.206^2} = 56.658.160.305.80$ m.

$$\varepsilon D = a(1-\varepsilon^2) = 5.79 \cdot 10^{10}(1-0.206^2) = 55.442.955.600.00$$ m.

$$D = \frac{a(1-\varepsilon^2)}{\varepsilon} = \frac{5.79 \cdot 10^{10}(1-0.206^2)}{0.206} = 269.140.561.165.00$$ m.

The orbital period of the Earth ($P_T$) and Mercury ($P_M$) around the Sun in seconds are:

$$P_T = 3.16 \cdot 10^7 \text{ s.}$$

$$P_M = 7.60 \cdot 10^6 \text{ s.}$$

The number of turns that Mercury ($m_o$) makes around the Sun ($M_o$) in one century is, therefore:

$$N = 100 \frac{3.16 \cdot 10^7}{7.60 \cdot 10^6} = 415.79.$$  \hspace{1cm} 19.65

Theoretical angular moment of Mercury:

$$L^2 = r^2 \left( \frac{d\phi}{dt} \right)^2 = GM_o a(1-\varepsilon^2) = 6.67 \cdot 10^{-11} \cdot 198.10 \cdot 5.79 \cdot 10^{10}(1-0.206^2) = 7,322,129,374,27.10^{30}$$

$$A = \frac{(GM_o m_o)^2}{m_o^2 \varepsilon^2 L^2} = \frac{(GM_o)^2}{\varepsilon^2 L^2} = \frac{\left(6.67 \cdot 10^{-11}\right)^2 \left(198.10^{30}\right)^2}{\left(3.0 \cdot 10^8\right)^2 \left(7.32 \cdot 10^{30}\right)} = 2,65.10^{-8}.$$  \hspace{1cm} 19.66

$$B = \frac{(GM_o m_o)^2}{m_o^2 L^2} = \frac{(GM_o)^2}{L^2} = \frac{\left(6.67 \cdot 10^{-11}\right)^2 \left(198.10^{30}\right)^2}{\left(7.32 \cdot 10^{30}\right)^2} = 3,25.10^{-22}.$$  \hspace{1cm} 19.67

$$Q = \sqrt{1+A} = \sqrt{1+2.65.10^{-8}} = 1,000.000.013.23.$$  \hspace{1cm} 19.68

Applying the numeric data with several decimal numbers to the rest of the equation 19.63 we have:

$$\frac{AQ^2}{D^2} + \frac{(A+1)}{\varepsilon^2 D^2} - B = \frac{2.65.10^{-8}(1,000,000,013.23)^2}{(269.140.561.165.00)^2} + \frac{2.65.10^{-8} + 1}{(55.442.955.600.00)^2} = 3,25.10^{-22} = 8,976.10^{-30}$$  \hspace{1cm} 19.70

Result that we can consider null.
We will obtain the relativistic angular moment of the rest of the equation 19.63 in this applying the variables we have:

\[
\frac{AQ^2}{D^2} + \frac{(A+1)}{e^2 D^2} - B \frac{(GM_o)^2}{c^2 L^2 D^2} \left[ l + \frac{(GM_o)^2}{c^2 L^2} \right] + \frac{l}{e^2 D^2} \left[ l + \frac{(GM_o)^2}{c^2 L^2} \right] - \frac{(GM_o)^2}{L^2} = 0
\]

19.71

\[
e^2 L^2 (GM_o)^2 \left[ l + \frac{(GM_o)^2}{c^2 L^2} \right] + L^2 c^2 \left[ l + \frac{(GM_o)^2}{c^2 L^2} \right] - c^2 e^2 D^2 (GM_o)^2 = 0
\]

\[
e^2 L^2 (GM_o)^2 + e^2 L^2 (GM_o)^2 \left[ l + \frac{(GM_o)^2}{c^2 L^2} \right] + L^2 c^2 + L^2 c^2 \left( GM_o \right)^2 - c^2 e^2 D^2 (GM_o)^2 = 0
\]

\[
e^2 L^2 (GM_o)^2 + e^2 L^2 (GM_o)^2 \left[ l + \frac{(GM_o)^2}{c^2 L^2} \right] + L^2 c^2 + L^2 c^2 (GM_o)^2 - c^2 e^2 D^2 (GM_o)^2 = 0
\]

\[
c^2 L^2 + (l + e^2) (GM_o)^2 L^2 + e^2 \left( GM_o \right)^2 - c^2 e^2 D^2 (GM_o)^2 = 0
\]

19.72

\[
L^2 = \frac{-(l + e^2) (GM_o)^2 \pm \sqrt{(l + e^2)^2 (GM_o)^2} - 4e^2 \left( GM_o \right)^2 - c^2 e^2 D^2 (GM_o)^2}{2c^2}
\]

This last equation has the exclusive property of relating the speed \( c \) to the denominated relativistic angular moment that is smaller than the theoretical angular moment 19.66.

The variation of the relativistic angular moment in relation to the theoretical angular moment is very small and given by:

\[
\Delta L = \frac{7.32212927328.10^{30} - 7.3221293742.7.10^{30}}{7.3221293742.7.10^{30}} = -1.38.10^{-8} = \frac{-1}{72.503.509.00}
\]

19.74

That demonstrates the accuracy of the principle of constancy of the speed of the light.
In reality, the equation 19.06 provides a secular retrocession perihelion of Mercury, which is given by in

\[ \Delta \phi = 2\pi 415.79 \left( \frac{1}{Q} - 1 \right) = 2\pi 415.79 (-0.000.000.013.23) = -3.46.10^{-5} \text{rad}. \]  

Converting for the second we have:

\[ \Delta \phi = -3.46.10^{-5} \frac{180.000.3.600.00}{\pi} = -7.13^\circ. \]

This retrocession, is not expected in Newtonian theory is due to relativistic variation of mass and energy and is shrouded in total observed precession of 5599.
§§19 Advance of Mercury's perihelion of 42.79"

If we write the equation for the gravitational relativity energy $E_R$ covering the terms for the kinetic energy, the potential energy $E_p$ and the resting energy:

$$ E_R = m_o c^2 \left( \frac{1}{\sqrt{1 - \frac{u^2}{c^2}}} - 1 \right) + E_p + m_o c^2 = \frac{m_o c^2}{\sqrt{1 - \frac{u^2}{c^2}}} + E_p. \quad 19.77 $$

Being the conservative the gravitational force its energy is constant. Assuming then that in 19.77 when the radius tends to infinite, the speed and potential energy tends to zero, resulting then:

$$ E_R = \frac{m_o c^2}{\sqrt{1 - \frac{u^2}{c^2}}} + E_p = m_o c^2 \quad 19.78 $$

Writing the equation to the Newton's gravitation energy $E_N$ having the correspondent Newton's terms to the 19.77:

$$ E_N = \frac{m_o u^2}{2} - \frac{k}{r} + m_o c^2 = m_o c^2 \quad 19.79 $$

Where $\frac{m_o u^2}{2}$ is the kinetic energy, $-\frac{k}{r}$ the potential energy and $m_o c^2$ the resting energy or better saying the inertial energy.

From this 19.79 we have:

$$ \frac{m_o u^2}{2} - \frac{k}{r} + m_o c^2 \Rightarrow \frac{m_o u^2}{2} = \frac{k}{r} \Rightarrow u^2 = \frac{2k}{m_o r} \Rightarrow \frac{2Gm_o m_o}{m_o r} \Rightarrow u^2 = \frac{2Gm_o}{r} \quad 19.80 $$

Deriving 19.79 we have:

$$ \frac{dE_R}{dt} = \frac{\partial}{\partial t} \left( \frac{m_o u^2}{2} - \frac{k}{r} + m_o c^2 \right) = 0 $$

$$ \frac{m_o 2u du}{2 \ dt} + \frac{k \ dr}{r^2 \ dt} = 0 $$

$$ \frac{du}{dt} = -\frac{k \ dr}{m_o r^2 \ dt} = \frac{-Gm_o \ dr}{r^2 \ dt} $$

$$ \frac{du}{dr} = \frac{-Gm_o}{r^2} \ dt \quad 19.81 $$
Making the relativity energy $19.78$ equal to the Newton’s energy $19.79$ we have:

$$E_R = E_N \Rightarrow \frac{m_o c^2}{\sqrt{1 - \frac{u^2}{c^2}}} + E_p = \frac{m_o u^2}{2} - \frac{k}{r} + m_o c^2 \quad 19.82$$

$$ \frac{m_o c^2}{m_o \sqrt{1 - \frac{u^2}{c^2}}} + E_p = \frac{m_o u^2}{m_o 2} - \frac{GM_o m_o}{m_o r} + \frac{m_o c^2}{m_o} \quad 19.83$$

In that denoting the relativity potential ($\varphi$) as:

$$\varphi = \frac{E_p}{m_o} \quad 19.84$$

We have:

$$\frac{c^2}{\sqrt{1 - \frac{u^2}{c^2}}} + \varphi = \frac{u^2}{2} - \frac{GM_o}{r} + c^2 \quad 19.85$$

In this one replacing the approximation:

$$\frac{1}{\sqrt{1 - \frac{u^2}{c^2}}} \approx 1 + \frac{u^2}{2c^2} \quad 19.86$$

We have:

$$\varphi = \frac{u^2}{2} - \frac{GM_o}{r} + c^2 - c^2 \left(1 + \frac{u^2}{2c^2}\right)$$

That simplified results in the Newton’s potential:

$$\varphi = \frac{u^2}{2} - \frac{GM_o}{r} + c^2 - \frac{u^2}{2} - \frac{GM_o}{r} = - \frac{GM_o}{r} \quad 19.87$$

Replacing $19.84$ and the relativity potential $19.85$ in the relativity energy $19.78$:

$$E_R = \frac{m_o c^2}{\sqrt{1 - \frac{u^2}{c^2}}} + m_o \left\{ \frac{u^2}{2} - \frac{GM_o}{r} + c^2 - \frac{c^2}{\sqrt{1 - \frac{u^2}{c^2}}} \right\} \quad 19.88$$

We have the Newton’s energy $19.79$:

$$E_n = \frac{m_o u^2}{2} - \frac{GM_o m_o}{r} + m_o c^2$$
Deriving the relativity potential 19.85 we have the relativity gravitational acceleration modulus exactly as in the Newton’s theory:

\[ a = -\frac{d\varphi}{dr} \]

\[ a = \frac{-d\varphi}{dr} = \frac{-d}{dr}\left(\frac{u^2 - GM_o + c^2}{2 - \frac{GM_o}{r} + \frac{c^2}{\sqrt{1 - \frac{u^2}{c^2}}}}\right) \]

\[ a = -\frac{d}{dr}\left(\frac{u^2}{2} - \frac{GM_o}{r} + c^2\right) \cdot \frac{1}{\frac{2}{c^2} - \frac{1}{\sqrt{1 - \frac{u^2}{c^2}}}} \]

Where we have:

\[ -\frac{d}{dr}\left(\frac{u^2}{2} - \frac{GM_o}{r} + c^2\right) = -\frac{d}{dr}\left(\frac{E_n}{m_o}\right) = \text{zero}. \] Because the term to be derived is the Newton’s energy divided by \( m_o \) that is \( E_n = \frac{u^2}{2} - \frac{GM_o}{r} + c^2 \) that is constant, resulting then in:

\[ a = -\frac{d}{dr}\left(\frac{-\frac{c^2}{\sqrt{1 - \frac{u^2}{c^2}}}}{}\right) \]

\[ a = -\frac{u}{\sqrt{1 - \frac{u^2}{c^2}}} \cdot \frac{du}{dr} \]

In this one applying 19.81 we have:

\[ a = \frac{-1}{1 - \frac{u^2}{c^2}} \cdot \frac{GM_o}{r^2} \]

19.89

The vector acceleration is given by 19.05:

\[ \ddot{a} = \left[ \frac{d^2}{dt^2} - \frac{d^2}{dt^2} \right] \frac{d\phi}{dt} \cdot \left[ \frac{2}{dt^2} \frac{d\phi}{dt} + r \frac{d^2\phi}{dt^2} \right] \]
The relativity gravitational acceleration modulus 19.89 is equal to the component of the vector radius (\( \hat{r} \)) thus we have:

\[
a = \left[ \frac{d^2r}{dt^2} - r \left( \frac{d\phi}{dt} \right)^2 \right] = -\frac{1}{\left( 1 - \frac{u^2}{c^2} \right)} \frac{GM_o}{r^2} \quad \text{19.90}
\]

Being null the transversal acceleration we have:

\[
\left[ 2 \frac{dr}{dt} \frac{d\phi}{dt} + r \frac{d^2\phi}{dt^2} \right] \hat{\phi} = \text{zero} \quad \text{19.91}
\]

\[
2 \frac{dr}{dt} \frac{d\phi}{dt} + r \frac{d^2\phi}{dt^2} = \text{zero}
\]

That is equal to the derivative of the constant angular momentum \( L = r^2 \frac{d\phi}{dt} \) \text{19.92}

\[
\frac{dL}{dt} = \frac{d}{dt} \left( r^2 \frac{d\phi}{dt} \right) = 2r \frac{dr}{dt} \frac{d\phi}{dt} + r^2 \frac{d^2\phi}{dt^2} = \text{zero} \quad \text{19.93}
\]

Rewriting some equations already described we have:

\[
w = \frac{1}{r}
\]

\[
\frac{dw}{dr} = \frac{\partial w}{\partial r} \Rightarrow dw = -\frac{1}{r^2} dr
\]

\[
\frac{dw}{d\phi} = -\frac{1}{r^2} \frac{dr}{d\phi} \text{ or } \frac{dr}{d\phi} = -r^2 \frac{dw}{d\phi} \text{ and } \frac{dw}{dt} = -\frac{1}{r^2} \frac{dr}{dt}
\]

\[
\frac{dr}{dt} = \frac{d\phi}{dt} \frac{dr}{d\phi} = \frac{L}{r^2} \frac{dr}{d\phi} = \frac{-L}{r^2} \frac{dw}{d\phi} \Rightarrow \frac{dr}{dt} = \frac{-L}{r^2} \frac{dw}{d\phi}
\]

\[
\frac{d^2r}{dt^2} = \frac{d}{dt} \left( \frac{dr}{dt} \right) = \frac{d\phi}{dt} \frac{d}{dt} \frac{dr}{d\phi} \left( -\frac{L}{r^2} \right) = \frac{L}{r^2} \frac{d}{d\phi} \left( \frac{-L}{r^2} \right) = -\frac{L^2}{r^2} \frac{d^2w}{d\phi^2} \quad \text{19.94}
\]

From 19.90 we have:

\[
\left( 1 - \frac{3u^2}{2c^2} \right) \frac{d^2r}{dt^2} - r \left( \frac{d\phi}{dt} \right)^2 = -\frac{GM_o}{r^2}
\]

In this one we 19.94 the speed of 19.80 and the angular momentum we have:

\[
\left[ 1 - \frac{3}{2c^2} \left( \frac{2GM_o}{r} \right) \right] \left[ -\frac{L^2}{r^2} \frac{d^2w}{d\phi^2} - r \left( \frac{L}{r^2} \right)^2 \right] = -\frac{GM_o}{r^2}
\]

\[
\left( 1 - \frac{3GM_o}{c^2} \frac{1}{r} \left( \frac{d^2w}{d\phi^2} + \frac{1}{r} \right) \right) = \frac{GM_o}{L^2}
\]
\[
(1 - \frac{3Gm_1}{c^2 r}) \frac{d^2 w}{d\phi^2} + (1 - \frac{3Gm_0}{c^2 r}) \frac{1}{r} \frac{1}{L^2} = \frac{GM_o}{L^2}
\]

\[
\frac{d^2 w}{d\phi^2} - \frac{3Gm_0}{c^2} \frac{1}{r} \frac{1}{L^2} + \frac{1}{c^2} \frac{1}{r^2} \frac{1}{L^2} = \text{zero}
\]

\[
\frac{d^2 w}{d\phi^2} - A \frac{d^2 w}{d\phi^2} r + \frac{1}{r} - \frac{A}{r^2} = \text{zero}
\]

\[
\frac{d^2 w}{d\phi^2} - A \frac{d^2 w}{d\phi^2} w + w - A w^2 = \text{zero}
\]

\[
\frac{d^2 w}{d\phi^2} - A \frac{d^2 w}{d\phi^2} w - A w^2 + w - B = \text{zero}
\]

Where we have:

\[
A = \frac{3Gm_0}{c^2}, \quad B = \frac{GM_o}{L^2}
\]

The solution to the differential equation 19.95 is:

\[
w = \frac{1}{\epsilon D} [1 - \cos(\phi_Q + \phi_o)] \Rightarrow w = \frac{1}{\epsilon D} [1 - \cos(\phi_Q)].
\]

Where we consider \( \phi_o = \text{zero} \)

Then the radius is given by:

\[
r = \frac{1}{w} = \frac{\epsilon D}{1 - \epsilon \cos(\phi_Q)Q} \Rightarrow r = \epsilon D \frac{1}{1 - \epsilon \cos(\phi_Q)Q}
\]

Where \( \epsilon \) is the eccentricity and \( D \) the focus distance to the director.

Deriving 19.97 we have

\[
\frac{dw}{d\phi} = \frac{Q \cos(\phi_Q)}{D} \quad \text{and} \quad \frac{d^2 w}{d\phi^2} = \frac{Q^2 \cos(\phi_Q)}{D}
\]

Applying the derivatives in 19.95 we have:

\[
\frac{d^2 w}{d\phi^2} - A \frac{d^2 w}{d\phi^2} w - A w^2 + w - B = \text{zero}
\]

\[
\frac{Q^2 \cos(\phi_Q)}{D} - A \frac{Q^2 \cos(\phi_Q)}{D} \frac{1}{\epsilon D} [1 - \epsilon \cos(\phi_Q)] - \frac{A}{\epsilon D^2} [1 - \epsilon \cos(\phi_Q)]^2 + \frac{1}{\epsilon D} [1 - \epsilon \cos(\phi_Q)] - B = \text{zero}
\]

\[
\frac{Q^2 \cos(\phi_Q)}{D} - A \frac{Q^2 \cos(\phi_Q)}{D} \frac{1}{\epsilon D^2} [1 - 2 \epsilon \cos(\phi_Q) + \epsilon^2 \cos^2(\phi_Q)] + \frac{1 - \epsilon \cos(\phi_Q)}{\epsilon D} - B = \text{zero}
\]

\[
\frac{Q^2 \cos(\phi_Q)}{D} - A \frac{Q^2 \cos(\phi_Q)}{D} + A \frac{Q^2 \cos(\phi_Q)}{D} \epsilon \cos(\phi_Q) - \frac{A}{\epsilon D^2} - \frac{A}{\epsilon D} \epsilon \cos(\phi_Q) - B = \text{zero}
\]

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\[
\cos(\phi) \left( \frac{Q^2}{D^2} - \frac{A Q^2}{A^2 D} + \frac{2 A}{A^2} - \frac{1}{A^2} \right) + \frac{A Q^2 \cos^2(\phi)}{D^2} - \frac{A \cos^2(\phi)}{D^2} = \frac{A}{A^2 D^2} + \frac{1}{A} = B = \text{zero}
\]

\[
\cos(\phi) \left( \frac{Q^2}{A D} - \frac{A Q^2}{A^2 D} + \frac{2 A}{A^2} - \frac{1}{A^2} \right) + \frac{A Q^2 \cos^2(\phi)}{A D^2} - \frac{A \cos^2(\phi)}{A D^2} = \frac{A}{A^2 D^2} + \frac{1}{A A D} - \frac{1}{A} = \text{zero}
\]

\[
\cos(\phi) \left( \frac{Q^2}{D} - \frac{Q^2}{A D} + \frac{2}{A D} - \frac{1}{A^2} \right) + \frac{Q^2 \cos^2(\phi)}{D^2} - \frac{Q^2}{D^2} = \frac{1}{A} \frac{1}{D} + \frac{1}{A^2 D} - \frac{B}{A} = \text{zero}
\]

\[
\cos^2(\phi) \left( \frac{Q^2}{D^2} - 1 \right) + \cos^2(\phi) \left( \frac{Q^2}{A} - \frac{Q^2}{A D} + \frac{2}{A D} - \frac{1}{A^2} \right) = \frac{1}{A} \frac{1}{D} + \frac{1}{A^2 D} - \frac{B}{A} = \text{zero}
\]

The coefficient of the squared co-cosine can be considered null because \( Q \approx 1 \) and \( D^2 \) is a very large number:

\[
\cos^2(\phi) \left( \frac{Q^2}{D^2} - 1 \right) = \text{zero}
\]

Resulting from the equation 19.100:

\[
\cos(\phi) \left( \frac{Q^2}{A} - \frac{Q^2}{A D} + \frac{2}{A D} - \frac{1}{A^2} \right) - \frac{1}{A} \frac{1}{D} - \frac{B}{A} = \text{zero}
\]

Due to the unicity of the equation 19.102 we must have the only solution that makes it null simultaneously the parenthesis and the rest of the equation, that is, we must have a unique solution for both the following equations:

\[
\frac{Q^2}{A} - \frac{Q^2}{A D} + \frac{2}{A D} - \frac{1}{A^2} = \text{zero} \quad \text{and} \quad \frac{1}{A} \frac{1}{D} + \frac{1}{A A D} - \frac{B}{A} = \text{zero}
\]

These equations can be written as:

\[
[a = b] \Rightarrow \frac{1}{A} \frac{1}{D} = \frac{1}{A} \left( \frac{1}{A D} - \frac{2}{A} \right)
\]

\[
[a = c] \Rightarrow \frac{1}{A} \frac{1}{D} = \frac{\epsilon DB}{A}
\]

In these ones the common term \( a = \frac{1}{A} - \frac{1}{A D} \) must have a single solution then we have:

\[
[b = c] \Rightarrow \frac{1}{A} \frac{1}{D} = \frac{\epsilon DB}{A}
\]

With 19.96 and the theoretical momentum we have:

\[
A = \frac{3GM_o}{c^2} \quad B = \frac{GM_o}{L^2} \quad L^2 = \epsilon DGM_o \quad \epsilon DB = \frac{\epsilon DGM_o}{L^2} = 1
\]
It is applied in 19.105 and 19.106 resulting in:

\[ \begin{align*}
[a = c] & \Rightarrow \frac{1}{A} - \frac{1}{\varepsilon D} = \frac{1}{A} \\
[b = c] & \Rightarrow \frac{1}{Q^2} \left( \frac{1}{A} - \frac{2}{\varepsilon D} \right) = \frac{1}{A}
\end{align*} \]

19.108

19.109

From 19.108 we have the mistake made in 19.105:

\[ \frac{1}{A} - \frac{1}{\varepsilon D} = \frac{1}{A} \Rightarrow \frac{1}{\varepsilon D} \approx \text{zero} \]

19.110

\[ -\frac{1}{\varepsilon D} = \frac{-1}{55.442.955.600.00} = -1.80 \cdot 10^{-11} \approx \text{zero} \]

19.111

From 19.109 we have \( Q \):

\[ \frac{1}{Q^2} \left( \frac{1}{A} - \frac{2}{\varepsilon D} \right) = \frac{1}{A} \Rightarrow Q^2 = 1 - \frac{2A}{\varepsilon D} \Rightarrow Q^2 = 1 - \frac{2 \cdot 3GM_c}{c^2} \]

19.112

It is applied in 19.104 resulting in 19.110:

\[ \frac{1}{A} - \frac{1}{\varepsilon D} = \frac{1}{Q^2} \left( \frac{1}{A} - \frac{2}{\varepsilon D} \right) \Rightarrow \frac{1}{A} - \frac{1}{\varepsilon D} = \frac{1}{A} \left( \frac{1}{Q^2} - 2 \frac{2A}{\varepsilon D} \right) \Rightarrow \frac{1}{A} - \frac{1}{\varepsilon D} = \frac{1}{A} \Rightarrow \frac{1}{\varepsilon D} \approx \text{zero} \]

19.113

From 19.112 we have:

\[ Q = \sqrt{1 - \frac{6GM_c}{\varepsilon Dc^2}} = \sqrt{1 - \frac{6(6,67 \cdot 10^{-11})(1,98 \cdot 10^{10})}{55.442.955.600.00(3.10^8)^2}} = 0,999.999.920.599 \]

That corresponds to the advance of Mercury's perihelion in one century of:

\[ \sum \Delta \phi = \Delta \phi \cdot 415,79 \left( \frac{1}{Q} \right) \cdot 1.296.000.00 \cdot 415,79 = 42,79" \]

19.114

Calculated in this way:

In one trigonometric turn we have \( 360 \times 60 \times 60 = 1.296.000.0000 \) seconds.

The angle \( \phi \) in seconds ran by the planet in one trigonometric turn is given by:

\[ \phi Q = 1.296.000.00 \Rightarrow \phi = \frac{1.296.000.00}{Q} \]

If \( Q > 1,00 \) we have a regression. \( \phi < 1.296.000.00 \).

If \( Q < 1,00 \) we have an advance. \( \phi > 1.296.000.00 \).
The angular variation in seconds in one turn is given by:

\[
\Delta \phi = \frac{1.296.000.000}{Q} - 1.296.000.00 = \left( \frac{1}{Q} - 1 \right) 1.296.000.000.
\]

If \( \Delta \phi < \text{zero} \) we have a regression.

If \( \Delta \phi > \text{zero} \) we have an advance.

In one century we have 415.79 turns that supply a total angular variation of:

\[
\sum \Delta \phi = \Delta \phi \cdot 415.79 = \left( \frac{1}{Q} - 1 \right) 1.296.000.000 \cdot 415.79 = 42.79''
\]

If \( \sum \Delta \phi < \text{zero} \) we have a regression.

If \( \sum \Delta \phi > \text{zero} \) we have an advance.

**§20 Inertia**

Imagine in an infinite universe totally empty, a point O' which is the beginning of the coordinates of the observer O'. In the cases of the observer O' being at rest or in uniform motion the law of inertia requires that the spherical electromagnetic waves with speed c issued by a source located at point O' is always observed by O', regardless of time, with spherical speed c and therefore the uniform motion and rest are indistinguishable from each other remain valid in both cases the law of inertia. To the observer O' the equations of electromagnetic theory describe the spread just like a spherical wave. The image of an object located in O' will always be centered on the object itself and a beam of light emitted from O' will always remain straight and perpendicular to the spherical waves.

Imagine another point O what will be the beginning of the coordinates of the observer which has the same properties as described for the inertial observer O'.

Obviously two imaginary points without any form of interaction between them remain individually and together perfectly meeting the law of inertia even though there is a uniform motion between them only detectable due to the presence of two observers who will be considered individually in rest, setting in motion the other referential.

The intrinsic properties of these two observers are described by the equations of relativistic transformations.

Note: the infinite universe is one in which any point can be considered the central point of this universe.

(§ 20 electronic translation)

**§20 Inertia (clarifications)**

Imagine in a totally empty infinite universe a single point O. Due to the uniqueness properties of O a radius of light emitted from O must propagate with velocity c. If this ray propagates in a straight line, then O is defined as the origin of an inertial frame because it is either at rest or in a uniform rectilinear motion. However, in the hypothesis of propagation of the light ray being a curve the movement of O must be interpreted as the origin of an accelerated frame. Therefore the propagation of a ray of light is sufficient to demonstrate whether O is the origin of an inertial frame or accelerated frame.

Now imagine if in the universe described above for the inertial reference frame O there is another inertial frame O' that does not have any kind of physical interaction with O. In the absence of any interaction between O and O' the uniqueness properties are inviolable for both points and rays of light emitted from O and O' have the same velocity c. It is impossible for the velocity of light emitted from O to be different from the velocity of light emitted from O' because each reference exists as if the other did not exist. Being O and O' the origin of inertial frames the propagation of light rays occurs in a straight line with velocity c and the relations between times t and t' of each frame are given by table I.
§21 Advance of Mercury's perihelion of 42.79" calculated with the Undulating Relativity

Assuming $u_x = v$

$$u_x = v$$

$$u'x' = \frac{u_x - v}{\sqrt{1 + \frac{v^2}{c^2} - 2vu_x}} = \frac{v - v}{\sqrt{1 + \frac{v^2}{c^2} - 2vv'}} \Rightarrow u'x' = \text{zero}$$

$$v = \frac{v'}{\sqrt{1 + \frac{v'^2}{c^2}}}$$

$$dt = dt' \sqrt{1 + \frac{v^2}{c^2} - 2vu_x} = dt' \sqrt{1 + \frac{v^2}{c^2} - 2vv'} \Rightarrow dt = dt' \sqrt{1 + \frac{v^2}{c^2}}$$

$$dt' = dt \sqrt{1 - \frac{v^2}{c^2}}$$

$$v = \frac{v'}{\sqrt{1 + \frac{v'^2}{c^2}}}$$

$$v' = \frac{-v}{\sqrt{1 - \frac{v^2}{c^2}}}$$

$$vdt = v'dt'$$

$$vdt' = d\hat{r} - r\hat{\phi}; \quad \hat{r}d\hat{r} - r\hat{\phi}d\hat{\phi} = dr$$

$$\dot{\hat{r}}d\hat{r} = dr\hat{r} + rd\hat{r} = -d\hat{p}$$

$$d\hat{p} = -dr\hat{r} - rd\hat{r} = -d\hat{r}$$

$$\dot{\hat{p}}d\hat{r} = d\hat{r}\hat{r} + r\hat{\phi}d\hat{\phi} = \dot{\hat{p}}$$

$$\dot{\hat{\phi}}d\hat{\phi} = \frac{dr}{dt} + r\frac{d\phi}{dt}; \quad \hat{v} = \frac{dr}{dt} + r\frac{d\phi}{dt}$$

$$d\hat{\phi} = d\hat{\phi}$$

$$\dot{\hat{v}}d\hat{v} = \frac{dr}{dt} + r\frac{d\phi}{dt}$$

$$\dot{\hat{v'}}d\hat{v'} = \frac{dr}{dt'} + r\frac{d\phi}{dt'}$$

$$\hat{v'} = \frac{dr}{dt'} + r\frac{d\phi}{dt'}$$
\[ \ddot{a} = \frac{d^2 \ddot{r}}{dt^2} = \frac{d (r \ddot{r})}{dt} = \left[ \frac{d^2 r}{dt^2} - r \left( \frac{d \phi}{dt} \right)^2 \right] \dot{r} + \left( \frac{2 dr \, d\phi}{dt \, dt} + r \frac{d^2 \phi}{dt^2} \right) \phi \]

\[ \ddot{a} = \frac{d^2 \ddot{r}}{dt^2} = \frac{d (r \ddot{r})}{dt} = \left[ \frac{d^2 r}{dt^2} - r \left( \frac{d \phi}{dt} \right)^2 \right] \dot{r} + \left( \frac{2 dr \, d\phi}{dt \, dt} + r \frac{d^2 \phi}{dt^2} \right) \phi \]

\[ -\ddot{v} = \frac{\ddot{v}}{\sqrt{1 - \frac{v^2}{c^2}}} \]

\[ -\ddot{a} = \frac{d (-\ddot{v})}{dt^2} = \frac{d}{dt^2} \left( \frac{\ddot{v}}{\sqrt{1 - \frac{v^2}{c^2}}} \right) = \frac{d}{dt} \left( \frac{\ddot{v}}{\sqrt{1 - \frac{v^2}{c^2}}} \right) \frac{d}{dt} \left( \frac{\ddot{v}}{\sqrt{1 - \frac{v^2}{c^2}}} \right) \]

\[ -\ddot{a} = \frac{d \ddot{v}}{dt^2} = \sqrt{1 + \frac{v^2}{c^2}} \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}} \left[ \frac{1 - \frac{v^2}{c^2}}{dt^2} \frac{d \ddot{v}}{dt} - \ddot{v} \frac{1}{2} \left( \frac{1 - \frac{v^2}{c^2}}{c^2} \right)^{\frac{3}{2}} \left( \frac{2 v \, dv}{c^2 \, dt} \right) \right] \]

\[ -\ddot{a} = \frac{d \ddot{v}}{dt^2} = \sqrt{1 + \frac{v^2}{c^2}} \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}} \left( \frac{1 - \frac{v^2}{c^2}}{dt^2} \frac{d \ddot{v}}{dt} + \frac{1}{2} \frac{v \, dv \, \ddot{v}}{c^2 \, dt \, c^2} \right) \]

\[ -\ddot{a} = \frac{d \ddot{v}}{dt^2} = \sqrt{1 + \frac{v^2}{c^2}} \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}} \left( \frac{1 - \frac{v^2}{c^2}}{dt^2} \frac{d \ddot{v}}{dt} + \frac{1}{2} \frac{v \, dv \, \ddot{v}}{c^2 \, dt \, c^2} \right) \]

\[ -\ddot{a} = \frac{d \ddot{v}}{dt^2} = \sqrt{1 + \frac{v^2}{c^2}} \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}} \left[ \left( \frac{1 - \frac{v^2}{c^2}}{c^2} \right) \frac{d \ddot{v}}{dt} + \frac{v \, dv \, \ddot{v}}{dt \, c^2} \right] \]

\[ -m \ddot{a} = \frac{-m \ddot{r}}{\sqrt{1 + \frac{v^2}{c^2}}} \frac{d \ddot{v}}{dt^2} = \frac{m \ddot{r}}{\sqrt{1 + \frac{v^2}{c^2}}} \left[ \left( \frac{1 - \frac{v^2}{c^2}}{c^2} \right) \frac{d \ddot{v}}{dt} + \frac{v \, dv \, \ddot{v}}{dt \, c^2} \right] \]

\[ \ddot{F} = -m \ddot{a} = \frac{-m \ddot{r}}{\sqrt{1 + \frac{v^2}{c^2}}} \frac{d \ddot{v}}{dt^2} \]
\[ \ddot{F} = \frac{m_b}{\left(1 - \frac{v^2}{c^2}\right)^{\frac{3}{2}}} \left( \frac{1}{1 - \frac{v^2}{c^2}} \right) \left( \frac{d\vec{v}}{dt} + \frac{d\vec{v}}{dt} + \frac{\vec{v}}{c^2} \right) \]  
\[ (= 19.06) \]  

\[ \ddot{F} = -m \ddot{a} = -\frac{m_b}{\sqrt{1 + \frac{v^2}{c^2}}} \frac{d\vec{v}}{dt} = \ddot{F} = \frac{m_b}{\left(1 - \frac{v^2}{c^2}\right)^{\frac{3}{2}}} \left( \frac{1}{1 - \frac{v^2}{c^2}} \right) \left( \frac{d\vec{v}}{dt} + \frac{d\vec{v}}{dt} + \frac{\vec{v}}{c^2} \right) \]  

\[ E_k = \int \ddot{F}(-d\vec{r}) = \int \ddot{F}d\vec{r} = \int \frac{k}{r^2} \ddot{F} d\vec{r} \]  

\[ E_k = \int \ddot{F}(-d\vec{r}) = \int \ddot{F}d\vec{r} = \int \frac{-m_b}{\sqrt{1 + \frac{v^2}{c^2}}} \frac{d\vec{v}}{dt} (-d\vec{r}) = \int \frac{m_b}{\left(1 - \frac{v^2}{c^2}\right)^{\frac{3}{2}}} \left( \frac{1}{1 - \frac{v^2}{c^2}} \right) \left( \frac{d\vec{v}}{dt} + \frac{d\vec{v}}{dt} + \frac{\vec{v}}{c^2} \right) d\vec{r} = \int \frac{-k}{r^2} \ddot{F} d\vec{r} \]  

\[ E_k = \int \frac{m_b}{\sqrt{1 + \frac{v^2}{c^2}}} \frac{d\vec{v}}{dt} \left( \frac{1}{1 - \frac{v^2}{c^2}} \right) \left( \frac{d\vec{v}}{dt} + \frac{d\vec{v}}{dt} + \frac{\vec{v}}{c^2} \right) = \int \frac{-k}{r^2} \ddot{F} d\vec{r} \]  

\[ E_k = \int \frac{m_b}{\sqrt{1 + \frac{v^2}{c^2}}} \left( \frac{1}{1 - \frac{v^2}{c^2}} \right) \left( \frac{d\vec{v}}{dt} + \frac{d\vec{v}}{dt} + \frac{\vec{v}}{c^2} \right) = \int \frac{-k}{r^2} \ddot{F} d\vec{r} \]  

\[ E_k = \int \frac{m_b}{\sqrt{1 + \frac{v^2}{c^2}}} \left( \frac{1}{1 - \frac{v^2}{c^2}} \right) \left( \frac{d\vec{v}}{dt} + \frac{d\vec{v}}{dt} + \frac{\vec{v}}{c^2} \right) = \int \frac{-k}{r^2} \ddot{F} d\vec{r} \]  

\[ E_k = \int \frac{m_b}{\sqrt{1 + \frac{v^2}{c^2}}} \left( \frac{1}{1 - \frac{v^2}{c^2}} \right) \left( \frac{d\vec{v}}{dt} + \frac{d\vec{v}}{dt} + \frac{\vec{v}}{c^2} \right) = \int \frac{-k}{r^2} \ddot{F} d\vec{r} \]  

\[ E_k = m c^2 \sqrt{1 + \frac{v^2}{c^2}} \frac{k}{r} \text{ constant} \]  
\[ E_k = \frac{m c^2}{\sqrt{1 - \frac{v^2}{c^2}}} \frac{k}{r} \text{ constant} \]  

\[ E_k = \frac{m c^2}{\sqrt{1 - \frac{v^2}{c^2}}} \frac{k}{r} - \frac{m c^2}{2} \frac{k}{r} \]  
\[ E_k = \frac{m c^2}{\sqrt{1 - \frac{v^2}{c^2}}} \frac{k}{r} - \frac{m c^2}{\infty} = m c^2 \]  

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\[
\frac{1}{\sqrt{1-v^2/c^2}} = \frac{E_n}{m_c^2} + \frac{k}{m_c^2} \frac{1}{r} \quad H = \frac{E_n}{m_c^2} \quad A = \frac{k}{m_c^2} = \frac{GM_m m_c}{m_c^2} = \frac{GM_o}{c^2}
\]

\[
\frac{1}{\sqrt{1-v^2/c^2}} = H + A \frac{1}{r} \quad \frac{1}{(1-v^2/c^2)^{\frac{3}{2}}} = \left( H + A \frac{1}{r} \right)^{\frac{3}{2}}
\]

\[
\ddot{L} = \ddot{t} \times \ddot{\hat{v}} = r \dot{r} \times \left( \frac{d\dot{r}}{dt} + r \frac{d\phi}{dt} \right) = r^2 \frac{d\phi}{dt} (\dot{r} \times \dot{\phi}) = r^2 \frac{d\phi}{dt} \dot{\hat{r}}
\]

\[
\ddot{L} = \ddot{r} \times \ddot{\hat{v}} = \ddot{r} \times \frac{-v}{\sqrt{1+v^2/c^2}} = r \dot{r} \times \frac{-1}{\sqrt{1+v^2/c^2}} \left[ \left( \frac{d\dot{r}}{dt} + r \frac{d\phi}{dt} \right) \right] = \frac{1}{(1-v^2/c^2)^{\frac{3}{2}}} r^2 \frac{d\phi}{dt} (\dot{r} \times \dot{\phi}) = r^2 \frac{d\phi}{dt} \dot{\hat{r}}
\]

\[
\ddot{L} = r^2 \frac{d\phi}{dt} \dot{\hat{r}} = L \ddot{\hat{r}} = \text{constant} \quad L = r^2 \frac{d\phi}{dt}
\]

\[
dE_k = \frac{m_v v' v'}{\sqrt{1+v'^2/c^2}} = \frac{m_v v_d v}{r^2} dr = -k \frac{k}{r^2} \dot{r} \ddot{r}
\]

\[
dE_k = \frac{m_b v d\dot{v}}{dt} = \frac{m_b v d}{dt} \frac{d\dot{v}}{dt} = -k \frac{k}{r^2} \dot{r} \ddot{r}
\]

\[
\ddot{F} = \frac{m_b}{\left( \frac{1-v^2}{c^2} \right)^{\frac{3}{2}}} \left[ \left( \frac{d^2 \phi}{dt^2} - r \left( \frac{d\phi}{dt} \right)^2 \right) \right] \ddot{r} + \left( \frac{d^2 \phi}{dt^2} - r \left( \frac{d\phi}{dt} \right)^2 \right) \ddot{\phi} = -k \frac{k}{r^2} \ddot{r}
\]

\[
\ddot{F}_\phi = \frac{m_b}{\left( \frac{1-v^2}{c^2} \right)^{\frac{3}{2}}} \left( \frac{d^2 \phi}{dt^2} - r \left( \frac{d\phi}{dt} \right)^2 \right) \ddot{\phi} = \text{zero}
\]

\[
\ddot{F}_r = \frac{m_b}{\left( \frac{1-v^2}{c^2} \right)^{\frac{3}{2}}} \left( \frac{d^2 \phi}{dt^2} - r \left( \frac{d\phi}{dt} \right)^2 \right) \ddot{r} = -k \frac{k}{r^2} \ddot{r}
\]

\[
\frac{d^2 \phi}{dt^2} = \frac{L}{r^2} \quad \frac{dr}{dt} = -L \frac{dw}{d\phi} \quad \frac{d^2 r}{dt^2} = -\frac{L^2}{r^2} \frac{d^2 w}{d\phi^2} \quad \frac{d^2 \phi}{dt^2} = \frac{2L^2}{r^3} \frac{dw}{d\phi}
\]
\[
\vec{F}_i = \frac{m_v}{(1 - \frac{v^2}{c^2})^2} \left[ \frac{-I_x^2}{r^2} \frac{d^2 w}{d\phi^2} + \frac{L}{r^2} \right] = \frac{-k}{r^2} \hat{r}
\]

\[
\frac{1}{(1 - \frac{v^2}{c^2})^2} \left( \frac{-I_x^2}{r^2} \frac{d^2 w}{d\phi^2} - \frac{I_y^2}{r^2} \right) = \frac{-GM_o}{r^2}
\]

\[
\frac{1}{(1 - \frac{v^2}{c^2})^2} \left( \frac{d^2 w}{d\phi^2} + \frac{1}{r} \left( \frac{-I_y^2}{r^2} \right) \right) = \frac{-GM_o}{r^2}
\]

\[
\frac{1}{(1 - \frac{v^2}{c^2})^2} \left( \frac{d^2 w}{d\phi^2} + \frac{1}{r} \right) = \frac{GM_o}{L^2}
\]

\[
(H + A \frac{1}{r}) \left( \frac{d^2 w}{d\phi^2} + \frac{1}{r} \right) = \frac{GM_o}{L^2}
\]

\[
(H + 3A \frac{1}{r}) \left( \frac{d^2 w}{d\phi^2} + \frac{1}{r} \right) = \frac{GM_o}{L^2}
\]

\[
H \frac{d^2 w}{d\phi^2} + Hw + 3A \frac{d^2 w}{d\phi^2} + 3A \frac{1}{r^2} = \frac{GM_o}{L^2}
\]

\[
H \frac{d^2 w}{d\phi^2} + Hw + 3A \frac{d^2 w}{d\phi^2} + 3A w^2 - \frac{GM_o}{L^2} = 0
\]

\[
H = \frac{E_r}{m_v c^2}, \quad A = \frac{k}{m_v c^2} = \frac{GM_o m_b}{m_v c^2}, \quad B = \frac{GM_o}{L^2}
\]

\[
H \frac{d^2 w}{d\phi^2} + Hw + 3A \frac{d^2 w}{d\phi^2} + 3A w^2 - B = 0
\]

\[
w = \frac{1}{r} \left[ 1 + \frac{\varepsilon D}{\varepsilon D} \right] \left[ 1 + \frac{\varepsilon \cos(\phi)}{\varepsilon D} \right], \quad \frac{d w}{d\phi} - \frac{Q \cos(\phi)}{D} = \frac{-Q^2 \cos(\phi)}{D}
\]

\[
H \frac{\cos(\phi)}{D} \left[ 1 + \frac{\varepsilon \cos(\phi)}{\varepsilon D} \right] + H \frac{1}{\varepsilon D} \left[ 1 + \frac{\varepsilon \cos(\phi)}{\varepsilon D} \right] + 3A \frac{1}{\varepsilon D} \left[ 1 + \frac{\varepsilon \cos(\phi)}{\varepsilon D} \right]^2 = 0
\]

\[
\frac{-Q^2 H \cos(\phi)}{D} + H \frac{1}{\varepsilon D} + H \frac{1}{\varepsilon D} + 3A \frac{1}{\varepsilon D} \left[ 1 + \frac{\varepsilon \cos(\phi)}{\varepsilon D} \right] + 3A \frac{1}{\varepsilon D} \left[ 1 + \frac{\varepsilon \cos(\phi)}{\varepsilon D} \right] = 0
\]

\[
\frac{-Q^2 H \cos(\phi)}{D} + H \frac{1}{\varepsilon D} + H \frac{\cos(\phi)}{D} \frac{3A}{\varepsilon D} - \frac{3A}{\varepsilon D} - \frac{3A}{\varepsilon D} \frac{\varepsilon \cos(\phi)}{D} + \frac{3A}{\varepsilon D} \frac{\varepsilon \cos(\phi)}{D} = 0
\]

\[
+ 3A \frac{1}{\varepsilon D} + 3A \frac{1}{\varepsilon D} \frac{2 \varepsilon \cos(\phi)}{\varepsilon D} + 3A \frac{1}{\varepsilon D} \frac{\varepsilon \cos(\phi)}{D} = 0
\]
\[-Q^2 \frac{\cos(\phi)}{D} + H \frac{1}{\varepsilon D} + H \frac{\cos(\phi)}{D} - 3Q^2A \frac{\cos(\phi)}{D} - 3Q^2A \frac{\cos^2(\phi)}{D^2} + \frac{3A}{\varepsilon D^2} + \frac{6A \cos(\phi)}{D} + 3A \frac{\cos(\phi)}{D^2} - B = \text{zero}\]

\[-Q^2 \frac{\cos^2(\phi)}{D} + H \frac{\cos(\phi)}{D} - 3Q^2A \frac{\cos(\phi)}{D} + 6A \frac{\cos(\phi)}{D} - \frac{3A}{\varepsilon D^2} + H = \text{zero}\]

\[-3Q^2A \frac{\cos^2(\phi)}{D^2} + 3A \frac{\cos^2(\phi)}{D^2} + H \frac{1}{\varepsilon D} + \frac{3A}{\varepsilon D^2} - B = \text{zero}\]

\[\left(-Q^2 + H - \frac{3Q^2A}{\varepsilon D} + \frac{6A}{\varepsilon D}\right) \frac{\cos(\phi)}{D} + \left(-3Q^2A + 3A\right) \frac{\cos^2(\phi)}{D^2} + \frac{H}{3A} + \frac{3A}{3AeD} - B = \text{zero}\]

\[\left(-3Q^2A + 3A\right) \frac{\cos^2(\phi)}{3AD^2} + \left(-Q^2 + H - \frac{3Q^2A}{\varepsilon D} + \frac{6A}{\varepsilon D}\right) \frac{\cos(\phi)}{D} + \frac{H}{3AeD} + \frac{1}{3A} = \text{zero}\]

\[1 - Q^2 \frac{\cos^2(\phi)}{D^2} = \text{zero}\]

\[Q^2 \approx 1\]

\[1 - Q^2 \frac{\cos^2(\phi)}{D^2} = \text{zero}\]

\[\frac{\cos(\phi)}{D} = \text{zero} \Rightarrow \frac{H}{3AeD} + \frac{1}{\varepsilon D^2} - B = \text{zero}\]

\[\frac{\cos(\phi)}{D} \neq \text{zero} \Rightarrow -Q^2 + H - \frac{Q^2}{3A} = \text{zero}\]

\[-Q^2 + H - Q^2 = \text{zero}\]

\[\frac{H}{3AeD} + \frac{1}{\varepsilon D^2} - B = \text{zero}\]

\[\begin{cases} a=b \Rightarrow H + \frac{1}{\varepsilon D} = \frac{1}{Q^2} \left(\frac{H}{3AeD} + \frac{1}{3A} \right) \\ a=c \Rightarrow H + \frac{1}{\varepsilon D} = \frac{\varepsilon DB}{3A} \end{cases}\]

\[Q^2 = 1 \quad \frac{H}{m_c e^2} = \frac{m_c c^2}{m_y} = 1 \quad \varepsilon DB = \frac{\varepsilon D G M}{L} = \frac{\varepsilon D G M}{\varepsilon D G M_o} = 1\]

\[\begin{cases} a=b \Rightarrow H + \frac{1}{\varepsilon D} = \frac{1}{3AeD} \left(\frac{H}{3AeD} + \frac{1}{3A} \right) = \frac{1}{eD} \Rightarrow \frac{1}{\varepsilon D} = \frac{1}{\varepsilon D} = \text{zero} \\ a=c \Rightarrow \frac{1}{3AeD} + \frac{1}{3A} = \frac{1}{\varepsilon D} \Rightarrow \frac{1}{\varepsilon D} = \text{zero} \end{cases}\]

\[b=c \Rightarrow \frac{1}{Q^2} \left(\frac{H}{3AeD} + \frac{1}{3AeD} \right) = \varepsilon DB = \frac{\varepsilon D G M_o}{L^2} = \frac{\varepsilon D G M_o}{\varepsilon D G M_o} = 1\]

\[\varepsilon DB = \frac{\varepsilon D G M_o}{L^2} = \frac{\varepsilon D G M_o}{\varepsilon D G M_o} = 1\]
\[ b=c \Rightarrow \frac{1}{Q}(\frac{H}{3A} + \frac{2}{\varepsilon D}) = \frac{1}{3A} \]

\[ Q^2 = H + \frac{6A}{\varepsilon D} \]

\[ Q = Q(H) \] The regression is a function of positive energy that governs the movement.

\[ H = \frac{E_B}{m_c^2} = \frac{m_B c^2}{m_c^2} = 1 \]

\[ Q^2 = 1 + \frac{6A}{\varepsilon D} \text{ Regression} \]

\[ a=b \Rightarrow \frac{1}{3A} + \frac{1}{\varepsilon D} = \frac{1}{1 + \frac{6A}{\varepsilon D}}\left(\frac{1}{3A} + \frac{2}{\varepsilon D}\right) \Rightarrow \frac{1}{\varepsilon D} = \text{zero} \]

\[ 3A^2\varepsilon D\left(-\frac{Q^2 H}{3A} + \frac{H}{3A} - \frac{Q^2}{\varepsilon D} + \frac{2}{\varepsilon D}\right) = \text{zero} \]

\[ 3A^2\varepsilon^2D^2\left(\frac{H}{3A\varepsilon D} + \frac{1}{\varepsilon^2D^2} - \frac{B}{3A}\right) = \text{zero} \]

\[ H = \frac{E_B}{m_c^2} \quad A = \frac{GM_o}{c^2} \quad B = \frac{GM_o}{L^2} \]

\[ -Q^2H\varepsilon D + H\varepsilon D - Q^23A + 6A = \text{zero} \]

\[ H\varepsilon D + 3A - \varepsilon D(\varepsilon DB) = \text{zero} \]

\[ -Q^2(-3A + \varepsilon D) - 3A + \varepsilon D - Q^23A + 6A = \text{zero} \]

\[ H\varepsilon D = -3A + \varepsilon D \]

\[ Q^23A - Q^2\varepsilon D + \varepsilon D - Q^23A + 3A = \text{zero} \]

\[ -Q^2\varepsilon D + \varepsilon D + 3A = \text{zero} \]

\[ Q^2 = 1 + \frac{3A}{\varepsilon D} \]

This regression is not governed by the positive energy.
\[ \ddot{v} = \frac{-v'}{\sqrt{1 + \frac{v'^2}{c^2}}} \]

\[ \ddot{a} = \frac{d\ddot{v}}{dt} = \frac{d}{dt}\left(\frac{-v'}{\sqrt{1 + \frac{v'^2}{c^2}}}\right) = \frac{dt'}{dt} \frac{d}{d\tau}\left(\frac{-v'}{\sqrt{1 + \frac{v'^2}{c^2}}}\right) = \sqrt{1 - \frac{v'^2}{c^2}} \frac{d}{d\tau}\left(\frac{-v'}{\sqrt{1 + \frac{v'^2}{c^2}}}\right) \]

\[ \ddot{a} = \frac{d\ddot{v}}{dt} = \sqrt{1 - \frac{v'^2}{c^2}} \left[ \frac{-1}{\sqrt{1 + \frac{v'^2}{c^2}}} \frac{dv'}{dt} - v' \frac{d}{d\tau}\left(\frac{1}{\sqrt{1 + \frac{v'^2}{c^2}}}\right) \right] \]

\[ \ddot{a} = \frac{d\ddot{v}}{dt} = \sqrt{1 - \frac{v'^2}{c^2}} \left[ \frac{-1}{\sqrt{1 + \frac{v'^2}{c^2}}} \frac{dv'}{dt} - v' \frac{1}{2} \left(\frac{1}{\sqrt{1 + \frac{v'^2}{c^2}}}\right)^{\frac{1}{2} - \frac{1}{2}} \right] \]

\[ \ddot{a} = \frac{d\ddot{v}}{dt} = \sqrt{1 - \frac{v'^2}{c^2}} \left[ \frac{1}{\sqrt{1 + \frac{v'^2}{c^2}}} \frac{dv'}{dt} - \frac{1}{\sqrt{1 + \frac{v'^2}{c^2}}} \frac{dv'}{dt} \right] \]

\[ \ddot{a} = \frac{d\ddot{v}}{dt} = \sqrt{1 - \frac{v'^2}{c^2}} \left[ \frac{1}{\sqrt{1 + \frac{v'^2}{c^2}}} \frac{dv'}{dt} - \frac{1}{\sqrt{1 + \frac{v'^2}{c^2}}} \frac{dv'}{dt} \right] \]

\[ m\ddot{a} = \frac{m'\ddot{a}}{1 - \frac{v'^2}{c^2}} = \frac{m_b}{\sqrt{1 - \frac{v'^2}{c^2}}} \frac{dv'}{dt} = -\frac{m_b}{\sqrt{1 - \frac{v'^2}{c^2}}} \left[ \frac{1}{\sqrt{1 + \frac{v'^2}{c^2}}} \frac{dv'}{dt} \right] \]

\[ \ddot{F} = \frac{m\ddot{a}}{\sqrt{1 - \frac{v'^2}{c^2}}} = \frac{m_b}{\sqrt{1 - \frac{v'^2}{c^2}}} \frac{dv'}{dt} \]

\[ \ddot{F} = \frac{m\ddot{a}}{\sqrt{1 - \frac{v'^2}{c^2}}} = \frac{m_b}{\sqrt{1 - \frac{v'^2}{c^2}}} \frac{dv'}{dt} = \ddot{F} = \frac{m_b}{\sqrt{1 - \frac{v'^2}{c^2}}} \left[ \frac{1}{\sqrt{1 + \frac{v'^2}{c^2}}} \frac{dv'}{dt} \right] \]

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\[ E_k = \int \ddot{r} \, d\bar{r} = \int \ddot{r}' \, (-d\bar{r}') = \int \frac{-k}{r^2} \bar{r} (-d\bar{r}') \]  

21.54

\[ E_k = \int \ddot{r} \, d\bar{r} = \int \ddot{r}' \, (-d\bar{r}') = \int \frac{m_b}{\sqrt{1 - v_e^2}} \, \frac{dv_e}{dt} \, d\bar{r}' = \int \frac{-m_b}{\sqrt{1 + \frac{v_e^2}{c^2}}} \left[ 1 + \frac{v_e^2}{c^2} \right] \frac{dv_e}{dt} = \int \frac{-m_b}{r^2} \ddot{r} \, d\bar{r}' \]  

21.55

\[ E_k = \int \frac{m_b}{\sqrt{1 - v_e^2}} \, \frac{dv_e}{dt} \, d\bar{r}' = \int \frac{m_b}{\sqrt{1 + \frac{v_e^2}{c^2}}} \left[ 1 + \frac{v_e^2}{c^2} \right] \frac{dv_e}{dt} \, d\bar{r}' = \int k \frac{\ddot{r} \, d\bar{r}'}{r^2} \]  

\[ E_k = \int m_b v \frac{dv}{dt} = \int \frac{m_b}{\sqrt{1 - v_e^2}} \, \frac{1}{\left( 1 + \frac{v_e^2}{c^2} \right)^2} \left( 1 + \frac{v_e^2}{c^2} \right) = \int \frac{m_b v \frac{dv}{dt}}{r^2} \, d\bar{r}' \]  

\[ E_k = \int m_b v \frac{dv}{dt} = \int \frac{m_b v' \frac{dv'}{dt}}{r^2} = \int \frac{m_b v \frac{dv}{dt}}{r^2} \, d\bar{r}' \]  

21.56

\[ E_k = \int \frac{m_b v' \frac{dv'}{dt}}{r^2} = \int \frac{m_b v' \frac{dv'}{dt}}{r^2} = -k \frac{dr}{r^2} \]  

21.57

\[ E_k = -m_c^2 \sqrt{1 - \frac{v^2}{c^2}} = - \frac{m_c^2}{\sqrt{1 + \frac{v^2}{c^2}}} = k + \text{constant} \]  

21.58

\[ E_k = -m_c^2 \sqrt{1 - \frac{v^2}{c^2}} \frac{k}{r} = \text{constant} \]  

\[ E_p = - \frac{m_c^2}{\sqrt{1 + \frac{v^2}{c^2}}} \frac{k}{r} = \text{constant} \]  

\[ E_k = - \frac{m_c^2}{\sqrt{1 + \frac{v^2}{c^2}}} \frac{k}{r} = -m_c^2 \frac{k}{r} \]  

21.59

\[ E_P = - \frac{m_c^2}{\sqrt{1 + \frac{v^2}{c^2}}} \end{equation} \]  

21.60

\[ H = \frac{E_p}{m_c^2} \]  

21.61
\[
-\frac{1}{\sqrt{1 + \frac{v^2}{c^2}}} = H + A \frac{1}{r} \\
\frac{1}{\left(1 + \frac{v^2}{c^2}\right)^{\frac{3}{2}}} = -\left(H + A \frac{1}{r}\right)^3
\]

\[
\vec{F} = \vec{P} \times \vec{v} = -r \hat{r} \times \left[ -\left(\frac{d\hat{r}}{dt} + r \frac{d\phi}{dt} \hat{\phi}\right) \right] = r^2 \frac{d\phi}{dt} \left(\hat{r} \times \hat{\phi}\right) = r^2 \frac{d\phi}{dt} \hat{k}
\]

\[
\vec{F} = \vec{P} \times \vec{v} = -r \hat{r} \times \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}} \left[ \left(\frac{d\hat{r}}{dt} + r \frac{d\phi}{dt} \hat{\phi}\right) \right] = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}} \left(\frac{d\phi}{dt} \right) \left(\hat{r} \times \hat{\phi}\right) = r^2 \frac{d\phi}{dt} \hat{k}
\]

\[
\vec{F} = r^2 \frac{d\phi}{dt} \hat{k} = L \hat{k} \\
L' = r^2 \frac{d\phi}{dt}' = L
\]

\[
dE_k = \frac{m_v v d\nu}{\sqrt{1 - \frac{v^2}{c^2}}} = \frac{m_v v' v'}{\left(1 + \frac{v^2}{c^2}\right)} \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}}
\]

\[
dE_k = \vec{P} \cdot v = \frac{m_v v' v'}{\left(1 + \frac{v^2}{c^2}\right)} \vec{P} = \frac{k}{r^2} \vec{P} = k \vec{P}
\]

\[
\vec{F}_n = \frac{m_v \vec{v}^2}{\left(1 + \frac{v^2}{c^2}\right)^{\frac{3}{2}}} = \frac{k \vec{F}}{r^2}
\]

\[
\vec{F}_n = \frac{m_v}{\left(1 + \frac{v^2}{c^2}\right)} \left[ \frac{d^2 \hat{r}}{dt^2} - r \left(\frac{d\phi}{dt}\right)^2 \right] = \frac{k}{r^2} \vec{r}
\]

\[
\vec{F}_\phi = -\frac{m_v}{\left(1 + \frac{v^2}{c^2}\right)} \left[ 2 \frac{d\hat{r}}{dt} \frac{d\phi}{dt} + r \frac{d^2 \phi}{dt^2} \right] = \text{zero}
\]

\[
\vec{F}_r = -\frac{m_v}{\left(1 + \frac{v^2}{c^2}\right)} \left[ \frac{d^2 \hat{r}}{dt^2} - r \left(\frac{d\phi}{dt}\right)^2 \right] = \frac{k}{r^2} \vec{r}
\]

\[
-\frac{1}{\left(1 + \frac{v^2}{c^2}\right)^{\frac{3}{2}}} \left[ \frac{d^2 \hat{r}}{dt^2} - r \left(\frac{d\phi}{dt}\right)^2 \right] = \frac{-GM}{r^2} \vec{r}
\]

\[
\frac{d\phi}{dt} = \frac{L'}{r^2} \\
\frac{d\hat{r}}{dt} = -r \frac{d\omega}{d\phi} \\
\frac{d^2 \hat{r}}{dt^2} = -\frac{L^2}{r^2} \frac{d^2 \omega}{d\phi^2} \\
\frac{d^2 \phi}{dt^2} = \frac{2L}{r^3} \frac{d\omega}{d\phi}
\]

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\[
\frac{1}{1+\nu^2/c^2} \left[ -\frac{L^2}{r^2} \frac{d^2 w}{d\phi^2} - r \left( \frac{L^2}{r^2} \right) \right] = -\frac{GM_o}{r^2}
\]

\[
\frac{1}{1+\nu^2/c^2} \left[ -\frac{L^2}{r^2} \frac{d^2 w}{d\phi^2} - r \left( \frac{L^2}{r^2} \right) \right] = -\frac{GM_o}{r^2}
\]

\[
\frac{1}{1+\nu^2/c^2} \left( \frac{d^2 w}{d\phi^2} + \frac{1}{r} \right) \left( \frac{-\frac{L^2}{r^2}}{r^2} \right) = -\frac{GM_o}{r^2}
\]

\[
\frac{1}{1+\nu^2/c^2} \left( \frac{d^2 w}{d\phi^2} + \frac{1}{r} \right) = \frac{GM_o}{L^2}
\]

\[-\left( H + A \frac{1}{r} \right) \left( \frac{d^2 w}{d\phi^2} + \frac{1}{r} \right) = \frac{GM_o}{L^2} \]

\[
\left( H + 3A \frac{1}{r} \right) \left( \frac{d^2 w}{d\phi^2} + \frac{1}{r} \right) = -\frac{GM_o}{L^2}
\]

\[
H \frac{d^2 w}{d\phi^2} + H \frac{1}{r} + 3A \frac{d^2 w}{d\phi^2} \frac{1}{r} + 3A \frac{1}{r^2} = -\frac{GM_o}{L^2}
\]

\[
H \frac{d^2 w}{d\phi^2} + Hw + 3A \frac{d^2 w}{d\phi^2} + 3Aw^2 + \frac{GM_o}{L^2} = 0
\]

\[
H = \frac{E_B}{m_r c^2} \quad A = \frac{GM_o}{c^2} \quad B = \frac{GM_o}{L^2}
\]

\[
H \frac{d^2 w}{d\phi^2} + Hw + 3A \frac{d^2 w}{d\phi^2} = 0
\]

\[
w = \frac{1}{r} \frac{1}{r} \left[ 1 + \epsilon \cos(\phi) \right] \quad \frac{dw}{d\phi} = -\frac{\epsilon \sin(\phi)}{D} \quad \frac{d^2 w}{d\phi^2} = -\frac{\epsilon^2 \cos(\phi)}{D}
\]

\[
-\frac{\epsilon^2 \cos(\phi)}{D} + H \frac{1}{r} \frac{1}{r} \left[ 1 + \epsilon \cos(\phi) \right] + 3A \frac{\epsilon^2 \cos(\phi)}{D} \frac{1}{r} \left[ 1 + \epsilon \cos(\phi) \right] + 3A \left[ \frac{1}{r} \left[ 1 + \epsilon \cos(\phi) \right] \right]^2 = 0
\]

\[
-\frac{\epsilon^2 \cos(\phi)}{D} + H \frac{1}{r} + \frac{1}{r} \epsilon \cos(\phi) - 3A \frac{\epsilon^2 A \cos(\phi)}{D} \frac{1}{r} \left[ 1 + \epsilon \cos(\phi) \right] + 3A \left[ \frac{1}{r} \left[ 1 + \epsilon \cos(\phi) \right] \right]^2 = 0
\]

\[
-\frac{\epsilon^2 \cos(\phi)}{D} + H \frac{1}{r} + \frac{1}{r} \epsilon \cos(\phi) - 3A \frac{\epsilon^2 A \cos(\phi)}{D} \frac{1}{r} \left[ 1 + \epsilon \cos(\phi) \right] + 3A \left[ \frac{1}{r} \left[ 1 + \epsilon \cos(\phi) \right] \right]^2 = 0
\]

\[
+ \frac{3A}{\epsilon^2 D^2} + \frac{3A}{\epsilon^2 D^2} \epsilon^2 \cos(\phi) + 3A \frac{3A}{\epsilon^2 D^2} \epsilon^2 \cos^2(\phi) = 0
\]

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\[-Q^2 H \frac{\cos(\phi)}{D} + H \frac{1}{\varepsilon D} + H \frac{\cos(\phi)}{D} - \frac{3Q^2 A \cos(\phi)}{\varepsilon D} - \frac{3Q^2 A \cos'(\phi)}{D^2} + \]
\[+ \frac{3A}{\varepsilon D^2} + \frac{6A \cos(\phi)}{D} + \frac{3A \cos'(\phi)}{D^2} + B = \text{zero} \]

\[-Q^2 H \frac{\cos(\phi)}{D} + H \frac{\cos(\phi)}{D} - \frac{3Q^2 A \cos(\phi)}{\varepsilon D} + \frac{6A \cos(\phi)}{D} - \frac{3Q^2 A \cos'(\phi)}{D^2} + \]
\[+ \frac{1}{\varepsilon D} + \frac{3A}{\varepsilon D^2} + B = \text{zero} \]

\[-3Q^2 A \frac{\cos'(\phi)}{D^2} + \frac{3A \cos'(\phi)}{D^2} + H \frac{1}{\varepsilon D} + \frac{3A}{\varepsilon D^2} + B = \text{zero} \]

\[-3Q^2 A + 3A \frac{\cos'(\phi)}{D} \left( \frac{3A}{\varepsilon D^2} \right) + \left( -Q^2 H + H \frac{Q^2}{D^2} + \frac{2}{\varepsilon D} \right) \frac{\cos(\phi)}{D} + H \frac{1}{3A \varepsilon D} + \frac{1}{\varepsilon D^2} + B = \text{zero} \]

\[Q^2 \approx 1 \quad \left( 1 - Q^2 \right) \frac{\cos'(\phi)}{D^2} = \text{zero} \]

\[-Q^2 H + \frac{Q^2}{3A} + \frac{2}{\varepsilon D} \frac{\cos(\phi)}{D} + \frac{H}{3A \varepsilon D} + \frac{1}{\varepsilon D^2} + \frac{B}{3A} = \text{zero} \]

\[\frac{\cos(\phi)}{D} = \text{zero} \Rightarrow \frac{H}{3A \varepsilon D} + \frac{1}{\varepsilon D^2} + \frac{B}{3A} = \text{zero} \]

\[\frac{\cos(\phi)}{D} \neq \text{zero} \Rightarrow -Q^2 H + \frac{Q^2}{3A} + \frac{2}{\varepsilon D} = \text{zero} \]

\[-Q^2 H + \frac{1}{3A} + \frac{1}{Q^2} \left( \frac{H}{3A} + \frac{2}{\varepsilon D} \right) = \text{zero} \quad H + \frac{1}{3A} = \frac{\text{EDB}}{3A} \]

\[Q^2 = 1 \quad H = \frac{E_0}{m_c^2} = -\frac{m_e c^2}{m_c^2} = -1 \quad \text{EDB} = \frac{\text{EDG}_M}{L^2} = \frac{\text{EDGM}}{\varepsilon DGM} = 1 \]

\[\text{[a=b]} \Rightarrow \frac{H}{3A} + \frac{1}{3A} = \frac{Q}{3A} + \frac{2}{3A} = \text{zero} \quad \frac{H}{3A} + \frac{1}{3A} = \frac{\text{EDB}}{3A} \]

\[\text{[a=c]} \Rightarrow \frac{H}{3A} + \frac{1}{3A} = \frac{\text{EDB}}{3A} \]

\[\text{[b=c]} \Rightarrow -\frac{1}{3A} + \frac{1}{3A} - \frac{1}{3A} \Rightarrow \frac{1}{3A} = \text{zero} \]

\[\varepsilon \text{DB} = \frac{\text{EDG}_{M_{\text{ osc}}}}{L^2} = \frac{\text{EDGM}}{\varepsilon DGM} = 1 \]

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\[ [b=c] \Rightarrow \frac{1}{Q^2} \left( \frac{H}{3A} + \frac{2}{\epsilon D} \right) = -\frac{1}{3A} \quad Q^2 = -H - \frac{6A}{\epsilon D} \]

\[ Q = Q(H) \] The advance is a function of negative energy that governs the movement

\[ H = \frac{E_r}{m c^2} = -\frac{m c^2}{m c^2} = -1 \quad Q^2 = \left(1 - \frac{6A}{\epsilon D}\right) \Rightarrow Q^2 = 1 - \frac{6A}{\epsilon D} \] Advance

\[ [a=b] \Rightarrow -\frac{1}{3A} + \frac{1}{\epsilon D} = \frac{1}{(1 - \frac{6A}{\epsilon D})} \left( \frac{1}{3A} + \frac{2}{\epsilon D} \right) \Rightarrow \frac{1}{\epsilon D} = \text{zero} \]

\[ H = \frac{E_r}{m c^2} \quad A = \frac{GM_o}{c^2} \quad B = \frac{GM_o}{L^2} \]

\[ -\frac{Q^2 H}{3A} + \frac{H}{3A} - \frac{Q^2}{\epsilon D} + \frac{2}{\epsilon D} = \text{zero} \quad \frac{H}{3A eD} + \frac{1}{\epsilon^2 D^2} + \frac{B}{3A} = \text{zero} \]

\[ 3A eD \left( -\frac{Q^2 H}{3A} + \frac{H}{3A} - \frac{Q^2}{\epsilon D} + \frac{2}{\epsilon D} \right) = \text{zero} \quad 3A e^2 D^2 \left( \frac{H}{3A eD} + \frac{1}{\epsilon^2 D^2} + \frac{B}{3A} \right) = \text{zero} \]

\[ -Q^2 H eD + HeD - Q^2 3A + 6A = \text{zero} \quad HeD + 3A + \epsilon D(eDB) = \text{zero} \]

\[ \epsilon DB = \frac{\epsilon DGM_o}{L^2} = \frac{\epsilon DGM_o}{\epsilon DGM_o} = 1 \quad HeD = -3A - eD \]

\[ -Q^2 (-3A - eD) - 3A - eD - Q^2 3A + 6A = \text{zero} \]

\[ Q^2 3A + Q^2 eD - eD - Q^2 3A + 3A = \text{zero} \]

\[ Q^2 eD - eD + 3A = \text{zero} \quad Q^2 = 1 - \frac{3A}{\epsilon D} \]

This advance is not governed by negative energy

\[ -Q^2 H eD + HeD - Q^2 3A + 6A = \text{zero} \]

\[ -Q^2 (-3A - eD) + HeD - Q^2 3A + 6A = \text{zero} \]

\[ Q^2 3A + Q^2 eD + HeD - Q^2 3A + 6A = \text{zero} \]

\[ Q^2 eD + HeD + 6A = \text{zero} \quad Q^2 = -H - \frac{6A}{\epsilon D} \]

\[ \left( -\frac{Q^2 H}{3A} + \frac{H}{3A} - \frac{Q^2}{\epsilon D} + \frac{2}{\epsilon D} \right) \cos(\phi) + \frac{H}{3A eD} + \frac{1}{\epsilon^2 D^2} + \frac{B}{3A} = \text{zero} \]

\[ 3A e^2 D^2 \left[ \left( -\frac{Q^2 H}{3A} + \frac{H}{3A} - \frac{Q^2}{\epsilon D} + \frac{2}{\epsilon D} \right) \cos(\phi) + \frac{H}{3A eD} + \frac{1}{\epsilon^2 D^2} + \frac{B}{3A} \right] = \text{zero} \]

\[ \epsilon D \left( -\frac{Q^2 H 3A eD}{3A} + \frac{H 3A eD}{3A} - \frac{Q^2 3A eD}{\epsilon D} + \frac{2}{\epsilon D} \right) \cos(\phi) + \frac{H 3A e^2 D^2}{3A eD} + \frac{3A e^2 D^2}{\epsilon^2 D^2} + \frac{B 3A e^2 D^2}{3A} = \text{zero} \]
\[ \varepsilon D(-Q^2 H eD + H eD - Q^2 3A + 6A) \frac{\cos(\phi)}{D} + H eD + 3A + \varepsilon D(eDB) = 0 \]

\[ \varepsilon DB = \frac{\varepsilon DGM_o}{I^2} = \frac{\varepsilon DGM_o}{I^2} = 1 \]

\[ H = \frac{E_o}{m_j c^2} = \frac{m_j c^2}{-1} \]

\[ \varepsilon D(-Q^2 H eD + H eD - Q^2 3A + 6A) \frac{\cos(\phi)}{D} - 3A + \varepsilon D = 0 \]

\[ (-Q^2 H eD + H eD - Q^2 3A + 6A) \frac{\cos(\phi)}{D} + 3A = 0 \]

\[ Q^2 = 1 - \frac{3A}{\varepsilon D} \]

\[ \left[ \left(1 - \frac{3A}{\varepsilon D}\right) H eD + H eD - \left(1 - \frac{3A}{\varepsilon D}\right) 3A + 6A \right] \frac{\cos(\phi)}{D} + 3A = 0 \]

\[ (-H eD + H eD 3A + H eD - 3A + 3A 6A + 3A) \frac{\cos(\phi)}{D} + 3A = 0 \]

\[ \left(-H eD + H eD 3A + H eD - 3A + 6A 9A^2 + 6A \right) \frac{\cos(\phi)}{D} + 3A = 0 \]

\[ (H eD + 9A^2 + 3A) \frac{\cos(\phi)}{D} + 3A = 0 \]

\[ H = \frac{E_o}{m_j c^2} = \frac{m_j c^2}{-1} \]

\[ (-3A + 9A^2 + 3A) \frac{\cos(\phi)}{D} + 3A = 0 \]

\[ \frac{9A^2 \cos(\phi)}{\varepsilon D} + \frac{3A}{\varepsilon D} = 0 \quad \frac{\cos(\phi)}{D} + \frac{1}{3A} = 0 \]

\[ (-Q^2 H eD + H eD - Q^2 3A + 6A) \frac{\cos(\phi)}{D} + 3A = 0 \]

\[ Q^2 = 1 - \frac{6A}{\varepsilon D} \]

\[ \left[ \left(1 - \frac{6A}{\varepsilon D}\right) H eD + H eD - \left(1 - \frac{6A}{\varepsilon D}\right) 3A + 6A \right] \frac{\cos(\phi)}{D} + 3A = 0 \]

\[ (-H eD + H eD 6A + H eD - 3A + 3A 6A + 6A) \frac{\cos(\phi)}{D} + 3A = 0 \]

\[ (-H eD + H eD + 6A + H eD - 3A + 6A) \frac{\cos(\phi)}{D} + 3A = 0 \]
\[
\left( H6A + \frac{18A^2}{\varepsilon D} + 3A \right) \frac{\cos(\phi Q)}{D} + \frac{3A}{\varepsilon D} = 0
\]

\[
H = \frac{E_s}{m c^2} = -\frac{m_e}{m c^2} = -1
\]

\[
\left( -6A + \frac{18A^2}{\varepsilon D} + 3A \right) \frac{\cos(\phi Q)}{D} + \frac{3A}{\varepsilon D} = 0
\]

\[
\frac{1}{3A} \left[ \left( -3A + \frac{18A^2}{\varepsilon D} \right) \cos(\phi Q) + \frac{3A}{\varepsilon D} \right] = 0
\]

\[
\left( -1 + 5A \right) \frac{\cos(\phi Q)}{D} + \frac{1}{\varepsilon D} = 0
\]

\[
-\left( 1 - 5A \right) \frac{\cos(\phi Q)}{D} + \frac{1}{\varepsilon D} = 0
\]

\[
-\cos(\phi Q) + \frac{1}{\varepsilon D} = 0
\]

\[
-\cos(\phi Q) + \frac{1}{\varepsilon D} = 0
\]

\[
2.93
\]

\[
-\cos(\phi Q) + \frac{1}{\varepsilon D} = 0
\]

\[
2.91
\]

\[
Q^2 = 1
\]

\[
H = \frac{E_s}{m c^2} = -\frac{m_e}{m c^2} = -1
\]

\[
(\varepsilon D - \varepsilon D - 3A + 6A) \frac{\cos(\phi Q)}{D} + \frac{3A}{\varepsilon D} = 0
\]

\[
(3A) \frac{\cos(\phi Q)}{D} + \frac{3A}{\varepsilon D} = 0
\]

\[
\frac{\cos(\phi Q)}{D} + \frac{1}{\varepsilon D} = 0
\]

\[
2.94
\]

\[
Q^2 = 1 - \frac{6A}{\varepsilon D}
\]

\[
Q^2 = 1
\]

\[
Q^2 = 1 - \frac{3A}{\varepsilon D}
\]

\[
\frac{Q^2 \cos(\phi Q) + 1}{\varepsilon D} = 0
\]

\[
\frac{\cos(\phi Q) + 1}{\varepsilon D} = 0
\]

\[
2.95
\]
Energy Newtonian (E_N)

\[ E_N = \frac{m_o u^2}{2} - \frac{k}{r} \]

\[ u^2 = \left( \frac{dr}{dt} \right)^2 + \left( r \frac{d\phi}{dt} \right)^2 = \left( \frac{dr}{dt} \right)^2 + \frac{L^2}{r^2} \]

\[ E_N = \frac{m_o}{2} \left[ \left( \frac{dr}{dt} \right)^2 + \frac{L^2}{r^2} \right] - \frac{k}{r} \]

\[ \frac{2E_N}{m_o} = \left( \frac{dr}{dt} \right)^2 + \frac{L^2}{r^2} - \frac{2k}{m_o r} \]

\[ \left( \frac{dr}{dt} \right)^2 + \frac{L^2}{r^2} \right) - \frac{2k}{m_o r} \frac{2E_N}{m_o} = 0 \]

\[ \frac{d\phi}{dt} = \frac{L}{r^2} \quad \frac{dr}{dt} = -L \frac{dw}{d\phi} \quad \frac{d^2r}{dt^2} = -\frac{L^2}{r^2} \frac{d^2w}{d\phi^2} \quad \frac{d^2\phi}{dt^2} = \frac{2L^2}{r^3} \frac{dw}{d\phi} \]

\[ -L \left( \frac{dw}{d\phi} \right)^2 + \frac{L^2}{r^2} \frac{2k}{m_o r} \frac{2E_N}{m_o} = 0 \]

\[ \left( \frac{dw}{d\phi} \right)^2 + \frac{1}{r^2} \frac{2k}{m_o L^2} \frac{2E_N}{m_o L^2} = 0 \]

\[ \left( \frac{dw}{d\phi} \right)^2 + \frac{2k}{m_o L^2} \frac{2E_N}{m_o L^2} = 0 \]

\[ \left( \frac{dw}{d\phi} \right)^2 + \frac{2k}{m_o L^2} \frac{2E_N}{m_o L^2} = 0 \]

\[ x = \frac{2k}{m_o L^2} \quad y = \frac{2E_N}{m_o L^2} \]

\[ \left( \frac{dw}{d\phi} \right)^2 + \frac{2k}{m_o L^2} \frac{2E_N}{m_o L^2} = 0 \]

\[ w = \frac{1}{r} \left[ 1 + \varepsilon \cos(\phi) \right] \quad \frac{dw}{d\phi} = \frac{-Q \sin(\phi)}{D} \quad \frac{d^2w}{d\phi^2} = \frac{-Q^2 \cos(\phi)}{D} \]

\[ \left[ \frac{-Q \sin(\phi)}{D} \right]^2 + \left( \frac{1}{\varepsilon D} \left[ 1 + \varepsilon \cos(\phi) \right] \right)^2 \frac{-x}{\varepsilon D} \left[ 1 + \varepsilon \cos(\phi) \right] = 0 \]

\[ \frac{Q^2}{D^2} \left[ 1 - \cos^2(\phi) \right] + \frac{1}{\varepsilon^2 D^2} \left[ 1 + 2 \varepsilon \cos(\phi) + \varepsilon^2 \cos^2(\phi) \right] - x \frac{1}{\varepsilon D} - x \frac{1}{\varepsilon D} \varepsilon \cos(\phi) = 0 \]
\[
\frac{Q^2}{D^2} - \frac{Q^2}{D^2} \cos^2(\phi\psi) + \frac{1}{\varepsilon^2 D^2} + \frac{1}{\varepsilon^2 D^2} 2 \cos(\phi\psi) + \frac{1}{\varepsilon^2 D^2} \varepsilon^2 \cos^2(\phi\psi) - \frac{x}{\varepsilon D} - \frac{x \cos(\phi\psi)}{D} = 0
\]

\[
\frac{Q^2}{D^2} - \frac{Q^2}{D^2} \cos^2(\phi\psi) + \frac{1}{\varepsilon^2 D^2} + \frac{2 \cos(\phi\psi)}{\varepsilon D} + \frac{\cos^2(\phi\psi)}{\varepsilon D} - \frac{x}{\varepsilon D} - \frac{x \cos(\phi\psi)}{D} = 0
\]

\[
\frac{\cos^2(\phi\psi)}{D^2} - \frac{Q^2}{D^2} \frac{\cos^2(\phi\psi)}{D^2} + \frac{2 \cos(\phi\psi)}{\varepsilon D} - \frac{x}{\varepsilon D} - \frac{\cos^2(\phi\psi)}{D} + \frac{Q^2}{D^2} - \frac{1}{\varepsilon^2 D^2} - \frac{x}{\varepsilon D} = 0
\]

\[
(1-Q^2) \frac{\cos^2(\phi\psi)}{D^2} + \left( \frac{2}{\varepsilon D} - x \right) \frac{\cos(\phi\psi)}{D} + \frac{Q^2}{D^2} - \frac{1}{\varepsilon^2 D^2} - \frac{x}{\varepsilon D} = 0
\]

\[
Q^2 \approx 1, \quad (1-Q^2) \frac{\cos^2(\phi\psi)}{D^2} = 0
\]

\[
\left( \frac{2}{\varepsilon D} - x \right) \frac{\cos(\phi\psi)}{D} + \frac{1}{\varepsilon D} - \frac{x}{\varepsilon D} = 0
\]

\[
\frac{\cos(\phi\psi)}{D} = \frac{1}{\varepsilon D} + \frac{1}{\varepsilon^2 D^2} - \frac{x}{\varepsilon D} = 0
\]

\[
x = \frac{2k}{m \, L^2}, \quad y = \frac{2E_N}{m \, L^2}
\]

\[
\frac{2}{\varepsilon D} - x = 0 \Rightarrow x = \frac{2}{\varepsilon D} = \frac{2k}{m \, L^2} \Rightarrow \frac{1}{\varepsilon D} = \frac{\varepsilon D M \, m}{m \, L^2} \Rightarrow L^2 = \varepsilon D M
\]

\[
\frac{\varepsilon^2 D^2}{D^2} + \frac{\varepsilon^2 D^2}{\varepsilon D} - \frac{\varepsilon^2 D^2}{\varepsilon D} - \frac{x}{\varepsilon^2 D^2} = 0
\]

\[
\varepsilon^2 + 1 - \varepsilon D x - \varepsilon^2 D^2 y = 0
\]

\[
\varepsilon D x = \varepsilon D \frac{2}{\varepsilon D} \Rightarrow \varepsilon D x = 2
\]

\[
\varepsilon^2 D^2 y = \varepsilon^2 D^2 \frac{2E_N}{m \, L^2} = \varepsilon^2 D^2 \frac{2E_N}{m \, L^2} = \frac{2 \varepsilon D E_N}{k}
\]

\[
\varepsilon^2 + 1 - \frac{2 \varepsilon D E_N}{k} = 0
\]

\[
E_N = \frac{k}{2 \varepsilon D} \left( \varepsilon^2 - 1 \right)
\]

\[
\frac{1}{a} = \frac{1}{\varepsilon D} \left( 1 - \varepsilon^2 \right)
\]

\[
E_N = \frac{-k}{2a}
\]
\[ t = \frac{t'}{\sqrt{1 - \frac{v^2}{c^2}}} \quad t > t' \]

\[ t = t_1 + t_2 = \frac{L}{c - v} + \frac{L}{c + v} = \frac{2L}{c} \left( \frac{1}{1 - \frac{v^2}{c^2}} \right) \quad t' = \frac{2L'}{c} \]

\[ t = \frac{2L}{c} \left( \frac{1}{1 - \frac{v^2}{c^2}} \right) = \frac{2L'}{c} \left( \frac{1}{1 - \frac{v^2}{c^2}} \right) \Rightarrow L = L' \sqrt{1 - \frac{v^2}{c^2}} \quad L' > L \]

This is the spatial deformation.

The length \( L' \) at rest in the reference frame of the observer \( O' \) is greater than the length \( L \) that is moving with velocity relative \( v \) on reference frame the observer \( O \).

Now compute to the observer \( O' \) the distance \( d' = vt' \) between \( O \leftrightarrow O' \):

\[ d' = vt' = v \frac{2L'}{c} \]

Thus we obtain the velocity \( v \):

\[ d' = v \frac{2L'}{c} \Rightarrow v = \frac{cd'}{2L'} \]

Now compute to the observer \( O \) the distance \( d = vt \) between \( O \leftrightarrow O' \):

\[ d = vt = v(t_1 + t_2) = v \frac{2L}{c} \left( \frac{1}{1 - \frac{v^2}{c^2}} \right) \]

Thus we obtain the velocity \( v \):

\[ d = v \frac{2L}{c} \left( \frac{1}{1 - \frac{v^2}{c^2}} \right) \Rightarrow v = \frac{cd}{2L} \left( 1 - \frac{v^2}{c^2} \right) \]

The speed \( v \) is the same to both observers so we have:

\[ v = \frac{cd}{2L'} = \frac{cd}{2L} \left( 1 - \frac{v^2}{c^2} \right) \]

Where applying the relation \( L = L' \sqrt{1 - \frac{v^2}{c^2}} \) we obtain:

\[ \frac{cd}{2L'} = \frac{cd}{2L'} \sqrt{1 - \frac{v^2}{c^2}} \left( 1 - \frac{v^2}{c^2} \right) \Rightarrow d' = d \sqrt{1 - \frac{v^2}{c^2}} \quad d > d' \]

Where the distance \( d \) and \( d' \) varies inversely with the distances \( L \) and \( L' \).
In general, we obtain (14.2, 14.4):

\[ d^a = \frac{\left(1 - \frac{vux}{c^2}\right)}{\sqrt{1 - \frac{v^2}{c^2}}} \]  or  \[ d = \frac{d^a \left(1 + \frac{vux}{c}\right)}{\sqrt{1 - \frac{v^2}{c^2}}} \]

\[ u'x' = \text{zero} \]
\[ d = \frac{d^a \left(1 + \frac{v(0)}{c}\right)}{\sqrt{1 - \frac{v^2}{c^2}}} \]
\[ d = \frac{d^a}{\sqrt{1 - \frac{v^2}{c^2}}} \]

\[ u'x' = c \]
\[ d = \frac{d^a \left(1 + \frac{v}{c}\right)}{\sqrt{1 - \frac{v^2}{c^2}}} \]
\[ d = \frac{d^a \left(1 + \frac{v}{c}\right)}{\sqrt{1 - \frac{v^2}{c^2}}} \]

\[ u'x' = -v \]
\[ d = \frac{d^a \left(1 + \frac{v(-v)}{c^2}\right)}{\sqrt{1 - \frac{v^2}{c^2}}} \]
\[ d = d^a \left(1 - \frac{v^2}{c^2}\right) \]

\[ ux = v \]
\[ d^a = \frac{d^a \left(1 - \frac{v(v)}{c^2}\right)}{\sqrt{1 - \frac{v^2}{c^2}}} \]
\[ d^a = d^a \left(1 - \frac{v^2}{c^2}\right) \]

\[ ux = c \]
\[ d^a = \frac{d^a \left(1 - \frac{v}{c}\right)}{\sqrt{1 - \frac{v^2}{c^2}}} \]
\[ d^a = \frac{d^a \left(1 - \frac{v}{c}\right)}{\sqrt{1 - \frac{v^2}{c^2}}} \]

\[ ux = \text{zero} \]
\[ d^a = \frac{d^a \left(1 - \frac{v(0)}{c^2}\right)}{\sqrt{1 - \frac{v^2}{c^2}}} \]
\[ d^a = \frac{d}{\sqrt{1 - \frac{v^2}{c^2}}} \]
Variables with line \( t', v', x', y', z' \) etc... They are used in §21.

Geometry of space and time in the plan \( xy \rightarrow y \perp x \).

\[
y = f(x) \\
x = ct' \\
y = \int ds' = \int \sqrt{d\tau'.d\tau''} \\
\int ds' = f(ct') \\
dx = c dt' \\
dy = ds' = \sqrt{d\tau'.d\tau''} \\
\ddot{r} = x\hat{i} + y\hat{j} = c dt' \hat{i} + ds' \hat{j} \\
\ddot{r}' = x'\hat{i} + y'\hat{j} \\
d\ddot{r} = dx\hat{i} + dy\hat{j} = c dt' \hat{i} + ds' \hat{j} \\
d\ddot{r}' = dx'\hat{i} + dy'\hat{j} \\
dr = \frac{\ddot{r}.d\ddot{r}'}{r} = dx + \frac{y}{r} dy \\
\ddot{v} = c + \ddot{v}' \\
\ddot{v}' = v'\hat{j} \\
\ddot{a} = \frac{d\ddot{v}}{dt'} = \frac{d\ddot{c}}{dt'} + \ddot{v}' \\
\dddot{c} = \text{zero} \\
\ddot{a} = \ddot{a}' \\
d\dddot{s}' = d\ddot{r}.d\ddot{r}' = (dx\hat{i} + dy\hat{j})(dx'\hat{i} + dy'\hat{j}) = (cdt' \hat{i} + ds' \hat{j})(cdt' \hat{i} + ds' \hat{j}) = dx^2 + dy^2 = c^2 dt'^2 + ds'^2 \\
ds = \sqrt{c^2 dt'^2 + ds'^2} \\
\ddot{v} = \frac{ds'}{dt'} = \sqrt{c^2 + \frac{(ds')^2}{dt'}} > c \\
\ddot{v}' = \frac{ds'}{dt'} = \sqrt{\frac{(ds')^2}{dt'} - c^2} = \sqrt{v'^2 - c^2}
\[ K = \left| \frac{d\phi}{ds} \right| \rightarrow \phi \equiv \angle \] theoretical curve

\[ \tan \phi = \frac{dy}{dx} \quad \phi = \arctan \frac{dy}{dx} \quad \frac{d\phi}{dx} = \frac{d^2y}{dx^2} = \frac{1}{1 + \left(\frac{dy}{dx}\right)^2} \]

\[ \frac{ds}{dx} = \sqrt{1 + \left(\frac{dy}{dx}\right)^2} = \sqrt{1 + \frac{1}{c^2} \left(\frac{ds'}{dt'}\right)^2} \]

\[ K = \frac{d\phi}{ds} = \frac{\frac{d\phi}{ds}}{\frac{ds}{dx}} = \frac{\frac{1}{c^2} \frac{d^2s'}{dt'^2}}{\sqrt{1 + \frac{1}{c^2} \left(\frac{ds'}{dt'}\right)^2}} = \frac{\frac{1}{c^2} \frac{d^2s'}{dt'^2}}{\sqrt{1 + \frac{1}{c^2} \left(\frac{ds'}{dt'}\right)^2}} \left[1 + \frac{1}{c^2} \left(\frac{ds'}{dt'}\right)^2\right]^{\frac{3}{2}} \]

\[ \frac{ds'}{dt'} \frac{d\phi}{ds} = v' K = \frac{v' \frac{d\phi}{ds}}{\left[1 + \frac{1}{c^2} \left(\frac{v'}{c}\right)^2\right]^2} = \frac{1}{c^2} \frac{v' \frac{d\phi}{ds}}{\left[1 + \frac{1}{c^2} \left(\frac{v'}{c}\right)^2\right]^2} \]

\[ \frac{dE_k}{dt'} = \frac{\frac{m_r v_d v}{\sqrt{1 - \frac{v^2}{c^2}}}}{\left(1 + \frac{v'^2}{c^2}\right)^{\frac{3}{2}}} = -\frac{k_r}{r^2} dr = \frac{k_r}{r^2} r d\tilde{r} \]

\[ \frac{dE_k}{dt'} = \tilde{F}', \tilde{v}' = \frac{m_r c^2 \frac{v_d}{c} \frac{d\phi}{ds}}{\left(1 + \frac{v'^2}{c^2}\right)^{\frac{3}{2}}} = \frac{k_r}{r^2} \tilde{r} d\tilde{r}' = \frac{k_r}{r^2} \tilde{r}' \tilde{v}' \]

\[ \frac{dE_k}{dt'} = \tilde{F}', \tilde{v}' = m_r c^2 \tilde{v}' \frac{d\phi}{ds} = \frac{k_r}{r^2} \tilde{r}' \tilde{v}' \]

\[ \tilde{F}' = m_r c^2 \frac{d\phi}{ds} = \frac{k_r}{r^2} \tilde{r} \]

\[ K = \frac{d\phi}{ds} = \frac{k_r}{m_r c^2 r^2} \]

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Variational Principle

\[ E_k = \frac{m_o c^2}{\sqrt{1 - \frac{v^2}{c^2}}} = \frac{k}{r} + \text{constant} \]

\[ E_k = \frac{m_o v^2}{\sqrt{1 - \frac{v^2}{c^2}}} + m_o c^2 \sqrt{1 - \frac{v^2}{c^2}} = \frac{m_o c^2}{\sqrt{1 - \frac{v^2}{c^2}}} = \frac{k}{r} + \text{constant} \]

\[ \frac{m_o v^2}{\sqrt{1 - \frac{v^2}{c^2}}} - m_o c^2 \sqrt{1 - \frac{v^2}{c^2}} = m_o c^2 \]

\[ p = \frac{d}{dv} \left( -m_o c^2 \sqrt{1 - \frac{v^2}{c^2}} \right) = \frac{m_o v}{\sqrt{1 - \frac{v^2}{c^2}}} \]

\[ L = -m_o c^2 \sqrt{1 - \frac{v^2}{c^2}} + \frac{k}{r} \text{ Lagrangeana.} \]

\[ p v - L = m_o c^2 \quad L = p v - m_o c^2 = -m_o c^2 \sqrt{1 - \frac{v^2}{c^2}} + \frac{k}{r} \]

Variational Principle

\[ \mathcal{A} = \int_{t_1}^{t_2} L \left( x(t), x'(t), t \right) dt \]

\[ \dot{x} = \frac{dx}{dt} = u_x \text{ This is the velocity component in x axis.} \]

\[ \delta \mathcal{A} = \int_{t_1}^{t_2} L \left( x, \dot{x}, t \right) dt = 0 \text{ Variation of the action along the X axis.} \]

Building the variable \( x' = x + \epsilon \eta \) in the range \( t_1 \leq t \leq t_2 \) we have seen this when \( \epsilon \to 0 \Rightarrow x' = x \) and where \( \epsilon \neq 0 \) we will have the conditions:

\[ \frac{d\eta}{dt} = 0 \quad \eta = \eta(t) \quad \eta(t_1) = 0 \quad \eta(t_2) = 0 \quad \frac{d\eta}{d\epsilon} = 0 \quad \dot{\eta} = \frac{d\eta}{dt} \]

\[ x' = x + \epsilon \eta \quad \dot{x}' = \dot{x} + \epsilon \dot{\eta} \quad \frac{dx'}{d\epsilon} = \eta \quad \frac{d\dot{x}'}{d\epsilon} = \dot{\eta} \quad \frac{dx}{d\epsilon} = 0 \quad \frac{d\dot{x}}{d\epsilon} = 0 \]

Then we have a new function \( I(\epsilon) = \int_{t_1}^{t_2} G \left( x + \epsilon \eta, \dot{x} + \epsilon \dot{\eta}, t \right) dt = \int_{t_1}^{t_2} F \left( x', \dot{x}', t \right) dt \) and where:

\[ \epsilon = 0 \Rightarrow x' = x \Rightarrow \dot{x}' = \dot{x} \Rightarrow F = \int_{t_1}^{t_2} L \left( x, \dot{x}, t \right) dt \]

\[ \epsilon \neq 0 \Rightarrow x' \neq x \Rightarrow \dot{x}' \neq \dot{x} \Rightarrow F \neq \int_{t_1}^{t_2} L \left( x, \dot{x}, t \right) dt \]
So we have $I(\varepsilon)=\int_{t_1}^{t_2} F[x(\varepsilon),\dot{x}(\varepsilon),t]dt$ that provides derived:

$$\delta I(\varepsilon) = 2\int_{t_1}^{t_2} \frac{\partial F}{\partial x'} dx' + \int_{t_1}^{t_2} \frac{\partial F}{\partial x''} dt + \int_{t_1}^{t_2} \frac{\partial F}{\partial \eta} \delta \eta dt + \int_{t_1}^{t_2} \frac{\partial F}{\partial \eta'} \delta \eta' dt = 0$$

$$\frac{d}{dt}\left(\frac{\partial F}{\partial x'}\eta\right) = \frac{d}{dt}\left(\frac{\partial F}{\partial x''}\right)\eta + \frac{\partial F}{\partial \eta} \frac{d\eta}{dt} + \frac{\partial F}{\partial \eta'} \frac{d\eta'}{dt} = \frac{d}{dt}\left(\frac{\partial F}{\partial x'}\right)\eta$$

$$\delta I(\varepsilon) = \int_{t_1}^{t_2} \frac{\partial F}{\partial x'} \delta \eta dt + \int_{t_1}^{t_2} \frac{\partial F}{\partial x''} dt + \int_{t_1}^{t_2} \frac{d}{dt}\left(\frac{\partial F}{\partial \eta}\right) \delta \eta dt = 0$$

$$\delta I(\varepsilon) = \int_{t_1}^{t_2} \frac{\partial F}{\partial x'} \delta \eta dt + \int_{t_1}^{t_2} \frac{d}{dt}\left(\frac{\partial F}{\partial x''}\right) \delta \eta' dt = 0$$

$$\delta I(\varepsilon) = \int_{t_1}^{t_2} \frac{\partial F}{\partial x'} \delta \eta dt + \int_{t_1}^{t_2} \frac{d}{dt}\left(\frac{\partial F}{\partial \eta}\right) \delta \eta dt = 0$$

$$\delta I(\varepsilon) = \int_{t_1}^{t_2} \frac{\partial F}{\partial x'} \delta \eta dt + \int_{t_1}^{t_2} \frac{d}{dt}\left(\frac{\partial F}{\partial \eta'}\right) \delta \eta' dt = 0$$

$$\varepsilon = \text{zero} \rightarrow x' = x \rightarrow x'' = \dot{x} \rightarrow F = L \Rightarrow \frac{\partial L}{\partial \dot{x}} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) = 0$$

$$\frac{\partial L}{\partial \dot{x}} = 2m_o c^2 \sqrt{1 - \frac{v^2}{c^2}} + k$$

$$\frac{\partial}{\partial \dot{x}}\left(-m_o c^2 \sqrt{1 - \frac{v^2}{c^2}} + k \right) = \frac{d}{dt}\left[\frac{\partial}{\partial \dot{x}}\left(-m_o c^2 \sqrt{1 - \frac{v^2}{c^2}} + k \right)\right]$$

$$\frac{\partial}{\partial \dot{x}}\left(-m_o c^2 \sqrt{1 - \frac{v^2}{c^2}}\right) = 0 \quad \frac{\partial}{\partial \dot{x}}\left(\frac{k}{r}\right) = 0 \quad v = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2} = \sqrt{x^2 + y^2 + z^2}$$

$$\frac{\partial}{\partial \dot{x}}\left(\frac{k}{r}\right) = \frac{d}{dt}\left[\frac{\partial}{\partial \dot{x}}\left(-m_o c^2 \sqrt{1 - \frac{v^2}{c^2}}\right)\right]$$

This is the X axis component

$$\frac{\partial}{\partial \dot{x}}\left(\frac{k}{r}\right) = k\frac{\partial}{\partial \dot{x}}\left(r^{-1}\right) = k(-1)r^{-1-1} \frac{\partial}{\partial \dot{x}} = -k \frac{x}{r^2} = -k \frac{x}{r^3} \quad r^2 = x^2 + y^2 + z^2$$

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\[
\frac{\partial}{\partial x} \left(-m_0 c^2 \sqrt{1-\frac{v^2}{c^2}}\right) = -m_0 c^2 \frac{1}{2} \left(1-\frac{v^2}{c^2}\right)^{1-1} \left(2\frac{dv}{c} \frac{dx}{d\chi}\right) = \frac{m_0 v}{\sqrt{1-\frac{v^2}{c^2}}} d\left(\sqrt{\chi^2 + \dot{\chi}^2 + \ddot{\chi}^2}\right)
\]

\[
\frac{\partial}{\partial x} \left(-m_0 c^2 \frac{1}{2} \left(1-\frac{v^2}{c^2}\right)^{1-1} \left(2\frac{dv}{c} \frac{dx}{d\chi}\right)\right) = \frac{m_0 v}{\sqrt{1-\frac{v^2}{c^2}}} \frac{d}{d\chi} \left(\sqrt{\chi^2 + \dot{\chi}^2 + \ddot{\chi}^2}\right) = \frac{m_0 v}{\sqrt{1-\frac{v^2}{c^2}}} \frac{d\chi}{d\chi} = m_0 \dot{\chi}
\]

\[
\frac{d}{dt} \left(\frac{m_0 \dot{\chi}}{\sqrt{1-\frac{v^2}{c^2}}}\right) = m_0 \frac{dx}{dt} \frac{1}{\sqrt{1-\frac{v^2}{c^2}}} - \dot{\chi} \frac{d}{dt} \left(\frac{1}{\sqrt{1-\frac{v^2}{c^2}}} - \frac{2v}{c} \frac{dx}{d\chi}\right)
\]

\[
\frac{d}{dt} \left(\frac{m_0 \dot{\chi}}{\sqrt{1-\frac{v^2}{c^2}}}\right) = m_0 \frac{dx}{dt} \frac{1}{\sqrt{1-\frac{v^2}{c^2}}} - \dot{\chi} \frac{d}{dt} \left(\frac{1}{\sqrt{1-\frac{v^2}{c^2}}} - \frac{2v}{c} \frac{dx}{d\chi}\right)
\]

\[
\frac{d}{dt} \left(\frac{m_0 \dot{\chi}}{\sqrt{1-\frac{v^2}{c^2}}}\right) = m_0 \frac{dx}{dt} \frac{1}{\sqrt{1-\frac{v^2}{c^2}}} - \dot{\chi} \frac{d}{dt} \left(\frac{1}{\sqrt{1-\frac{v^2}{c^2}}} - \frac{2v}{c} \frac{dx}{d\chi}\right)
\]

\[
-k \frac{\ddot{x}}{r^3} = \frac{m_0}{\left(1-\frac{v^2}{c^2}\right)^{\frac{3}{2}}} \left(\frac{1-\frac{v^2}{c^2}}{c^3} \ddot{x}\right) + v \frac{dv}{dt} \frac{\ddot{x}}{c^3}
\]

\[
-k \frac{\ddot{y}}{r^3} = \frac{m_0}{\left(1-\frac{v^2}{c^2}\right)^{\frac{3}{2}}} \left(\frac{1-\frac{v^2}{c^2}}{c^3} \ddot{y}\right) + v \frac{dv}{dt} \frac{\ddot{y}}{c^3}
\]

\[
-k \frac{\ddot{z}}{r^3} = \frac{m_0}{\left(1-\frac{v^2}{c^2}\right)^{\frac{3}{2}}} \left(\frac{1-\frac{v^2}{c^2}}{c^3} \ddot{z}\right) + v \frac{dv}{dt} \frac{\ddot{z}}{c^3}
\]

\[
-k \frac{\ddot{x}}{r^3} - k \frac{\ddot{y}}{r^3} - k \frac{\ddot{z}}{r^3} = -k \left(\dot{x} + \ddot{y} + \ddot{z}\right) = \frac{-k \ddot{r}}{r^3}
\]

\[
\frac{m_0}{\left(1-\frac{v^2}{c^2}\right)^{\frac{3}{2}}} \left(\frac{1-\frac{v^2}{c^2}}{c^3} \ddot{x}\right) + v \frac{dv}{dt} \frac{\ddot{x}}{c^3} + \frac{m_0}{\left(1-\frac{v^2}{c^2}\right)^{\frac{3}{2}}} \left(\frac{1-\frac{v^2}{c^2}}{c^3} \ddot{y}\right) + v \frac{dv}{dt} \frac{\ddot{y}}{c^3} + \frac{m_0}{\left(1-\frac{v^2}{c^2}\right)^{\frac{3}{2}}} \left(\frac{1-\frac{v^2}{c^2}}{c^3} \ddot{z}\right) + v \frac{dv}{dt} \frac{\ddot{z}}{c^3}
\]

\[
-k \frac{\ddot{x}}{r^3} - k \frac{\ddot{y}}{r^3} - k \frac{\ddot{z}}{r^3} = \frac{-k \ddot{r}}{r^3}
\]

\[
-k \frac{\ddot{x}}{r^3} - k \frac{\ddot{y}}{r^3} - k \frac{\ddot{z}}{r^3} = \frac{-k \ddot{r}}{r^3}
\]
\[\frac{m_o}{(1 - \frac{v^2}{c^2})^{\frac{3}{2}}} \left[ (1 - \frac{v^2}{c^2}) \ddot{x} + v \frac{dv}{dt} \ddot{x} + \left( 1 - \frac{v^2}{c^2} \right) \ddot{y} + v \frac{dv}{dt} \ddot{y} + \left( 1 - \frac{v^2}{c^2} \right) \ddot{z} + v \frac{dv}{dt} \ddot{z} \right] = -\frac{k \ddot{r}}{r^2}\]

\[\frac{m_o}{(1 - \frac{v^2}{c^2})^{\frac{3}{2}}} \left[ (1 - \frac{v^2}{c^2}) (\ddot{x} + \ddot{y} + \ddot{z}) + v \frac{dv}{dt} (\ddot{x} + \ddot{y} + \ddot{z}) \right] = -\frac{k \ddot{r}}{r^2}\]

\[\ddot{a} = \ddot{x} + \ddot{y} + \ddot{z} = \frac{d}{dt} (\ddot{x} + \ddot{y} + \ddot{z}) = \ddot{v} = \dddot{x} + \dddot{y} + \dddot{z}\]

\[\ddot{F} = \frac{m_o}{(1 - \frac{v^2}{c^2})^{\frac{3}{2}}} \left[ (1 - \frac{v^2}{c^2}) \frac{dv}{dt} + v \frac{dv}{dt} \ddot{v} \right] = -\frac{k \ddot{r}}{r^2} = 21.16\]

\[\ddot{F} = \frac{m_o}{(1 - \frac{v^2}{c^2})^{\frac{3}{2}}} \left[ (1 - \frac{v^2}{c^2}) \frac{dv}{dt} + v \frac{dv}{dt} \ddot{v} \right] = -\frac{k \ddot{r}}{r^2} = 21.19\]

\[\text{§24 Variational Principle Continuation}\]

\[E_k = m_o c^2 \sqrt{1 + \frac{v^2}{c^2}} = \frac{m_o c^2}{\sqrt{1 - \frac{v^2}{c^2}}} = \frac{k}{r} + \text{cons tan te}\]

\[E_k = m_o c^2 \sqrt{1 + \frac{v^2}{c^2}} = \frac{m_o v^2}{\sqrt{1 - \frac{v^2}{c^2}}} + m_o c^2 \sqrt{1 - \frac{v^2}{c^2}} = \frac{m_o c^2}{\sqrt{1 - \frac{v^2}{c^2}}} = \frac{k + \text{cons tan te}}{r}\]

\[E_k - \frac{k}{r} = m_o c^2 \sqrt{1 + \frac{v^2}{c^2}} - \frac{k}{r} = \frac{m_o v^2}{\sqrt{1 - \frac{v^2}{c^2}}} + m_o c^2 \sqrt{1 - \frac{v^2}{c^2}} - \frac{k}{r} = \frac{m_o c^2}{\sqrt{1 - \frac{v^2}{c^2}}} - \frac{k}{r} - \frac{k}{r} = \text{cons tan te}\]

\[E_k - \frac{k}{r} = m_o c^2 \sqrt{1 + \frac{v^2}{c^2}} - \frac{k}{r} = \frac{m_o v^2}{\sqrt{1 - \frac{v^2}{c^2}}} = \left( -m_o c^2 \sqrt{1 - \frac{v^2}{c^2}} + \frac{k}{r} \right) = m_o c^2 = \text{cons tan te}\]

\[T' = m_o c^2 \sqrt{1 + \frac{v^2}{c^2}} \quad T = -m_o c^2 \sqrt{1 - \frac{v^2}{c^2}} \quad E_p = -\frac{k}{r} \quad p v = \frac{m_o v^2}{\sqrt{1 - \frac{v^2}{c^2}}} \]

\[p v = \frac{m_o v}{\sqrt{1 - \frac{v^2}{c^2}}} v = v' p' \quad \text{p} = p' \sqrt{1 + \frac{v'^2}{c^2}} \quad \text{p}' = p' \sqrt{1 + \frac{v^2}{c^2}}\]

\[E_R = E_k + E_p = T + E_p = p v - (T - E_p)\]

\[E_k = T' \quad E_k = p v - T \quad T' = p v - T \quad T = p' v' - T'\]
\[ L' = T' + E_p \quad L = T - E_p \]

\[ E_R = E_k + E_p = L' = pv - L \]

\[ L' = pv - L \quad L = pv' - L' \quad L + L' = pv = pv' \]

\[ p' = \frac{dT}{dv'} = \frac{d}{dv'} \left( m_o c^2 \sqrt{1 + \frac{v'^2}{c^2}} \right) = \frac{m_o v'}{\sqrt{1 + \frac{v'^2}{c^2}}} = m_o v \]

\[ p = \frac{dT}{dv} = \frac{d}{dv} \left( -m_o c^2 \sqrt{1 - \frac{v^2}{c^2}} \right) = \frac{m_o v}{\sqrt{1 - \frac{v^2}{c^2}}} \]

\[ \ddot{d} = dx \hat{i} + dy \hat{j} + dz \hat{k} = -dx \hat{i} - dy \hat{j} - dz \hat{k} = -\ddot{r} \]

\[ \vec{v} = \ddot{r} = dx \frac{\dot{x}}{dt'} + dy \frac{\dot{y}}{dt'} + dz \frac{\dot{z}}{dt'} = -\frac{1}{\sqrt{1 - \frac{v^2}{c^2}}} \left( dx \hat{i} + dy \hat{j} + dz \hat{k} \right) = -\frac{1}{\sqrt{1 - \frac{v^2}{c^2}}} \ddot{r} \]

\[ \dot{x}' = v_x = \frac{dx'}{dt'} = -\frac{1}{\sqrt{1 - \frac{v^2}{c^2}}} \frac{dx}{dt} = -\frac{\dot{x}}{\sqrt{1 - \frac{v^2}{c^2}}} \]

\[ \dot{v}' = \ddot{v} = \frac{dv'}{dt'} = -\frac{1}{\sqrt{1 - \frac{v^2}{c^2}}} \frac{dv}{dt} = -\frac{\ddot{v}}{\sqrt{1 - \frac{v^2}{c^2}}} \]

\[ \dot{p}' = \frac{dp'}{dx'} = \frac{d}{dx'} \left( m_o c^2 \sqrt{1 + \frac{v'^2}{c^2}} \right) = \frac{m_o \dot{x}'}{\sqrt{1 + \frac{v'^2}{c^2}}} = -m_o \ddot{x} \]

\[ p_x = \frac{dT}{dx} = \frac{d}{dx} \left( -m_o c^2 \sqrt{1 - \frac{v^2}{c^2}} \right) = \frac{m_o \dot{x}}{\sqrt{1 - \frac{v^2}{c^2}}} \]

\[ \ddot{r} = x \hat{i} + y \hat{j} + z \hat{k} = -\dot{x} \hat{i} - \dot{y} \hat{j} - \dot{z} \hat{k} = -\ddot{r} \]

\[ x' = -x \quad y' = -y \quad z' = -z \]

\[ \frac{\partial v'}{\partial x} = -1 \quad \frac{\partial v'}{\partial y} = -1 \quad \frac{\partial v'}{\partial z} = -1 \]

\[ \frac{\partial L}{\partial x} - \frac{d}{dt} \left( \frac{\partial L}{\partial \dot{x}} \right) = 0 \]

\[ \frac{\partial L}{\partial x} - \frac{d}{dt} \left( \frac{\partial L}{\partial \dot{x}} \right) = \frac{\partial x}{\partial \dot{x}} \frac{d}{dt} \frac{d}{dt} \frac{\partial L}{\partial \dot{x}} = 0 \quad L = pv' - L' \quad \frac{\partial L}{\partial x} = \frac{\partial T}{\partial \dot{x}} = p_x = -m_o \ddot{x} \]

\[ -\frac{\partial}{\partial x} (p'v' - L') - \frac{m_o}{\sqrt{1 + \frac{v'^2}{c^2}}} \frac{d}{dt'} (p'v' - L') - \frac{d}{dt'} \left( \frac{m_o}{\sqrt{1 + \frac{v'^2}{c^2}}} \frac{d}{dt'} \right) = 0 \]

\[ \frac{\partial p'}{\partial x} = 0 \quad \frac{\partial v'}{\partial x} = 0 \quad L' = m_o c^2 \sqrt{\frac{1 + \frac{v'^2}{c^2}}{r}} \]

\[ \frac{\partial L'}{\partial x} = \frac{m_o}{\sqrt{1 + \frac{v'^2}{c^2}}} \frac{d}{dt'} = 0 \quad r^2 = \ddot{r} \cdot \ddot{r} = (-\ddot{r}) \cdot (-\ddot{r}) = x'^2 + y'^2 + z'^2 = x^2 + y^2 + z^2 \]

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\[
\frac{\partial L'}{\partial x'} = \frac{\partial}{\partial x'} \left( m_o c^2 \sqrt{1 + \frac{v'^2}{c^2}} - \frac{k}{r} \right) = -k \frac{\partial}{\partial x'} \left( r^{-1} \right) = -k(1) r^{-1-1} \frac{\partial}{\partial x'} = k \frac{1}{r^2} x' = k \frac{x'}{r^2}
\]

\[
\frac{\partial L'}{\partial x'} + \frac{m_o}{\sqrt{1 + \frac{v'^2}{c^2}}} \frac{dx'}{dt} = m_o \frac{\dot{x}'}{r^3} + \frac{m_o \dot{x}'}{\sqrt{1 + \frac{v'^2}{c^2}}} = 0
\]

\[
\frac{m_o \ddot{x}'}{1 + \frac{v'^2}{c^2}} = -k \frac{x'}{r^3} - k \frac{\dot{x}' + y' + z' k}{r^3}, k = \frac{r^3 (x' + y' + z' k)}{r^3} = k \frac{r^3}{r^3} = k \frac{r^2}{r^2}
\]

\[
\frac{m_o \ddot{x}'}{1 + \frac{v'^2}{c^2}} + \frac{m_o \dot{y}'}{1 + \frac{v'^2}{c^2}} + \frac{m_o \dot{z}'}{1 + \frac{v'^2}{c^2}} \hat{k} = \frac{-m_o \ddot{a}'}{r^2} = -k \frac{\dot{r}'}{r^2}
\]

\[
\frac{m_o \ddot{a}'}{1 + \frac{v'^2}{c^2}} = \frac{k \dot{r}}{r^2}, \quad -\dot{r}' = \dot{r}, \quad -\frac{m_o \ddot{a}'}{r^2} = \frac{k \dot{r}'}{r^2} = 21.19
\]

\[\text{§25 Logarithmic spiral}\]

\[H \frac{d^2 w}{d\phi^2} + Hw + 3A \frac{d^2 w}{d\phi^2} + 3Aw^2 - B = 0 \quad r = e^{a\phi} \]

\[w = \frac{1}{r} = \frac{1}{\varepsilon D} [1 + \varepsilon \cos(\phi Q)] \quad \frac{dw}{d\phi} = -Q \sin(\phi Q) \quad \frac{d^2 w}{d\phi^2} = -\frac{Q^2 \cos(\phi Q)}{D} \]

\[w = \frac{1}{r} = \frac{1}{e^{-a\phi}} \quad \frac{dw}{d\phi} = -ae^{-a\phi} \quad \frac{d^2 w}{d\phi^2} = a^2 e^{-a\phi} \]

\[Ha^2 e^{-a\phi} + He^{-a\phi} + 3Aa^2 e^{-a\phi} + 3A(e^{-a\phi})^2 - B = 0\]

\[Ha^2 e^{-a\phi} + He^{-a\phi} + 3Aa^2 e^{-2a\phi} + 3Ae^{-2a\phi} - B = 0\]

\[(1 + a^2)He^{-a\phi} + (1 + a^2)Bae^{-2a\phi} - B = 0\]

\[(1 + a^2)Bae^{-2a\phi} + (1 + a^2)He^{-a\phi} - B = 0\]

\[(1 + a^2)BAe^{-a\phi} + (1 + a^2)Hw - B = 0\]

\[3Aw^2 + Hw - \frac{B}{(1 + a^2)} = 0\]

\[w = e^{-a\phi} = \frac{1}{r} = -H \pm \sqrt{\frac{H^2 + 4.3A}{(1 + a^2)}} = -H \pm \frac{1}{6A} \sqrt{\frac{H^2 + 12AB}{(1 + a^2)}}\]

\[3A \left[ \frac{-H + \frac{1}{6A} \sqrt{H^2 + 12AB}}{(1 + a^2)} \right]^2 + H \left[ \frac{-H + \frac{1}{6A} \sqrt{H^2 + 12AB}}{(1 + a^2)} \right] - \frac{B}{(1 + a^2)} = 0\]

122/172
\[
3A \left[ \left( \frac{-H}{6A} \right)^2 \pm 2 \left( \frac{-H}{6A} \right) \frac{1}{6A} \sqrt{H^2 + \frac{12AB}{(1+a^2)}} + \left( \frac{1}{6A} \right) \sqrt{H^2 + \frac{12AB}{(1+a^2)}} \right]^2 - \frac{-H^2 \pm H}{6A} \sqrt{H^2 + \frac{12AB}{(1+a^2)}} - \frac{B}{(1+a^2)} = \text{zero}
\]

\[
3A \left[ \left( \frac{-H}{6A} \right)^2 \pm 2 \left( \frac{-H}{6A} \right) \frac{1}{6A} \sqrt{H^2 + \frac{12AB}{(1+a^2)}} + \left( \frac{1}{36A^2} \right) \left( H^2 + \frac{12AB}{(1+a^2)} \right) \right]^2 - \frac{-H^2 \pm H}{6A} \sqrt{H^2 + \frac{12AB}{(1+a^2)}} - \frac{B}{(1+a^2)} = \text{zero}
\]

\[
3A \left[ \left( \frac{H^2}{36A^2} \right) \pm - \frac{H}{18A^2} \sqrt{H^2 + \frac{12AB}{(1+a^2)}} + \left( \frac{1}{12A} \right) \left( H^2 + \frac{12AB}{(1+a^2)} \right) \right] - \frac{-H^2 \pm H}{6A} \sqrt{H^2 + \frac{12AB}{(1+a^2)}} - \frac{B}{(1+a^2)} = \text{zero}
\]

\[
\frac{H^2}{12A} \left( \frac{H^2}{12A} + \frac{1}{12A} \right) - \frac{H}{6A} \frac{B}{(1+a^2)} = \text{zero}
\]

\[
\frac{H^2}{12A} + \frac{H}{12A} \frac{B}{(1+a^2)} - \frac{H}{6A} \frac{B}{(1+a^2)} = \text{zero}
\]

Section 25 Logarithmic Spiral (Continuation)

\[
- \left( H + A \frac{1}{r} \right)^3 \left( \frac{d^2 w}{d\phi^2} + \frac{1}{r} \right) = \frac{GM_o}{L^2}
\]

\[
\left( H + A \frac{1}{r} \right)^3 \left( \frac{d^2 w}{d\phi^2} + \frac{1}{r} \right) = - \frac{GM_o}{L^2}
\]

\[
\left( H + A \frac{1}{r} \right)^3 \left( \frac{d^2 w}{d\phi^2} + \frac{1}{r} \right) = -B
\]

\[
H = \frac{E_r}{m_o c^2}, \quad A = \frac{GM_o}{c^2}, \quad B = \frac{GM_o}{L^2}
\]

\[
\left( H + A \frac{1}{r} \right)^3 \left( \frac{d^2 w}{d\phi^2} + \frac{1}{r} \right) + B = \text{zero}
\]

\[
H^3 + 3H^2 A \frac{1}{r} + 3HA^2 \frac{1}{r^2} + A^3 \frac{1}{r} \left( \frac{d^2 w}{d\phi^2} + \frac{1}{r} \right) + B = \text{zero}
\]

\[
H^3 + 3H^2 A \frac{1}{r} + 3HA^2 \frac{1}{r^2} + A^3 \frac{1}{r} \approx H^3 + 3H^2 A \frac{1}{r}
\]

\[
3HA^2 \frac{1}{r^2} + A^3 \frac{1}{r} \approx \text{zero}
\]
\[
\begin{align*}
\left( H^3 + 3AH^2 \frac{1}{r} \left( \frac{d^2 w}{d\phi^2} + \frac{1}{r} \right) \right) + B &= 0 \\
H^3 + 3AH^2 w \left( \frac{d^2 w}{d\phi^2} + w \right) + B &= 0 \\
H^3 \frac{d^2 w}{d\phi^2} + H^3 w + 3AH^2 \frac{d^2 w}{d\phi^2} w + 3AH^2 w^2 + B &= 0
\end{align*}
\]

\[
w = \frac{1}{r} \cos(\phi) \left[ 1 + \epsilon \cos(\phi) \right] \\
dw = -Q \sin(\phi) \\
\frac{d^2 w}{d\phi^2} = -Q^2 \cos(\phi) \\
21.38
\]

\[
\begin{align*}
H^3 \left[ -Q^2 \cos(\phi) + \frac{1}{\epsilon D} \left( 1 + \epsilon \cos(\phi) \right) + 3AH^2 \left[ -Q^2 \cos(\phi) \right] \frac{1}{\epsilon D} \left( 1 + \epsilon \cos(\phi) \right) + \\
+ 3AH^2 \left[ \frac{1}{\epsilon^2 D^2} \left( 1 + 2 \epsilon \cos(\phi) + \epsilon^2 \cos^2(\phi) \right) \right] \right] + B &= 0
\end{align*}
\]

\[
\begin{align*}
-H^3 Q^2 \cos(\phi) + H^3 \frac{1}{\epsilon D} + H^3 \frac{1}{\epsilon D} \epsilon \cos(\phi) + 3AH^2 \left[ -Q^2 \cos(\phi) \right] \frac{1}{\epsilon D} + 3AH^2 \left[ -Q^2 \cos(\phi) \right] \frac{1}{\epsilon D} \epsilon \cos(\phi) + \\
+ 3AH^2 \left[ \frac{1}{\epsilon^2 D^2} \left( 1 + 2 \epsilon \cos(\phi) + \epsilon^2 \cos^2(\phi) \right) \right] \right] + B &= 0
\end{align*}
\]

\[
\begin{align*}
-H^3 Q^2 \cos(\phi) + H^3 \frac{1}{\epsilon D} + H^3 \frac{1}{\epsilon D} \cos(\phi) - 3AH^2 Q^2 \cos(\phi) - 3AH^2 Q^2 \cos^2(\phi) + \\
+ 3AH^2 \left[ \frac{1}{\epsilon^2 D^2} \left( 1 + 2 \epsilon \cos(\phi) + \epsilon^2 \cos^2(\phi) \right) \right] \right] + B &= 0
\end{align*}
\]

\[
\begin{align*}
-H^3 Q^2 \cos(\phi) + H^3 \frac{1}{\epsilon D} + H^3 \frac{1}{\epsilon D} \cos(\phi) - 3AH^2 Q^2 \cos(\phi) - 3AH^2 Q^2 \cos^2(\phi) + \\
+ 3AH^2 \left[ \frac{1}{\epsilon^2 D^2} \left( 1 + 2 \epsilon \cos(\phi) + \epsilon^2 \cos^2(\phi) \right) \right] \right] + B &= 0
\end{align*}
\]

\[
\begin{align*}
-H^3 Q^2 \cos(\phi) + H^3 \frac{1}{\epsilon D} + H^3 \frac{1}{\epsilon D} \cos(\phi) - 3AH^2 Q^2 \cos(\phi) - 3AH^2 Q^2 \cos^2(\phi) + \\
+ 3AH^2 \left[ \frac{1}{\epsilon^2 D^2} \left( 1 + 2 \epsilon \cos(\phi) + \epsilon^2 \cos^2(\phi) \right) \right] \right] + B &= 0
\end{align*}
\]

\[
\begin{align*}
-H^3 Q^2 \cos(\phi) + H^3 \frac{1}{\epsilon D} + H^3 \frac{1}{\epsilon D} \cos(\phi) - 3AH^2 Q^2 \cos(\phi) - 3AH^2 Q^2 \cos^2(\phi) + \\
+ 3AH^2 \left[ \frac{1}{\epsilon^2 D^2} \left( 1 + 2 \epsilon \cos(\phi) + \epsilon^2 \cos^2(\phi) \right) \right] \right] + B &= 0
\end{align*}
\]

\[
\begin{align*}
-H^3 Q^2 \cos(\phi) + H^3 \frac{1}{\epsilon D} + H^3 \frac{1}{\epsilon D} \cos(\phi) - 3AH^2 Q^2 \cos(\phi) - 3AH^2 Q^2 \cos^2(\phi) + \\
+ 3AH^2 \left[ \frac{1}{\epsilon^2 D^2} \left( 1 + 2 \epsilon \cos(\phi) + \epsilon^2 \cos^2(\phi) \right) \right] \right] + B &= 0
\end{align*}
\]

\[
\begin{align*}
\frac{h}{3A} \frac{1}{\epsilon D} + \frac{H}{3A} \frac{1}{\epsilon D} \cos(\phi) - \frac{Q^2 \cos^2(\phi)}{\epsilon D} + \frac{1}{3A} \frac{1}{\epsilon D} + \frac{2 \cos(\phi) + \cos^2(\phi)}{\epsilon D} + \frac{B}{3A} &= 0
\end{align*}
\]
\[
\begin{align*}
\frac{\cos^2(\phi Q)}{D^2} - \frac{Q^2 \cos^2(\phi Q)}{D^2} - \frac{H^2 \cos(\phi Q)}{3A} + \frac{H \cos(\phi Q)}{3A} - \frac{Q^2 \cos(\phi Q)}{D^2} + \\
\frac{2}{\epsilon D} \frac{\cos(\phi Q)}{D} + \frac{H}{3A} \epsilon D + \frac{1}{\epsilon^2 D^2} + \frac{B}{3AH^2} = & \text{ zero} \\
\left(1 - Q^2\right) \frac{\cos^2(\phi Q)}{D^2} + \left(1 - \frac{Q^2}{3A} + \frac{H^2}{3A} + \frac{Q^2}{3A} + \frac{2}{\epsilon D} \right) \frac{\cos(\phi Q)}{D} + \frac{H}{3A} \epsilon D + \frac{1}{\epsilon^2 D^2} + \frac{B}{3A} = & \text{ zero} \\
\left(1 - Q^2\right) \frac{\cos^2(\phi Q)}{D^2} + \left(1 - \frac{Q^2}{3A} + \frac{H^2}{3A} + \frac{Q^2}{3A} + \frac{2}{\epsilon D} \right) \frac{\cos(\phi Q)}{D} - \frac{1}{3A} \epsilon D + \frac{1}{\epsilon^2 D^2} + \frac{B}{3A} = & \text{ zero} \\
\left(1 - Q^2\right) \frac{\cos^2(\phi Q)}{D^2} + \left(1 - \frac{Q^2}{3A} + \frac{H^2}{3A} + \frac{Q^2}{3A} + \frac{2}{\epsilon D} \right) \frac{\cos(\phi Q)}{D} + \frac{1}{\epsilon^2 D^2} = & \text{ zero} \\
\left(1 - \frac{6A}{\epsilon D}\right) \frac{\cos^2(\phi Q)}{D^2} + \left(1 - \frac{6A}{\epsilon D}\right) \frac{\cos(\phi Q)}{D} + \frac{1}{\epsilon^2 D^2} = & \text{ zero} \\
\left(1 - \frac{6A}{\epsilon D}\right) \frac{\cos^2(\phi Q)}{D^2} + \left(1 - \frac{6A}{\epsilon D}\right) \frac{\cos(\phi Q)}{D} + \frac{1}{\epsilon^2 D^2} = & \text{ zero} \\
\left(1 - \frac{6A}{\epsilon D}\right) \frac{\cos^2(\phi Q)}{D^2} + \left(1 - \frac{6A}{\epsilon D}\right) \frac{\cos(\phi Q)}{D} + \frac{1}{\epsilon^2 D^2} = & \text{ zero} \\
\left(1 - \frac{6A}{\epsilon D}\right) \frac{\cos^2(\phi Q)}{D^2} + \left(1 - \frac{6A}{\epsilon D}\right) \frac{\cos(\phi Q)}{D} + \frac{1}{\epsilon^2 D^2} = & \text{ zero} \\
\end{align*}
\]
\[(6A) \left( \frac{1}{\varepsilon D} \right) \cos^2(\phi Q) + \left( -1 + \frac{6A}{\varepsilon D} \right) \frac{1}{\varepsilon D} \cos(\phi Q) + \left( \frac{1}{\varepsilon D} \right) \left( \frac{1}{\varepsilon D} \right) = 0 \]

\[(6A) \frac{\cos^2(\phi Q)}{D^2} + \left( -1 + \frac{6A}{\varepsilon D} \right) \frac{\cos(\phi Q)}{D} + \frac{1}{\varepsilon D} = 0 \]

\[
\frac{\cos(\phi Q)}{D} = \frac{-\left[ -1 + \frac{6A}{\varepsilon D} \right] \pm \sqrt{\left( -1 + \frac{6A}{\varepsilon D} \right)^2 - 4 \cdot \left( \frac{6A}{\varepsilon D} \right)^2}}{2.6A}
\]

\[
\frac{\cos(\phi Q)}{D} = \frac{-\left[ -1 + \frac{6A}{\varepsilon D} \right] \pm \sqrt{(-1)^2 + 2(-1) \cdot \frac{6A}{\varepsilon D} + \left( \frac{6A}{\varepsilon D} \right)^2 - \frac{24A}{\varepsilon}}}{12A}
\]

\[
\frac{\cos(\phi Q)}{D} = \frac{-\left[ -1 + \frac{6A}{\varepsilon D} \right] \pm \sqrt{\frac{1 - 12A + (\frac{6A}{\varepsilon D})^2 - 24A}{12A}}}{12A}
\]

\[
\frac{\cos(\phi Q)}{D} = \frac{-\left[ -1 + \frac{6A}{\varepsilon D} \right] \pm \sqrt{\frac{1 - 36A + 36A^2}{12A}}}{12A}
\]

\[
\sqrt{1 - \frac{36A}{\varepsilon^2 D^2}} \approx \sqrt{1 - \frac{36A}{\varepsilon^2 D^2}} = \frac{36A^2}{\varepsilon^2 D^2} \approx \text{zero}
\]

\[
A = \frac{GM_o}{c^2}
\]

\[
36A^2 = \frac{36}{\varepsilon^2 D^2} \left( \frac{GM_o}{c^2} \right)^2 = \frac{36}{(55.442.955.600)^2} \left[ \frac{6.67 \times 10^{-11} \times 1.989.10^{30}}{(2.99792458.10^8)^2} \right]^2 = 2.55 \times 10^{-14}
\]

\[
\frac{\cos(\phi Q)}{D} = \frac{-\left[ -1 + \frac{6A}{\varepsilon D} \right] \pm \sqrt{\frac{1 - 36A}{\varepsilon D}}}{12A}
\]

\[
\frac{\cos(\phi Q)}{D} = \frac{-\left[ -1 + \frac{6A}{\varepsilon D} \right] + \sqrt{\frac{1 - 36A}{\varepsilon D}}}{12A} = \frac{-\left[ -1 + \frac{6A}{\varepsilon D} \right] - \sqrt{\frac{1 - 36A}{\varepsilon D}}}{12A} = \frac{-\left[ -1 + \frac{18A}{\varepsilon D} \right]}{12A} = \frac{-\left[ 1 + \frac{18A}{\varepsilon D} \right]}{12A}
\]

\[
-\frac{\cos(\phi Q)}{D} + \frac{1}{\varepsilon D} = 0
\]
zero \times r(\phi Q) < \infty \rightarrow M_o \neq zero \rightarrow Q = \sqrt{1 - \frac{6A}{\varepsilon D}} - \frac{\cos(\phi Q)}{D} + \frac{1}{\varepsilon D} = zero

\left(1 - Q^2\right) \frac{\cos^2(\phi Q)}{D^2} + \left(\frac{Q^2}{3A} - \frac{1}{3A} - \frac{Q^2}{\varepsilon D} + \frac{2}{\varepsilon D}\right) \frac{\cos(\phi Q)}{D} + \frac{1}{\varepsilon^2 D^2} = zero

r = \infty \rightarrow M_o = zero \rightarrow Q = 1

Q = \sqrt{1 - \frac{6A}{\varepsilon D}} = \sqrt{1 - \frac{6}{\varepsilon D} \left(\frac{GM_o}{c^2}\right)} = \sqrt{1 - \frac{6}{\varepsilon D} \left(G(\text{zero})\right)} = 1

(1-1) \frac{\cos^2(\phi Q)}{D^2} + \left(\frac{1}{3A} - \frac{1}{3A} - \frac{1}{\varepsilon D} + \frac{2}{\varepsilon D}\right) \frac{\cos(\phi Q)}{D} + \frac{1}{\varepsilon^2 D^2} = zero

\frac{1}{\varepsilon D} \frac{\cos(\phi Q)}{D} + \frac{1}{\varepsilon D} = zero

r = \infty \rightarrow M_o = zero \rightarrow Q = 1 \rightarrow w = \frac{1}{r = \infty} = \frac{1}{\varepsilon D} \left[1 + \varepsilon \cos(\phi Q)\right] = \frac{\cos(\phi Q)}{D} + \frac{1}{\varepsilon D} = zero

The presence of Q in the formula \( r = r(\phi Q) = \frac{\varepsilon D}{1 + \varepsilon \cos(\phi Q)} \), allows it to also describe a spiral.

The presence of Q in the formula \( r = r(\phi Q) = \frac{\varepsilon D}{1 + \varepsilon \cos(\phi Q)} \), allows it to also describe a spiral.

§25 Logarithmic Spiral Continuation II

\left(1 - Q^2\right) \frac{\cos^2(\phi Q)}{D^2} + \left(\frac{Q^2}{3A} - \frac{1}{3A} - \frac{Q^2}{\varepsilon D} + \frac{2}{\varepsilon D}\right) \frac{\cos(\phi Q)}{D} + \frac{1}{\varepsilon^2 D^2} = zero

zero < r(\phi Q) < \infty \rightarrow M_o \neq zero \rightarrow Q = \sqrt{1 - \frac{12A}{\varepsilon D}} - \frac{1}{\varepsilon D} \sqrt{1 - \frac{6A}{\varepsilon D}}
\[
\left[1 - \left(1 - \frac{12A}{\varepsilon D}ight) \cos^2(\phi Q) \right] + \left[ 1 + \left(1 - \frac{12A}{\varepsilon D} \right) \cos(\phi Q) \right] + \frac{2}{\varepsilon^2 D^2} = 0
\]

\[
\left[1 - \frac{6A}{\varepsilon D} \right] \cos^2(\phi Q) + \left[ 1 + \left(1 - \frac{12A}{\varepsilon D} \right) \cos(\phi Q) \right] + \frac{2}{\varepsilon^2 D^2} \left(1 - \frac{6A}{\varepsilon D} \right) = 0
\]

\[
\left(1 - \frac{6A}{\varepsilon D} - 12A \right) \cos^2(\phi Q) + \left[ 1 + \left(1 - \frac{12A}{\varepsilon D} \right) \cos(\phi Q) \right] + \frac{1}{\varepsilon^2 D^2} \left(1 - \frac{6A}{\varepsilon D} \right) = 0
\]

\[
\frac{6A}{\varepsilon D} \cos^2(\phi Q) + \left(1 - \frac{1}{\varepsilon D} \right) \cos(\phi Q) + \frac{1}{\varepsilon^2 D^2} - \frac{1}{\varepsilon} 6A = 0
\]

\[
\cos(\phi Q) = \frac{-\frac{1}{\varepsilon D} \pm \sqrt{\left(-\frac{1}{\varepsilon D}\right)^2 - 4 \frac{6A}{\varepsilon D} \left(\frac{1}{\varepsilon^2 D^2} - \frac{1}{\varepsilon} 6A\right)}}{2 \frac{6A}{\varepsilon D}}
\]

\[
\cos(\phi Q) = \frac{1 + \sqrt{-\frac{1}{\varepsilon^2 D^2} - \frac{24A}{\varepsilon D} \left(-\frac{1}{\varepsilon^2 D^2} - \frac{1}{\varepsilon} 6A\right)}}{\frac{12A}{\varepsilon D}}
\]

\[
\cos(\phi Q) = \frac{1 + \sqrt{-\frac{1}{\varepsilon^2 D^2} - \frac{24A}{\varepsilon D} \left(-\frac{1}{\varepsilon^2 D^2} - \frac{1}{\varepsilon} 6A\right)}}{\frac{12A}{\varepsilon D}}
\]

\[
\cos(\phi Q) = \frac{1 + \sqrt{-\frac{1}{\varepsilon^2 D^2} - \frac{24A}{\varepsilon D} \left(-\frac{1}{\varepsilon^2 D^2} - \frac{1}{\varepsilon} 6A\right)}}{\frac{12A}{\varepsilon D}}
\]

\[
\cos(\phi Q) = \frac{1 + \sqrt{-\frac{1}{\varepsilon^2 D^2} - \frac{24A}{\varepsilon D} \left(-\frac{1}{\varepsilon^2 D^2} - \frac{1}{\varepsilon} 6A\right)}}{\frac{12A}{\varepsilon D}}
\]
\[
\begin{align*}
\cos(\phi Q) &= \frac{1}{eD} - \left( \frac{1}{eD} - \frac{1}{eD} \frac{12A}{eD} \right) \\
\cos(\phi Q) &= \frac{1}{eD} - \frac{1 + 12A}{eD} \\
\frac{\cos(\phi Q)}{D} &= \frac{1}{eD} - \frac{1}{eD} \frac{12A}{eD} \\
\cos(\phi Q) &= \frac{1}{eD} \\
-\frac{\cos(\phi Q)}{D} + \frac{1}{eD} &= \text{zero}
\end{align*}
\]

\[
\begin{align*}
\text{zero} < r(\phi Q) < \infty \Rightarrow M_o \neq \text{zero} \Rightarrow Q &= \sqrt{\frac{1 - \frac{12A}{eD}}{\sqrt{1 - \frac{6A}{eD}}}} \\
\cos(\phi Q) &= \frac{1}{eD} - \frac{1}{eD} \frac{12A}{eD} \\
A &= \frac{GM_0}{c^2} \\
eD &= a(1 - e^2) = 57.909.227.000.00 \left[ (0.20563593^2 \right] = 55.460.469.568.40 \\
A &= \frac{GM_0}{c^2} = \frac{6.6740831.10^{-11}, 1.9891.10^{30}}{2,99792458.10^8} = 1.477,089,535.42 \\
Q &= \sqrt{\frac{1 - \frac{12A}{eD}}{1 - \frac{6A}{eD}}} = 0.999,999,920.1 \\
1.276.789.102.53^{14}
\end{align*}
\]

\[
\begin{align*}
\phi Q &= 1.296.000.00 \Rightarrow \phi = \frac{1.296.000.00}{Q} \\
\phi < 1 & \text{ Advance} \\
\phi > 1 & \text{ Regression}
\end{align*}
\]

\[
\begin{align*}
\Delta \phi &= \left( \frac{1}{Q} - 1 \right) 1.296.000.00 \\
\Delta \phi > 0 & \text{ Advance} \\
\Delta \phi < 0 & \text{ Regression}
\end{align*}
\]
\[ \Delta \phi = \frac{1}{\left( \frac{1-12A}{\varepsilon D} \right)^2 - 1} \approx 1.296.000,00 = 0,103.549.893.544'' \]

\[ \Delta \phi = \frac{1}{\left( \frac{1-6A}{\varepsilon D} \right)^2 - 1} \approx 1.296.000,00 = 0,103.549.876.997'' \]

\[ N = 100 \frac{PT}{PM} \approx 100 \frac{365.256.363.004}{87.969} = 415,210.316.139 \]

\[ \sum \Delta \phi = \Delta \phi N = 0,103.549.893.544 \times 415,210.316.139 = 42,994.984.034.7'' \]

\[ \sum \Delta \phi = \Delta \phi N = 0,103.549.876.997 \times 415,210.316.139 = 42,994.977.164.2'' \]

By definition \( \varepsilon > 0 \)

\[ \text{zero} < r(\phi Q) < \infty \rightarrow M_0 \# \text{zero} \rightarrow Q = \frac{1-12A}{\varepsilon D} \quad r = \infty \rightarrow M_0 = \text{zero} \rightarrow Q = 1 \]

\[ -\cos(\phi Q) + \frac{1}{\varepsilon D} = \text{zero} \rightarrow \varepsilon = -\frac{1}{\cos(\phi Q)} \quad \cos(\phi Q) + \frac{1}{\varepsilon D} = \text{zero} \rightarrow \varepsilon = -\frac{1}{\cos(\phi Q)} \]

Se If \( Q = 1 \)

\[
\begin{bmatrix}
\varepsilon = -\frac{1}{\cos(\phi - \pi)} \\
\varepsilon = -\frac{1}{\cos(\phi)}
\end{bmatrix}
\]

Energy Newtonian (\( E_n \))

\[
\left( 1-Q^2 \right) \cos^2(\phi Q) + \left( x - \frac{2}{\varepsilon D} \right) \cos(\phi Q) + \frac{Q^2}{\varepsilon D^2} + \frac{1}{\varepsilon D^2} - \frac{x}{\varepsilon D} - y = \text{zero}
\]

\[
r = \infty \rightarrow Q = 1 \rightarrow w = \frac{1}{r = \infty} = -\frac{1}{\varepsilon D} \left[ \frac{1+\varepsilon \cos(\phi Q)}{\varepsilon D} \right] = \frac{\cos(\phi Q)}{\varepsilon D} + \frac{1}{\varepsilon D} = \text{zero}
\]

\[
\left( 1-Q^2 \right) \left( \frac{1}{\varepsilon D} \right) \left( \frac{1}{\varepsilon D} \right) + \left( x - \frac{2}{\varepsilon D} \right) \left( \frac{1}{\varepsilon D} \right) + \frac{Q^2}{\varepsilon^2 D^2} + \frac{1}{\varepsilon^2 D^2} - \frac{x}{\varepsilon D} - y = \text{zero}
\]

\[
\left( 1-Q^2 \right) \left( \frac{1}{\varepsilon^2 D^2} \right) \left( \frac{1}{\varepsilon D} \right) + \frac{2}{\varepsilon D} + \frac{Q^2}{\varepsilon^2 D^2} + \frac{1}{\varepsilon^2 D^2} - \frac{x}{\varepsilon D} - y = \text{zero}
\]

\[
\frac{1}{\varepsilon^2 D^2} \left( \frac{Q^2}{\varepsilon D} + \frac{2}{\varepsilon D} + \frac{Q^2}{\varepsilon^2 D^2} + \frac{1}{\varepsilon^2 D^2} - \frac{x}{\varepsilon D} - y = \text{zero} \right)
\]

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\[- \frac{Q^2}{\varepsilon D} + \frac{Q^2}{\varepsilon D} + \frac{4}{\varepsilon D} - \frac{2x}{\varepsilon D} y = \text{zero} \quad Q^2 = 1 \]

\[- \frac{1}{\varepsilon D^2} + \frac{1}{\varepsilon D^2} + \frac{4}{\varepsilon D} - \frac{2x}{\varepsilon D} y = \text{zero} \]

\[- \frac{\varepsilon^2 D^2}{\varepsilon^2 D^2} + \frac{\varepsilon^2 D^2}{\varepsilon^2 D^2} + \frac{4\varepsilon^2 D^2}{\varepsilon D} - \frac{2x\varepsilon^2 D^2}{\varepsilon D} - \varepsilon^2 D^2 y = \text{zero} \]

\[- 1 + \varepsilon^2 + 4 - 2xD - \varepsilon^2 D^2 y = \text{zero} \]

\[x = \frac{2}{\varepsilon D} \quad y = \frac{2E_N}{m_o L^2} \quad L^2 = \varepsilon DGM \quad \frac{1}{a} = \frac{-1}{\varepsilon D} (\varepsilon^2 - 1) \]

\[- 1 + \varepsilon^2 + 4 - \frac{2}{\varepsilon D} \varepsilon D - \varepsilon^2 D^2 y = \text{zero} \]

\[- 1 + \varepsilon^2 - \varepsilon^2 D^2 2E_N = \text{zero} \quad - 1 + \varepsilon^2 - \varepsilon^2 D^2 \frac{2E_N}{m_o L^2} = \text{zero} \]

\[- 1 + \varepsilon^2 - \varepsilon D - \frac{2E_N}{GM_o m_o} = \text{zero} \quad \frac{1}{\varepsilon D} (\varepsilon^2 - 1) = \frac{2E_N}{k} \]

\[E_N = \frac{k}{2a} \]
Supposing $ux=v$

\[(2.3)\quad \frac{u'x'}{\sqrt{1+\frac{v'^2}{c^2}-\frac{2v'ux'}{c^2}}} = \frac{v-v}{\sqrt{1+\frac{v^2}{c^2}-\frac{2vv}{c^2}}} \Rightarrow u'x'=0\]

$ux=v$  $u'x'=0$

\[(1.17)\quad dt' = dt\sqrt{1+\frac{v^2}{c^2}-\frac{2vux}{c^2}} = dt\sqrt{1+\frac{v^2}{c^2}-\frac{2v}{c^2}} \Rightarrow dt' = dt\sqrt{1-\frac{v^2}{c^2}}\]

\[(1.22)\quad dt = dt'\sqrt{1+\frac{v'^2}{c^2}+\frac{2v'u'x'}{c^2}} = dt'\sqrt{1+\frac{v'^2}{c^2}+\frac{2v'(0)}{c^2}} \Rightarrow dt = dt'\sqrt{1+\frac{v^2}{c^2}}\]

\[
\sqrt{1-\frac{v^2}{c^2}}\sqrt{1+\frac{v'^2}{c^2}} = 1
\]

$v = \frac{v'}{\sqrt{1+\frac{v'^2}{c^2}}}$

$\frac{\sqrt{1-\frac{v^2}{c^2}}}{\sqrt{1+\frac{v^2}{c^2}}} = 1$

$v' = -\frac{v}{\sqrt{1-\frac{v^2}{c^2}}}$

$dt > dt'$  $v < v' \quad vdt = v'dt'$

\[(1.33)\quad \tilde{v} = \frac{-\tilde{v'}}{\sqrt{1+\frac{v'^2}{c^2}+\frac{2v'u'x'}{c^2}}} = \frac{-\tilde{v'}}{\sqrt{1+\frac{v'^2}{c^2}+\frac{2v'(0)}{c^2}}} \Rightarrow \tilde{v} = \frac{-\tilde{v'}}{\sqrt{1+\frac{v^2}{c^2}}}
\]

\[(1.34)\quad \tilde{v}' = \frac{-\tilde{v'}}{\sqrt{1+\frac{v^2}{c^2}-\frac{2vux}{c^2}}} = \frac{-\tilde{v'}}{\sqrt{1+\frac{v^2}{c^2}-\frac{2v}{c^2}}} \Rightarrow \tilde{v}' = \frac{-\tilde{v'}}{\sqrt{1-\frac{v^2}{c^2}}}
\]

$\tilde{r} = r\tilde{r} = -\tilde{r}$  $\tilde{r} = -r\tilde{r} = -\tilde{r}$  $|\tilde{r}| = |\tilde{r'}| = r$

\[
\tilde{d}r = dr\tilde{r} + r\tilde{dr} = -dP
\]

$\tilde{d}P = -dr\tilde{r} - r\tilde{dr} = -d\tilde{r}$

$\tilde{r}d\tilde{r} = dr\tilde{r} + r\tilde{dr} = dr$

$\tilde{r}dP = -dr\tilde{r} - r\tilde{dr} = -dr$

$\tilde{v} = \frac{dr}{dt} = \frac{dr\tilde{r}}{dt} + r\frac{d\phi}{dt}$

$\tilde{v}' = \frac{dr\tilde{r}}{dt} + r\frac{d\phi}{dt}$

$\tilde{v}^2 = \tilde{v}' = \left(\frac{dr}{dt}\right)^2 + \left(r\frac{d\phi}{dt}\right)^2$

$\tilde{v} = \frac{dP}{dt} = \frac{d(-\tilde{r})}{dt} = \left(\frac{dr\tilde{r}}{dt} + r\frac{d\phi}{dt}\right)$

$\tilde{v}' = \frac{dr\tilde{r}}{dt} + r\frac{d\phi}{dt}$

$\tilde{v}^2 = \tilde{v}' = \left(\frac{dr}{dt}\right)^2 + \left(r\frac{d\phi}{dt}\right)^2$
\[
\ddot{a} = \frac{d\ddot{v}}{dt} = \frac{d^2\ddot{v}}{dt^2} = \frac{d^2(r\dot{r})}{dt^2} = \left[ \frac{d^2r}{dt^2} + r \left( \frac{d\phi}{dt} \right)^2 \right] \ddot{\phi} + \left( \frac{2d\phi}{dt} + r \frac{d^2\phi}{dt^2} \right) \dot{\phi}
\]

\[
\ddot{a} = \frac{d\ddot{v}}{dt} = \frac{d^2\ddot{v}}{dt^2} = \frac{d^2(-r\ddot{r})}{dt^2} = \left[ \frac{d^2r}{dt^2} - r \left( \frac{d\phi}{dt} \right)^2 \right] \ddot{\phi} - \left( \frac{2d\phi}{dt} + r \frac{d^2\phi}{dt^2} \right) \dot{\phi}
\]

\[
\ddot{v} = \frac{-\ddot{r}}{\sqrt{1 + \frac{v^2}{c^2}}}
\]

\[
\ddot{a} = \frac{d\ddot{v}}{dt} = \frac{d}{dt} \left( \frac{-\ddot{r}}{\sqrt{1 + \frac{v^2}{c^2}}} \right) = \frac{-\ddot{r}}{\sqrt{1 + \frac{v^2}{c^2}}} \frac{d}{dt} \left( \frac{-\ddot{r}}{\sqrt{1 + \frac{v^2}{c^2}}} \right)
\]

\[
\ddot{a} = \frac{d\ddot{v}}{dt} = \sqrt{1 - \frac{v^2}{c^2}} \frac{1}{\sqrt{1 + \frac{v^2}{c^2}}} \left[ \frac{1}{\sqrt{1 + \frac{v^2}{c^2}}} \frac{d\ddot{v}}{dt'} - \ddot{v} \frac{d}{dt'} \left( \frac{1}{\sqrt{1 + \frac{v^2}{c^2}}} \right) \right]
\]

\[
\ddot{a} = \frac{d\ddot{v}}{dt} = \sqrt{1 - \frac{v^2}{c^2}} \frac{1}{\sqrt{1 + \frac{v^2}{c^2}}} \left[ \frac{1}{\sqrt{1 + \frac{v^2}{c^2}}} \frac{d\ddot{v}}{dt'} - \ddot{v} \frac{d}{dt'} \left( \frac{1}{\sqrt{1 + \frac{v^2}{c^2}}} \right) \right]
\]

\[
\ddot{a} = \frac{d\ddot{v}}{dt} = \sqrt{1 - \frac{v^2}{c^2}} \frac{1}{\sqrt{1 + \frac{v^2}{c^2}}} \left[ \frac{1}{\sqrt{1 + \frac{v^2}{c^2}}} \frac{d\ddot{v}}{dt'} - \ddot{v} \frac{d}{dt'} \left( \frac{1}{\sqrt{1 + \frac{v^2}{c^2}}} \right) \right]
\]

\[
\ddot{a} = \frac{d\ddot{v}}{dt} = \sqrt{1 - \frac{v^2}{c^2}} \frac{1}{\sqrt{1 + \frac{v^2}{c^2}}} \left[ \frac{1}{\sqrt{1 + \frac{v^2}{c^2}}} \frac{d\ddot{v}}{dt'} - \ddot{v} \frac{d}{dt'} \left( \frac{1}{\sqrt{1 + \frac{v^2}{c^2}}} \right) \right]
\]

\[
\ddot{a} = \frac{d\ddot{v}}{dt} = \sqrt{1 - \frac{v^2}{c^2}} \frac{1}{\sqrt{1 + \frac{v^2}{c^2}}} \left[ \frac{1}{\sqrt{1 + \frac{v^2}{c^2}}} \frac{d\ddot{v}}{dt'} - \ddot{v} \frac{d}{dt'} \left( \frac{1}{\sqrt{1 + \frac{v^2}{c^2}}} \right) \right]
\]

\[
\ddot{a} = \frac{d\ddot{v}}{dt} = \frac{m \ddot{a}}{\sqrt{1 - \frac{v^2}{c^2}}} = \frac{m}{\sqrt{1 - \frac{v^2}{c^2}}} \frac{d\ddot{v}}{dt}
\]

\[
\ddot{a} = \frac{d\ddot{v}}{dt} = \frac{m \ddot{a}}{\sqrt{1 - \frac{v^2}{c^2}}} = \frac{m}{\sqrt{1 - \frac{v^2}{c^2}}} \frac{d\ddot{v}}{dt}
\]
\[
\vec{F} = \frac{-m_b}{\left(1 + \frac{v^2}{c^2}\right)^{3/2}} \left[ \left(1 + \frac{v'}{c}ight) \frac{d\vec{v}'}{dt'} - \vec{v} \frac{d\vec{v}}{dt} \right]
\]

\[
\vec{F} = m\ddot{\vec{a}} = \frac{m_b}{\sqrt{1 - \frac{v^2}{c^2}}} \frac{d\vec{v}}{dt} = \vec{F}' = \frac{-m_b}{\left(1 + \frac{v^2}{c^2}\right)^{3/2}} \left[ \left(1 + \frac{v'}{c}ight) \frac{d\vec{v}'}{dt'} - \vec{v} \frac{d\vec{v}}{dt} \right]
\]

\[
E_k = \int \vec{F} \cdot d\vec{r} = \int \vec{F}' \cdot d\vec{r}' = \int \frac{-k}{r^2} \hat{\vec{r}} (-d\vec{P})
\]

\[
E_k = \int \vec{F} \cdot d\vec{r} = \int \vec{F}' \cdot d\vec{r}' = \int \frac{m_b}{\sqrt{1 - \frac{v^2}{c^2}}} \frac{d\vec{v}}{dt} = \int \frac{-m_b}{\left(1 + \frac{v^2}{c^2}\right)^{3/2}} \left[ \left(1 + \frac{v'}{c}ight) \frac{d\vec{v}'}{dt'} - \vec{v} \frac{d\vec{v}}{dt} \right] (-d\vec{P}) = \int \frac{-k}{r^2} \hat{\vec{r}} (-d\vec{P})
\]

\[
E_k = \int \frac{m_b}{\sqrt{1 - \frac{v^2}{c^2}}} \frac{d\vec{v}}{dt} = \int \frac{m_b}{\left(1 + \frac{v^2}{c^2}\right)^{3/2}} \left[ \left(1 + \frac{v'}{c}ight) \frac{d\vec{v}'}{dt'} - \vec{v} \frac{d\vec{v}}{dt} \right] = \int \frac{k}{r^2} \hat{\vec{r}} d\vec{P}
\]

\[
E_k = \int \frac{m_b}{\sqrt{1 - \frac{v^2}{c^2}}} \frac{d\vec{v}}{dt} = \int \frac{m_b}{\left(1 + \frac{v^2}{c^2}\right)^{3/2}} \left[ \left(1 + \frac{v'}{c}ight) \frac{d\vec{v}'}{dt'} - \vec{v} \frac{d\vec{v}}{dt} \right] = \int \frac{-k}{r^2} d\vec{r}
\]

\[
E_k = \int \frac{m_b v' \frac{d\vec{v}'}{dt}}{\sqrt{1 - \frac{v^2}{c^2}}} = \int \frac{m_b}{\left(1 + \frac{v^2}{c^2}\right)^{3/2}} \left(1 + \frac{v'}{c} \frac{v^2}{c^2} - \frac{v}{c} \frac{v'}{c} \right) = \int \frac{-k}{r^2} d\vec{r}
\]

\[
E_k = \int \frac{m_b v' \frac{d\vec{v}'}{dt}}{\sqrt{1 - \frac{v^2}{c^2}}} = \int \frac{m_b}{\left(1 + \frac{v^2}{c^2}\right)^{3/2}} \left(1 + \frac{v'}{c} \frac{v^2}{c^2} - \frac{v}{c} \frac{v'}{c} \right) = \int \frac{-k}{r^2} d\vec{r}
\]

\[
E_k = m_b c^2 \sqrt{1 - \frac{v^2}{c^2}} = \frac{-m_b c^2}{\sqrt{1 + \frac{v^2}{c^2}}} = k + \text{constante}
\]

\[
E_k = -m_b c^2 \sqrt{1 - \frac{v^2}{c^2}} = \frac{-m_b c^2}{\sqrt{1 + \frac{v^2}{c^2}}} = k + \text{constante}
\]
\[
E_r = \frac{-m_\infty c^2}{\sqrt{1 + \frac{v^2}{c^2}}} \cdot \frac{k - m_\infty v^2}{r} - k = -m_\infty c^2 \frac{k}{r} \quad E_r = \frac{-m_\infty c^2}{\sqrt{1 + \frac{(0)^2}{c^2}}} \cdot \frac{k}{\infty} = -m_\infty c^2
\]

\[
\frac{-1}{\sqrt{1 + \frac{v^2}{c^2}}} = \frac{E_r}{m_\infty c^2} + \frac{k}{m_\infty c^2} \frac{1}{r}
\]

\[
H = \frac{E_r}{m_\infty c^2} \quad A = \frac{k}{m_\infty c^2} = \frac{GM \beta}{m_\infty c^2} = \frac{GM}{c^2}
\]

\[
\frac{-1}{\sqrt{1 + \frac{v^2}{c^2}}} = H + A \frac{1}{r} \quad \frac{1}{\left(1 + \frac{v^2}{c^2}\right)^{3/2}} = -\left(H + A \frac{1}{r}\right)^3
\]

\[
\tilde{\mathbf{r}} = \mathbf{r} \times \mathbf{\tilde{v}} = -r \hat{\mathbf{r}} \times \left[-\frac{d\mathbf{r}}{dt} + r \frac{d\phi}{dt}\right] = r^2 \frac{d\phi}{dt} (\hat{\mathbf{r}} \times \hat{\mathbf{r}}) = r^2 \frac{d\phi}{dt} \hat{\mathbf{r}}
\]

\[
\tilde{\mathbf{v}} = \mathbf{v} \times \mathbf{\tilde{v}} = -\mathbf{v} \times -\mathbf{\tilde{v}} = r \hat{\mathbf{r}} \times \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}} \left[\left(\frac{dr}{dt} + r \frac{d\phi}{dt}\right) \hat{\mathbf{r}} \times \hat{\mathbf{r}}\right] = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}} \frac{d\phi}{dt} (\hat{\mathbf{r}} \times \hat{\mathbf{r}}) = \frac{d\phi}{dt} \hat{\mathbf{r}}
\]

\[
\tilde{L} = r^2 \frac{d\phi}{dt} \hat{\mathbf{r}} = L', \quad L' = r^2 \frac{d\phi}{dt}
\]

\[
dE_r = \frac{m_\infty v d\mathbf{v}}{\sqrt{1 - \frac{v^2}{c^2}}} \frac{1}{\left(1 + \frac{v^2}{c^2}\right)^{3/2}} = -k \frac{r^2}{r^2} \hat{\mathbf{r}} d\mathbf{r}
\]

\[
dE_k = \frac{m_\infty v'd\mathbf{v'}}{\sqrt{1 - \frac{v'^2}{c^2}}} \frac{1}{\left(1 + \frac{v'^2}{c^2}\right)^{3/2}} = \frac{k}{r^2} \frac{r^2}{r^2} \hat{\mathbf{r}} d\mathbf{r'} = \frac{k}{r^2} \hat{\mathbf{r}} d\mathbf{r'}
\]

\[
\tilde{F} = \frac{m_\infty \mathbf{\tilde{F}}}{\left(1 + \frac{v^2}{c^2}\right)^{3/2}} = k \frac{\hat{\mathbf{r}}}{r^2}
\]

\[
\tilde{F} = \frac{m_\infty}{\left(1 + \frac{v^2}{c^2}\right)^{3/2}} \left[\frac{\left(\frac{dr}{dt}^2 - r \left(\frac{d\phi}{dt}\right)^2\right)}{r^2} - \frac{2 \frac{dr}{dt} \frac{d\phi}{dt} + r \frac{d^2\phi}{dt^2}}{r^2} \right] = k \frac{\hat{\mathbf{r}}}{r^2}
\]

\[
\tilde{F} \frac{d\phi}{dt} = \frac{-m_\infty}{\left(1 + \frac{v^2}{c^2}\right)^{3/2}} \left(2 \frac{dr}{dt} \frac{d\phi}{dt} + r \frac{d^2\phi}{dt^2}\right) \hat{\mathbf{r}} = \text{zero}
\]
\[ \mathbf{F}_z = \frac{-m_o}{(1 + \frac{v^2}{c^2})^2} \left[ \frac{d^2 r}{dt^2} - r \left( \frac{d \phi}{dt} \right)^2 \right] = \frac{k \dot{r}}{r^2} \hat{r} \]

\[ \frac{1}{(1 + \frac{v^2}{c^2})^2} \left[ \frac{d^2 r}{dt^2} - r \left( \frac{d \phi}{dt} \right)^2 \right] \hat{r} = -\frac{GM_o}{r^2} \hat{r} \]

\[ \frac{d \phi}{dt} = \frac{L}{r^2} \]
\[ \frac{dr}{dt} = -L \frac{dw}{d \phi} \]
\[ \frac{d^2 r}{dt^2} = -L^2 \frac{d^2 w}{d \phi^2} \]
\[ \frac{d^2 \phi}{dt^2} = \frac{2L^2}{r^3} \frac{dw}{d \phi} \]

\[ \frac{1}{(1 + \frac{v^2}{c^2})^2} \left[ \frac{-L^2 \frac{d^2 w}{d \phi^2}}{r^2} \right] = -\frac{GM_o}{r^2} \]

\[ \frac{1}{(1 + \frac{v^2}{c^2})^2} \left( \frac{d^2 w}{d \phi^2} + \frac{1}{r} \right) = -\frac{GM_o}{L^2} \]

\[ \frac{1}{(1 + \frac{v^2}{c^2})^2} \left( \frac{d^2 w}{d \phi^2} + \frac{1}{r} \right) \frac{GM_o}{L^2} \]

\[ \left( H + A \frac{1}{r} \right) \left( \frac{d^2 w}{d \phi^2} + \frac{1}{r} \right) = \frac{GM_o}{L^2} \]

\[ \left( H + A \frac{1}{r} \right) \left( \frac{d^2 w}{d \phi^2} + \frac{1}{r} \right) = -\frac{GM_o}{L^2} \]

\[ \left( H + A \frac{1}{r} \right) \left( \frac{d^2 w}{d \phi^2} + \frac{1}{r} \right) = -B \]

\[ H = \frac{E_R}{m_o c^2} \]

\[ A = \frac{GM_o}{c^2} \]

\[ B = \frac{GM_o}{L^2} \]

\[ \left( H + A \frac{1}{r} \right) \left( \frac{d^2 w}{d \phi^2} + \frac{1}{r} \right) + B = 0 \]

§25 Logarithmic Spiral continued

\[ \left( H + A \frac{1}{r} \right) \left( \frac{d^2 w}{d \phi^2} + \frac{1}{r} \right) = \frac{GM_o}{L^2} \]
\[
\left( H^3 + 3H^2A \frac{1}{r} + 3HA \frac{1}{r}^2 + A^3 \frac{1}{r}^3 \right) \left( \frac{d^2w}{d\phi^2} + \frac{1}{r} \right) + B = 0
\]

\[
H^3 + 3H^2A \frac{1}{r} + 3HA \frac{1}{r}^2 + A^3 \frac{1}{r}^3 \simeq H^3 + 3H^2A \frac{1}{r} \\
3HA \frac{1}{r}^2 + A^3 \frac{1}{r}^3 \simeq 0
\]

\[
\left( H^3 + 3AH^2 \frac{1}{r} \right) \left( \frac{d^2w}{d\phi^2} + \frac{1}{r} \right) + B = 0
\]

\[
\left( H^3 + 3AH^2 \right) \left( \frac{d^2w}{d\phi^2} + w \right) + B = 0
\]

\[
H^3 \frac{d^2w}{d\phi^2} + H^3w + 3AH^2 \frac{d^2w}{d\phi^2}w + 3AH^2w^2 + B = 0
\]

\[
w = \frac{1}{r} = \frac{1}{\varepsilon D} \left[ 1 + \varepsilon \cos(\phi Q) \right] \\
dw = -\frac{Q \varepsilon \cos(\phi Q)}{D} \\
d^2w = -\frac{Q^2 \cos(\phi Q)}{\varepsilon D}
\]

The first hypothesis to obtain a particular solution of the differential equation is to assume the infinite radius \( r = \infty \), thus obtaining:

\[
w = \frac{1}{r} = \frac{1}{\varepsilon D} \left[ 1 + \varepsilon \cos(\phi Q) \right] = \text{zero} \Rightarrow \varepsilon \cos(\phi Q) = -1 \\
\frac{d^2w}{d\phi^2} = -\frac{Q^2 \cos(\phi Q)}{\varepsilon D} = -\frac{Q^2 \varepsilon \cos(\phi Q)}{\varepsilon D}
\]

\[
H^3 \frac{d^2w}{d\phi^2} + H^3w + 3AH^2 \frac{d^2w}{d\phi^2}w + 3AH^2w^2 + B = 0
\]

\[
w = \text{zero} \\
\frac{d^2w}{d\phi^2} = \frac{Q^2}{\varepsilon D} \\
H = \frac{E_R}{c^2} = -m_o c^2 = -1
\]

\[-1^3 \left( \frac{Q^2}{\varepsilon D} \right) + (-1)^3 \text{zero} + 3A(-1)^3 \left( \frac{Q^2}{\varepsilon D} \right) \text{zero} + 3A(-1)^3 \text{zero}^2 + B = 0
\]

\[-\left( \frac{Q^2}{\varepsilon D} \right) + B = \text{zero} \\
-\frac{\varepsilon D Q^2}{\varepsilon D} + \varepsilon DB = \text{zero}
\]

\[-Q^2 + 1 = \text{zero} \\
Q^2 = 1
\]

This result shows that in infinity the influence of the central mass is zero \( M_o = \text{zero} \).

The second hypothesis to obtain another particular solution of the differential equation is obtained by observing that the angle \( \phi Q \) of the equation \( \varepsilon \cos(\phi Q) = -1 \) indicates the direction of the infinite radius \( r = \infty \) where the influence of the central mass is zero \( M_o = \text{zero} \) and \( Q^2 = 1 \) therefore the direction of the center of mass is given by the angle \( \phi Q + \pi \) that replaced in the equation \( \varepsilon \cos(\phi Q) = -1 \) results in the new equation \( \varepsilon \cos(\phi Q + \pi) = -1 \) that indicates direction opposite the direction of the infinite radius which is the direction of the center of mass.

\[\varepsilon \cos(\phi Q + \pi) = -1 \quad \cos(\phi Q + \pi) = -\cos(\phi Q) \quad \varepsilon [-\cos(\phi Q)] = -1 \quad \varepsilon \cos(\phi Q) = 1\]
\[ w = \frac{1}{r} \left[ 1 + \epsilon \cos(\phi Q) \right] = \frac{1}{\epsilon D} (1 + 1) = \frac{2}{\epsilon D} \]
\[ \frac{d^2 w}{d\phi^2} = -\frac{Q^2 \cos(\phi Q)}{D} \]
\[ H = \frac{E_R}{m_e c^2} = -\frac{m_e c^2}{m_e c^2} = -1 \]
\[ H^3 \frac{d^2 w}{d\phi^2} + H^3 w + 3AH^2 \frac{d^2 w}{d\phi^2} w + 3AH^2 w^2 + B = 0 \]

\[ (-1)^9 \left( -\frac{Q^2}{\epsilon D} \right) + (-1)^9 \left( \frac{2}{\epsilon D} \right) + 3A(-1)^9 \left( -\frac{Q^2}{\epsilon D} \right) \left( \frac{2}{\epsilon D} \right) + 3A(-1)^9 \left( \frac{2}{\epsilon D} \right)^2 + B = 0 \]

\[ \left( -\frac{Q^2}{\epsilon D} \right) - \left( \frac{2}{\epsilon D} \right) + 3A \left( -\frac{Q^2}{\epsilon D} \right) \left( \frac{2}{\epsilon D} \right) + 3A \left( \frac{2}{\epsilon D} \right)^2 + B = 0 \]

\[ \frac{Q^2}{\epsilon D} = \frac{2}{\epsilon D} - \frac{6A Q^2}{\epsilon D} + \frac{12A}{\epsilon D} + B = 0 \]
\[ \frac{Q^2}{\epsilon D} = \frac{2}{\epsilon D} - \frac{6A Q^2}{\epsilon D} + \frac{12A}{\epsilon D} = 0 \]

\[ \frac{Q^2}{\epsilon D} - \frac{2}{\epsilon D} - \frac{6A Q^2}{\epsilon D} + \frac{12A}{\epsilon D} + 1 = 0 \]

\[ \frac{Q^2}{\epsilon D} - \frac{2}{\epsilon D} - \frac{6A Q^2}{\epsilon D} + \frac{12A}{\epsilon D} = 0 \]

\[ \frac{Q^2}{\epsilon D} - \frac{6A Q^2}{\epsilon D} + \frac{12A}{\epsilon D} = 0 \]

\[ \frac{Q^2}{\epsilon D} - \frac{6A Q^2}{\epsilon D} + \frac{12A}{\epsilon D} = 0 \]

Applying the results of the second hypothesis in the differential equation:

\[ H^3 \frac{d^2 w}{d\phi^2} + H^3 w + 3AH^2 \frac{d^2 w}{d\phi^2} w + 3AH^2 w^2 + B = 0 \]

\[ w = \frac{1}{r} \left[ 1 + \epsilon \cos(\phi Q) \right] \]
\[ \frac{dw}{d\phi} = -\frac{Q \cos(\phi Q)}{D} \]
\[ \frac{d^2 w}{d\phi^2} = -\frac{Q \cos(\phi Q)}{D} \]

21.38
\[-H^3Q^2 \cos(\phi) \frac{\cos(\phi)}{D} + H^3 \frac{1}{\varepsilon D} \cos(\phi) + 3AH^2 \left[ -\frac{Q^2 \cos(\phi)}{D} \right] \frac{1}{\varepsilon D} + 3AH^2 \left[ -\frac{Q^4 \cos(\phi)}{D} \right] \frac{1}{\varepsilon D} \varepsilon \cos(\phi) +
\]
\[+ 3AH^2 \left\{ \frac{1}{\varepsilon^2 D^2} \left[ 1 + 2\varepsilon \cos(\phi) + \varepsilon^2 \cos^2(\phi) \right] \right\} + B = 0 \]

\[-H^3Q^2 \cos(\phi) \frac{\cos(\phi)}{D} + H^3 \frac{1}{\varepsilon D} \cos(\phi) + 3AH^2 \left[ -\frac{Q^2 \cos(\phi)}{D} \right] \frac{1}{\varepsilon D} + 3AH^2 \left[ -\frac{Q^4 \cos(\phi)}{D} \right] \frac{1}{\varepsilon D} \varepsilon \cos(\phi) +
\]
\[+ 3AH^2 \left\{ \frac{1}{\varepsilon^2 D^2} \left[ 1 + 2\varepsilon \cos(\phi) + \varepsilon^2 \cos^2(\phi) \right] \right\} + B = 0 \]

\[-H^3Q^2 \cos(\phi) \frac{\cos(\phi)}{D} + H^3 \frac{1}{\varepsilon D} \cos(\phi) + 3AH^2 \left[ -\frac{Q^2 \cos(\phi)}{D} \right] \frac{1}{\varepsilon D} + 3AH^2 \left[ -\frac{Q^4 \cos(\phi)}{D} \right] \frac{1}{\varepsilon D} \varepsilon \cos(\phi) +
\]
\[+ 3AH^2 \left\{ \frac{1}{\varepsilon^2 D^2} \left[ 1 + 2\varepsilon \cos(\phi) + \varepsilon^2 \cos^2(\phi) \right] \right\} + B = 0 \]
\begin{align*}
(1-Q^2)\cos^2(\phi Q) &+ \frac{\left(\frac{Q^2}{3A} - \frac{1}{3A} - \frac{Q^2}{\varepsilon D} + \frac{2}{\varepsilon D}\right)\cos(\phi Q)}{D} - \frac{1}{3A\varepsilon D} + \frac{1}{\varepsilon^2 D^2} + \frac{\varepsilon DB}{3A\varepsilon D} = 0 \\
\varepsilon DB &= \frac{\varepsilon DGM_o}{L^2} = \frac{\varepsilon DGM_o}{1} = 1 \\
(1-Q^2)\cos^2(\phi Q) &+ \frac{\left(\frac{Q^2}{3A} - \frac{1}{3A} - \frac{Q^2}{\varepsilon D} + \frac{2}{\varepsilon D}\right)\cos(\phi Q)}{D} - \frac{1}{3A\varepsilon D} + \frac{1}{\varepsilon^2 D^2} + \frac{1}{3A\varepsilon D} = 0 \\
(1-Q^2)\cos^2(\phi Q) &+ \frac{\left(\frac{Q^2}{3A} - \frac{1}{3A} - \frac{Q^2}{\varepsilon D} + \frac{2}{\varepsilon D}\right)\cos(\phi Q)}{D} + \frac{1}{\varepsilon^2 D^2} = 0 \\
\text{zero} < r(\phi Q) < \infty \rightarrow M_o \neq \text{zero} &\rightarrow Q = \sqrt{\frac{1-12A}{1-6A}}
\end{align*}

\begin{align*}
[1 - \left(\frac{\frac{12A}{\varepsilon D}}{\frac{6A}{\varepsilon D}}\right)\cos^2(\phi Q)] &+ \left[\frac{1}{3A}\left(\frac{1-12A}{\varepsilon D}\right)\right] - \frac{1}{3A} - \frac{1}{\varepsilon D}\left(\frac{1-12A}{\varepsilon D} + \frac{2}{\varepsilon D}\right)\cos(\phi Q) + \frac{1}{\varepsilon^2 D^2} = 0 \\
[1 - \frac{6A}{\varepsilon D}\left(\frac{12A}{\varepsilon D}\right)\cos^2(\phi Q)] &+ \left[\frac{1}{3A}\left(\frac{1-12A}{\varepsilon D}\right)\right] - \frac{1}{3A} - \frac{1}{\varepsilon D}\left(\frac{1-12A}{\varepsilon D} + \frac{2}{\varepsilon D}\right)\cos(\phi Q) + \frac{1}{\varepsilon^2 D^2}\left(\frac{1-6A}{\varepsilon D}\right) = 0 \\
\left(\frac{6A}{\varepsilon D}\right)\cos^2(\phi Q) &+ \left(-\frac{1}{\varepsilon D}\right)\cos(\phi Q) + \frac{1}{\varepsilon^2 D^2} - \frac{1}{6A} = 0 \\
\cos(\phi Q) &= \frac{-1/\varepsilon D \pm \sqrt{(-1/\varepsilon D)^2 - 4\left(\frac{6A}{\varepsilon D}\right)\left(\frac{1}{\varepsilon^2 D^2} - \frac{1}{6A}\right)}}{2\frac{6A}{\varepsilon D}} \\
\cos(\phi Q) &= \frac{1 + \sqrt{\frac{1}{\varepsilon^2 D^2} - \frac{24A}{\varepsilon^2 D^2} - \frac{1}{\varepsilon^2 D^2} - \frac{1}{6A}}}{12A} \\
\cos(\phi Q) &= \frac{1 + \sqrt{\frac{1}{\varepsilon^2 D^2} - \frac{24A}{\varepsilon^2 D^2} + \frac{1}{\varepsilon^2 D^2} + \frac{1}{6A}}}{12A} \\
\cos(\phi Q) &= \frac{1 + \sqrt{\frac{1}{\varepsilon D} - \frac{24A}{\varepsilon D} + \frac{24A}{\varepsilon D} + \frac{6A}{\varepsilon D}}}{12A} \\
\cos(\phi Q) &= \frac{1 + \sqrt{\frac{1}{\varepsilon D} - \frac{24A + 24A}{\varepsilon D} + \frac{6A}{\varepsilon D}}}{12A}
\end{align*}
\[
\frac{\cos(\phi Q)}{D} = \frac{1}{\varepsilon D} \sqrt{1 - \frac{12A + 144A^2}{\varepsilon D}}
\]

\[
\frac{\cos(\phi Q)}{D} = \frac{1}{\varepsilon D} \left[ \left( 1 - \frac{12A}{\varepsilon D} \right)^2 \right]^{\frac{1}{2}}
\]

\[
\frac{\cos(\phi Q)}{D} = \frac{1}{\varepsilon D} \left( 1 - \frac{12A}{\varepsilon D} \right)
\]

\[
\frac{\cos(\phi Q)}{D} = \frac{1}{\varepsilon D} \left[ \left( 1 - \frac{12A}{\varepsilon D} \right)^2 \right]^{\frac{1}{2}}
\]

\[
\frac{\cos(\phi Q)}{D} = \frac{1}{\varepsilon D} \left( 1 - \frac{12A}{\varepsilon D} \right)
\]

\[
\frac{\cos(\phi Q)}{D} = \frac{1}{\varepsilon D} \left[ \left( 1 - \frac{12A}{\varepsilon D} \right)^2 \right]^{\frac{1}{2}}
\]

\[
\frac{\cos(\phi Q)}{D} = \frac{1}{\varepsilon D}
\]

Where applying the result of the second hypothesis \(\varepsilon \cos(\phi Q) = 1 \Rightarrow \cos(\phi Q) = \frac{1}{\varepsilon}\):

\[
\frac{1}{\varepsilon D} = \frac{1}{\varepsilon D}
\]

That it is an identity demonstrating that the result of the second hypothesis is correct.

\[
Q^2 = \frac{1 - \frac{12A}{\varepsilon D}}{\frac{1 - 6A}{\varepsilon D}} \approx 1 - \frac{6A}{\varepsilon D}
\]

\[
Q^2 = 1 - \frac{6A}{\varepsilon D}
\]

\[
A = \frac{\frac{GM_o}{c^2}}{2}
\]

\[
A = \frac{GM_o}{c^2} = 6.6740831.10^{-11}\cdot1.9891.10^{30}
\]

\[
= \left(2.99792458.10^8\right)^2 = 1.477,089,535.42
\]

\[
\varepsilon D = a(1 - e^2) = 57.909,227,000,000\left[1 - (0.20563593)^2\right] = 55.460,469,568,40
\]

\[
A = \frac{GM_o}{c^2} = 6.6740831.10^{-11}\cdot1.9891.10^{30}
\]

\[
= \left(2.99792458.10^8\right)^2 = 1.477,089,535.42
\]
\[
Q = \frac{\sqrt{1 - \frac{12A}{\varepsilon D}}}{\sqrt{1 - \frac{6A}{\varepsilon D}}} = 0.999,999,920.1
\]

\[
1,276.789.102.53^{14}
\]

\[
\phi Q = 1.296.000.00 \Rightarrow \phi = \frac{1.296.000.00}{Q} = 0.999,999,920.1
\]

\[
Q < 1 \text{ Advance} \quad Q > 1 \text{ Retrocess}
\]

\[
\Delta \phi = \left(\frac{1}{Q} - 1\right)1.296.000.00
\]

\[
\Delta \phi > 0 \text{ Advance} \quad \Delta \phi < 0 \text{ Retrocess}
\]

\[
\Delta \phi = \left[\frac{1}{\left(\frac{1 - \frac{12A}{\varepsilon D}}{\sqrt{1 - \frac{6A}{\varepsilon D}}}\right)^2} - 1\right]1.296.000.00 = 0.103,549,893.544''
\]

\[
\Delta \phi = \left[\frac{1}{\left(\frac{1 - \frac{6A}{\varepsilon D}}{\sqrt{1 - \frac{6A}{\varepsilon D}}}\right)^2} - 1\right]1.296.000.00 = 0.103,549,876.997''
\]

\[
N = 100, \frac{PT}{PM} = 100 \frac{365,256.363.004}{87,969} = 415,210.316.139
\]

\[
\sum \Delta \phi = \Delta \phi N = 0.103,549,893.544 \times 415,210.316.139 = 42,994.984.034.7''
\]

\[
\sum \Delta \phi = \Delta \phi N = 0.103,549,876.997 \times 415,210.316.139 = 42,994.977.164.2''
\]

**Newtonian Energy** \( E_N \)

\[
E_N = \frac{m_o u^2}{2} - \frac{k}{r}
\]

\[
u^2 = \left(\frac{dr}{dt}\right)^2 + \left(r \frac{d\phi}{dt}\right)^2 = \left(\frac{dr}{dt}\right)^2 + \frac{I_r^2}{r^2}
\]

\[
E_N = \frac{m_o}{2} \left[\left(\frac{dr}{dt}\right)^2 + \frac{I_r^2}{r^2}\right] - \frac{k}{r}
\]

\[
\frac{2E_N}{m_o} = \left(\frac{dr}{dt}\right)^2 + \frac{I_r^2}{r^2} - \frac{2k}{m_o} \frac{1}{r}
\]

\[
\left(\frac{dr}{dt}\right)^2 + \frac{I_r^2}{r^2} - \frac{2k}{m_o} \frac{1}{r} - \frac{2E_N}{m_o} = \text{zero}
\]
\[ \frac{d\phi}{dt} = \frac{L}{r^2} \quad \frac{dr}{dt} = -L \frac{dw}{d\phi} \quad \frac{d^2 r}{dt^2} = -\frac{L^2}{r^2} \frac{d^2 w}{d\phi^2} \quad \frac{d^3 \phi}{dt^3} = \frac{2L}{r^3} \frac{dw}{d\phi} \]

\[
\left(-L \frac{dw}{d\phi}\right)^2 + \frac{L^2}{r^2} \frac{2k_1}{m_o r} \frac{2E_N}{m_o} = \text{zero}
\]

\[
\left(\frac{dw}{d\phi}\right)^2 + \frac{1}{r^2} \frac{2k_1}{m_o L^2} \frac{2E_N}{m_o L^2} = \text{zero}
\]

\[
\left(\frac{dw}{d\phi}\right)^2 + \frac{1}{r^2} \frac{2k_1}{m_o L^2} \frac{2E_N}{m_o L^2} = \text{zero}
\]

\[
\left(\frac{dw}{d\phi}\right)^2 + w^2 - \frac{2k_1}{m_o L^2} \frac{2E_N}{m_o L^2} = \text{zero}
\]

\[
x = \frac{2k_1}{m_o L^2} \quad y = \frac{2E_N}{m_o L^2}
\]

\[
\left(\frac{dw}{d\phi}\right)^2 + w^2 - x w - y = \text{zero}
\]

\[
w = \frac{1}{r} = \frac{1}{\varepsilon D} \left[1 + \varepsilon \cos(\phi)\right]
\]

\[
\frac{dw}{d\phi} = -\frac{Q \sin(\phi)}{D} \quad \frac{d^2 w}{d\phi^2} = -\frac{Q^2 \cos(\phi)}{D}
\]

\[
\left[\frac{-Q \sin(\phi)}{D}\right]^2 + \left\{\frac{1}{\varepsilon D} \left[1 + \varepsilon \cos(\phi)\right]\right\}^2 - \frac{x}{\varepsilon D} \left[1 + \varepsilon \cos(\phi)\right] - y = \text{zero}
\]

\[
\frac{Q^2}{D^2} \left[1 - \cos^2(\phi)\right] + \frac{1}{\varepsilon^2 D^2} \left[1 + 2\varepsilon \cos(\phi) + \varepsilon^2 \cos^2(\phi)\right] - \frac{x}{\varepsilon D} \frac{1 + \varepsilon \cos(\phi)}{\varepsilon D} - y = \text{zero}
\]

\[
\frac{Q^2}{D^2} - \frac{Q^2}{D^2} \cos^2(\phi) + \frac{1}{\varepsilon^2 D^2} + \frac{1}{\varepsilon^2 D^2} 2\varepsilon \cos(\phi) + \frac{1}{\varepsilon^2 D^2} \varepsilon^2 \cos^2(\phi) - \frac{1}{\varepsilon^2 D^2} \frac{x}{\varepsilon D} \cos(\phi) - y = \text{zero}
\]

\[
\frac{Q^2}{D^2} - \frac{Q^2}{D^2} \cos^2(\phi) + \frac{1}{\varepsilon^2 D^2} + \frac{2}{\varepsilon D} \cos(\phi) + \frac{1}{\varepsilon^2 D^2} \cos^2(\phi) - \frac{x}{\varepsilon D} \cos(\phi) - \frac{x}{\varepsilon D} \cos(\phi) - \frac{x}{\varepsilon D} \cos(\phi) - y = \text{zero}
\]

\[
\frac{\cos^2(\phi)}{D^2} - \frac{Q^2}{D^2} \cos^2(\phi) + \frac{2}{\varepsilon D} \cos(\phi) + \frac{1}{\varepsilon^2 D^2} \cos^2(\phi) - \frac{x}{\varepsilon D} \cos(\phi) + \frac{Q^2}{D^2} - \frac{1}{\varepsilon^2 D^2} \frac{x}{\varepsilon D} - y = \text{zero}
\]

\[
\frac{1 - Q^2}{D^2} \frac{\cos^2(\phi)}{D^2} + \left(\frac{2}{\varepsilon D} \frac{x}{D} \right) \cos(\phi) + \frac{Q^2}{D^2} - \frac{1}{\varepsilon^2 D^2} \frac{x}{\varepsilon D} - y = \text{zero}
\]
Newtonian Energy $E_N$

\[ (1-Q^2)\frac{\cos^2(\phi Q)}{D^2} + \left( x - \frac{2}{\varepsilon D} \right) \frac{\cos(\phi Q)}{D} + \frac{Q^2}{D^2} + \frac{1}{\varepsilon^2 D^2} \frac{x}{\varepsilon D} - y = \text{zero} \]

\[ r = \infty \rightarrow Q = 1 \rightarrow w = \frac{1}{r} = \frac{1}{\varepsilon D} [1 + \varepsilon \cos(\phi Q)] = \frac{\cos(\phi Q)}{D} + \frac{1}{\varepsilon D} = \text{zero} \]

\[ (1-Q^2) \left( \frac{1}{\varepsilon D} \right) \left( \frac{1}{\varepsilon D} \right) + \left( x - \frac{2}{\varepsilon D} \right) \left( \frac{1}{\varepsilon D} \right) + \frac{Q^2}{D^2} + \frac{1}{\varepsilon^2 D^2} \frac{x}{\varepsilon D} - y = \text{zero} \]

\[ (1-Q^2) \left( \frac{1}{\varepsilon^2 D^2} \right) \left( \frac{1}{\varepsilon D} \right) - \frac{x}{\varepsilon D} + \frac{2}{\varepsilon^2 D^2} + \frac{Q^2}{D^2} + \frac{1}{\varepsilon^2 D^2} \frac{x}{\varepsilon D} - y = \text{zero} \]

\[ \frac{Q^2}{\varepsilon^2 D^2} + \frac{Q^2}{D^2} + \frac{2x}{\varepsilon D} - y = \text{zero} \quad \text{Q}^2 = 1 \]

\[ \frac{1}{\varepsilon^2 D^2} + \frac{1}{D^2} + \frac{4}{\varepsilon^2 D^2} - \frac{2x}{\varepsilon D} - y = \text{zero} \]

\[ \frac{\varepsilon^2 D^2}{\varepsilon^2 D^2} + \frac{\varepsilon^2 D^2}{D^2} + \frac{4\varepsilon^2 D^2}{\varepsilon D} - \frac{2x \varepsilon^2 D^2}{\varepsilon D} - \varepsilon^2 D^2 y = \text{zero} \]

\[ -1 + \varepsilon^2 + 4 - 2 \varepsilon D - \varepsilon^2 D^2 y = \text{zero} \]

\[ x = \frac{2}{\varepsilon D} \quad y = \frac{2E_N}{\varepsilon m_o L^2} \quad L^2 = \varepsilon D G M \quad \frac{1}{a} = \frac{1}{\varepsilon D} \left( \frac{e^2 - 1}{1/2a} \right) \]

\[ -1 + \varepsilon^2 + 4 - 2 \frac{\varepsilon D}{\varepsilon D} - \varepsilon^2 D^2 y = \text{zero} \quad -1 + \varepsilon^2 - \varepsilon^2 D^2 y = \text{zero} \]

\[ -1 + \varepsilon^2 - \varepsilon^2 D^2 \frac{2E_N}{\varepsilon m_o L^2} = \text{zero} \quad -1 + \varepsilon^2 - \varepsilon^2 D^2 \frac{2E_N}{\varepsilon m_o L^2} = \text{zero} \]

\[ -1 + \varepsilon^2 - \varepsilon D \frac{2E_N}{\varepsilon m_o L^2} = \text{zero} \quad \frac{1}{\varepsilon D} \left( \frac{e^2 - 1}{1/2a} \right) = \frac{2E_N}{k} \]

\[ E_N = \frac{k}{2a} \]
§27 Advancement of Perihelion of Mercury of 42.99” "contour Conditions"

Let us start from the equation expressing the equilibrium of forces:

\[ \vec{F} = \frac{m_0 \ddot{r}}{r^2} = \frac{k \ddot{r}}{r^2} \]

21.65

On the right side we have the gravitational force \( \frac{k \ddot{r}}{r^2} \) defined by Newton, on the left side we have the physical description of Force \( \ddot{r} = \frac{m_0 \ddot{r}}{r^2} \) of the Undulating Relativity.

The physical properties of equation 21.65 require its validity when its radius varies from a radius greater than zero to an infinite radius, so the radius varies from zero < \( r \leq \infty \), and so we have two distinct boundary conditions. The first boundary condition is when the radius is infinite \( r = \infty \) and the gravitational force is zero, which means that the particle is at rest with \( v' = \text{zero} \) and \( \ddot{r} = \text{zero} \) and the second boundary condition is when the radius is greater which is zero and smaller than infinity zero < \( r < \infty \) which means that the particle is in motion due to the influence of a gravitational force 21.65 with \( v' \neq \text{zero} \) and \( \ddot{r} \neq \text{zero} \).

In §26 following the calculations is substituted in 21.65, the equality, 21.62, 21.69 and

\[ H = \frac{E_r}{m_0 c^2}, \quad A = \frac{GM_o}{c^2}, \quad B = \frac{GM_o}{L^2}, \text{more } w = \frac{1}{r}. \]

After these substitutions we obtain the differential equation:

\[ H^3 \frac{d^2 w}{d\phi^2} + H^2 w + 3AH^2 \frac{d^2 w}{d\phi^2} w + 3AH^2 w^2 + B = \text{zero} \]

27.1

This equation has to be valid for the same boundary conditions as equation 21.65, that is, it has to be valid from a radius \( r \) greater than zero (\( r > 0 \)) to an infinite radius (\( 0 < r < \infty \)). Your solution is given by:

\[ w = \frac{1}{r} = \frac{1}{\varepsilon D} \left[ 1 + \varepsilon \cos(\phiQ) \right] \]

27.2

Which should cover the two contour conditions already described.

Applying solution 27.2 in differential equation 27.1 we have:

\[ H^3 \frac{d^2 w}{d\phi^2} + H^2 w + 3AH^2 \frac{d^2 w}{d\phi^2} w + 3AH^2 w^2 + B = \text{zero} \]

\[ w = \frac{1}{r} = \frac{1}{\varepsilon D} \left[ 1 + \varepsilon \cos(\phiQ) \right], \quad \frac{dw}{d\phi} = - \frac{Q \sin(\phiQ)}{D}, \quad \frac{d^2 w}{d\phi^2} = - \frac{Q^2 \cos(\phiQ)}{D} \]

21.38

\[ H^3 \left[ \frac{-Q^2 \cos(\phiQ)}{D} \right] + H^2 \frac{1}{\varepsilon D} [1 + \varepsilon \cos(\phiQ)] + 3AH^2 \left[ \frac{-Q^2 \cos(\phiQ)}{D} \right] \frac{1}{\varepsilon D} [1 + \varepsilon \cos(\phiQ)] + \]

\[ + 3AH^2 \left\{ \frac{1}{\varepsilon D} [1 + \varepsilon \cos(\phiQ)] \right\} \]

\[ - H^3 \frac{Q^2 \cos(\phiQ)}{D} + H^2 \frac{1}{\varepsilon D} + H^2 \frac{1}{\varepsilon D} [1 - \varepsilon \cos(\phiQ)] + 3AH^2 \left[ \frac{-Q^2 \cos(\phiQ)}{D} \right] \frac{1}{\varepsilon D} \]

\[ + 3AH^2 \left\{ \frac{1}{\varepsilon D^2} [1 + 2\varepsilon \cos(\phiQ) + \varepsilon^2 \cos^2(\phiQ)] \right\} + B = \text{zero} \]
\[-H^3Q^2 \frac{\cos(\phi Q)}{D} + H^3 \frac{\cos(\phi Q)}{\epsilon D} - 3AH^2Q^2 \frac{\cos(\phi Q)}{\epsilon D} - 3AH^2Q^2 \frac{\cos^2(\phi Q)}{D^2} +
\]
\[+ \frac{3AH^2}{\epsilon^2D^2} \left[1 + 2\epsilon \cos(\phi Q) + \epsilon^2 \cos^2(\phi Q)\right] = B = \text{zero}
\]
\[-H^3Q^2 \frac{\cos(\phi Q)}{D} + H^3 \frac{\cos(\phi Q)}{\epsilon D} - 3AH^2Q^2 \frac{\cos(\phi Q)}{\epsilon D} - 3AH^2Q^2 \frac{\cos^2(\phi Q)}{D^2} +
\]
\[+ \frac{3AH^2}{\epsilon^2D^2} + 3AH^2 \frac{2\epsilon \cos(\phi Q) + 3AH^2}{\epsilon^2D^2} \epsilon \cos^2(\phi Q) + B = \text{zero}
\]
\[-H^3Q^2 \frac{\cos(\phi Q)}{D} + H^3 \frac{\cos(\phi Q)}{\epsilon D} - 3AH^2Q^2 \frac{\cos(\phi Q)}{\epsilon D} - 3AH^2Q^2 \frac{\cos^2(\phi Q)}{D^2} +
\]
\[+ \frac{3AH^2}{\epsilon^2D^2} + 6AH^2 \frac{\cos(\phi Q)}{\epsilon D} + 3AH^2 \frac{\cos^2(\phi Q)}{D^2} + B = \text{zero}
\]
\[-H^3Q^2 \cos(\phi Q) + \frac{H^3}{3A} + \frac{H \cos(\phi Q)}{3A} - \frac{Q^2 \cos(\phi Q)}{D^2} - \frac{Q^2 \cos^2(\phi Q)}{D^2} +
\]
\[+ \frac{1}{\epsilon^2D^2} + \frac{2 \cos(\phi Q)}{\epsilon D} + \frac{\cos^2(\phi Q)}{D^2} + \frac{B}{3AH^2} = \text{zero}
\]
\[
\frac{\cos^2(\phi Q)}{D^2} - \frac{Q^2 \cos^2(\phi Q)}{D^2} + \frac{HQ^2 \cos(\phi Q)}{3A} + \frac{H \cos(\phi Q)}{3A} - \frac{Q^2 \cos(\phi Q)}{D^2} +
\]
\[+ \frac{2 \cos(\phi Q)}{\epsilon D} + \frac{H}{3A\epsilon D} + \frac{1}{\epsilon^2D^2} + \frac{B}{3AH^2} = \text{zero}
\]
\[
(1-Q^2)\frac{\cos^2(\phi Q)}{D^2} + \left( \frac{(-1)Q^2}{3A} + \frac{(-1)Q^2}{3A} + \frac{2}{\epsilon D} \right) \cos(\phi Q) + \frac{(-1)}{3A\epsilon D} + \frac{1}{\epsilon^2D^2} + \frac{B}{3A} = \text{zero}
\]
\[
H = \frac{E_R}{m_o c^2} = \frac{-m_o c^2}{m_o c^2} = -1
\]
\[
(1-Q^2)\frac{\cos^2(\phi Q)}{D^2} + \left( \frac{Q^2}{3A} - \frac{1}{3A} + \frac{2}{\epsilon D} \right) \cos(\phi Q) + \frac{1}{3A\epsilon D} + \frac{1}{\epsilon^2D^2} + \frac{B}{3A} = \text{zero}
\]
\[
(1-Q^2)\frac{\cos^2(\phi Q)}{D^2} + \left( \frac{Q^2}{3A} - \frac{1}{3A} + \frac{2}{\epsilon D} \right) \cos(\phi Q) + \frac{1}{3A\epsilon D} + \frac{1}{\epsilon^2D^2} + \frac{\epsilon DB}{3A\epsilon D} = \text{zero}
\]
\[
\epsilon DB = \frac{\epsilon D G M_o}{L^2} = \frac{\epsilon D G M_o}{\epsilon D G M_o} = 1
\]
\[
(1-Q^2) \frac{\cos^2(\phi Q)}{D^2} + \left( \frac{Q^2}{3A} - \frac{1}{3A} \frac{Q^2}{\varepsilon D} + \frac{2}{\varepsilon D} \right) \frac{\cos(\phi Q)}{D} - \frac{1}{3A \varepsilon D} + \frac{1}{\varepsilon^2 D^2} + \frac{1}{3A \varepsilon D} = 0
\]

\[
(1-Q^2) \frac{\cos^2(\phi Q)}{D^2} + \left( \frac{Q^2}{3A} - \frac{1}{3A} \frac{Q^2}{\varepsilon D} + \frac{2}{\varepsilon D} \right) \frac{\cos(\phi Q)}{D} + \frac{1}{\varepsilon^2 D^2} = 0
\] 27.3

This equation must have solution for the same two contour conditions of 21.65.

Solution of 27.3 for the first boundary condition which is when the radius is infinite \( r = \infty \), and the gravitational force is zero which means that the particle is at rest and we have \( v' = 0 \) and \( \dot{\alpha}' = 0 \).

Applying \( Q^2 = 1 \) in 27.3 we get:

\[
(1 - 1^2) \frac{\cos^2(\phi Q)}{D^2} + \left( \frac{1^2}{3A} - \frac{1}{3A} \frac{1^2}{\varepsilon D} + \frac{2}{\varepsilon D} \right) \frac{\cos(\phi Q)}{D} + \frac{1}{\varepsilon^2 D^2} = 0.
\]

\[
\frac{\cos(\phi)}{D} + \frac{1}{\varepsilon D} = 0 \quad \varepsilon = \frac{-1}{\cos(\phi)}
\] 27.4

Equation 27.4 is exactly equal to the result of equation 27.2 when the radius is infinite \( r = \infty \), \( w = 0 \) and \( Q = 1 \), as shown in 27.5:

\[
w = \frac{1}{r \omega} = \frac{1}{\varepsilon D} [1 + \varepsilon \cos(\phi Q)] = \frac{1}{\varepsilon D} [1 + \varepsilon \cos(\phi Q)] = \frac{\cos(\phi)}{D} + \frac{1}{\varepsilon D} = 0
\] 27.5

Therefore in 27.4 we have an exact result that describes how in infinity the eccentricity \( \varepsilon \) is related to the angle \( \phi \) of the infinite radius of the particle, being \( \varepsilon \geq 1 \) which means that the motion from infinity will be or parabolic with \( \varepsilon = 1 \) or hyperbolic with \( \varepsilon > 1 \). Note that by definition \( \varepsilon > 0 \).

Solution of 27.3 for the second boundary condition which is when the radius is greater than zero and less than infinity \( 0 < r < \infty \) which means that the particle is in motion due to the influence of a gravitational force with \( v' \neq 0 \) and \( \dot{\alpha}' \neq 0 \).

Applying \( Q = \sqrt{\frac{1-12A}{1-6A}} \) in 27.3 we have:

\[
(1-Q^2) \frac{\cos^2(\phi Q)}{D^2} + \left( \frac{Q^2}{3A} - \frac{1}{3A} \frac{Q^2}{\varepsilon D} + \frac{2}{\varepsilon D} \right) \frac{\cos(\phi Q)}{D} + \frac{1}{\varepsilon^2 D^2} = 0
\] 27.3

\[
\left[ 1 - \left( \frac{1}{3A} - \frac{1}{6A} \frac{Q^2}{\varepsilon D} \right) \frac{\cos^2(\phi Q)}{D^2} + \left( \frac{2}{3A} - \frac{1}{3A} \frac{1}{\varepsilon D} \right) \frac{\cos(\phi Q)}{D} + \frac{1}{\varepsilon^2 D^2} \right] = 0
\]

\[
\left[ 1 - \left( \frac{6A}{1-12A} \frac{Q^2}{\varepsilon D} \right) \frac{\cos^2(\phi Q)}{D^2} + \left( \frac{1}{3A} - \frac{1}{6A} \frac{Q^2}{\varepsilon D} \right) \frac{\cos(\phi Q)}{D} + \frac{1}{\varepsilon^2 D^2} \right] = 0
\]

\[
\left( \frac{6A}{1-12A} \frac{Q^2}{\varepsilon D} \right) \frac{\cos^2(\phi Q)}{D^2} + \left( \frac{1}{3A} - \frac{1}{6A} \frac{Q^2}{\varepsilon D} \right) \frac{\cos(\phi Q)}{D} + \frac{1}{\varepsilon^2 D^2} = 0
\]

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\[
\frac{\cos(\phi_Q)}{D} = \frac{-1 - 1}{\varepsilon D} \pm i \sqrt{\left(\frac{-1}{\varepsilon D}\right)^2 - 4 \frac{6A}{\varepsilon D} \frac{1}{\varepsilon D} - \frac{1}{\varepsilon D^2} \frac{1}{\varepsilon D} \frac{6A}{\varepsilon D}}
\]
\[
\frac{\cos(\phi_Q)}{D} = \frac{1 \pm \frac{1}{\varepsilon D}}{\varepsilon D} \sqrt{\frac{1}{\varepsilon D^2} - \frac{24A}{\varepsilon D} \frac{1}{\varepsilon D} \frac{6A}{\varepsilon D}}
\]
\[
\frac{\cos(\phi_Q)}{D} = \frac{1 \pm \frac{1}{\varepsilon D}}{\varepsilon D} \sqrt{\frac{1}{\varepsilon D^2} - \frac{24A}{\varepsilon D} \frac{1}{\varepsilon D} \frac{6A}{\varepsilon D}}
\]
\[
\frac{\cos(\phi_Q)}{D} = \frac{1 \pm \frac{1}{\varepsilon D}}{\varepsilon D} \sqrt{\frac{1}{\varepsilon D^2} - \frac{24A}{\varepsilon D} \frac{1}{\varepsilon D} \frac{6A}{\varepsilon D}}
\]
\[
\frac{\cos(\phi_Q)}{D} = \frac{1 \pm \frac{1}{\varepsilon D}}{\varepsilon D} \sqrt{\frac{1}{\varepsilon D^2} - \frac{24A}{\varepsilon D} \frac{1}{\varepsilon D} \frac{6A}{\varepsilon D}}
\]
\[
\frac{\cos(\phi_Q)}{D} = \frac{1 \pm \frac{1}{\varepsilon D}}{\varepsilon D} \sqrt{\frac{1}{\varepsilon D^2} - \frac{24A}{\varepsilon D} \frac{1}{\varepsilon D} \frac{6A}{\varepsilon D}}
\]
\[
\frac{\cos(\phi_Q)}{D} = \frac{1 \pm \frac{1}{\varepsilon D}}{\varepsilon D} \sqrt{\frac{1}{\varepsilon D^2} - \frac{24A}{\varepsilon D} \frac{1}{\varepsilon D} \frac{6A}{\varepsilon D}}
\]
\[
\frac{\cos(\phi_Q)}{D} = \frac{1 \pm \frac{1}{\varepsilon D}}{\varepsilon D} \sqrt{\frac{1}{\varepsilon D^2} - \frac{24A}{\varepsilon D} \frac{1}{\varepsilon D} \frac{6A}{\varepsilon D}}
\]
\[ - \cos(\theta) \frac{d}{eb} + \frac{1}{eb} = \text{zero} \]

\[ \varepsilon = \frac{1}{\cos(\theta)} \]

In the theory of conic for hyperbole we have \( \varepsilon = \frac{c}{a} \), equating to 27.6 we have \( \varepsilon = \frac{c}{a} = \frac{1}{\cos(\theta)} \). This results in \( a = c \cos(\theta) \) which is the correct formula, of the greater half axis of hyperbola.

Therefore in 27.6 we have an exact result that describes how in the course of \( \text{zero} < r < \infty \) the eccentricity \( \varepsilon \) is related to the angle \( \theta \) of the particle, being \( \varepsilon \geq 1 \) which means that the motion will be or parabolic with \( \varepsilon = 1 \) or hyperbolic with \( \varepsilon > 1 \). Note that by definition \( \varepsilon > \text{zero} \).

\[ \S 28 \text{ Simplified Perihelium Advance} \]

**Perihelion Retrogression \( Q > 1 \)**

Imagine that the sun and Mercury are two particles, with the Sun being at the origin of a coordinate system and Mercury lying at a point \( A \) on the \( xy \) plane. The vector radius \( \vec{r} = r\hat{r} \) connecting the origin to point \( A \) will describe Mercury's motion in the \( xy \) plane.

In the description of the movement of the planet Mercury to the observer \( O' \) corresponds to the variables with line for the observer \( O \) as without line being used a single radius \( \vec{r} = r\hat{r} \) and a single coordinate system for both observers.

Time \( t' \) is a function of time \( t \) that is \( t' = t'(t) \) and time \( t \) is a function of time \( t' \) that is \( t = t(t') \).

\[ \frac{dt}{dt'} = \sqrt{1 + \frac{v^2}{c^2}} \]

\[ \frac{dt'}{dt} = \sqrt{1 - \frac{v^2}{c^2}} \]

\[ \sqrt{1 - \frac{v^2}{c^2}} \frac{1 + \frac{v^2}{c^2}}{1} = 1 \]

\[ v' = \frac{v}{\sqrt{1 - \frac{v^2}{c^2}}} \]

\[ v = \frac{v'}{\sqrt{1 + \frac{v^2}{c^2}}} \]

\[ dt > dt' \]

\[ v' > v \]

\[ vdt = v'dt' \]

\[ \vec{r} = r\hat{r} \]

\[ d\vec{r} = dr\hat{r} + r\hat{r} \]

\[ \vec{r} \cdot d\vec{r} = dr\hat{r} \cdot \hat{r} + r\hat{r} \cdot d\hat{r} = dr \]

The radius can be considered a function of time \( t' \) or \( t(t') \) ie \( \vec{r} = \vec{r}(t') = \vec{r}[t(t')] \) or it can be considered a function of time \( t = t(t') \) ie \( \vec{r} = \vec{r}(t) = \vec{r}[t(t')] \).

\[ \vec{r} = \vec{r}(t') = \vec{r}(t(t')) \]

\[ \vec{r} = \vec{r}(t) = \vec{r}[t(t')] \]

\[ \vec{v}' = \frac{d\vec{r}}{dt} = \frac{dr}{dt} \hat{r} + r \frac{d\hat{r}}{dt} \]

\[ \vec{v} = \frac{d\vec{r}}{dt} = \frac{dr}{dt} \hat{r} + r \frac{d\hat{r}}{dt} \]

\[ \vec{v}' = \frac{d\vec{v}}{dt} = \frac{d\vec{r}}{dt} \frac{d\vec{r}}{dt} = \frac{d\vec{r}}{dt} \frac{d\vec{r}}{dt} = \frac{v}{\sqrt{1 - \frac{v^2}{c^2}}} \]

\[ \vec{v} = \frac{d\vec{v}}{dt} = \frac{d\vec{r}}{dt} \frac{d\vec{r}}{dt} = \frac{d\vec{r}}{dt} \frac{d\vec{r}}{dt} = \frac{\vec{v}}{\sqrt{1 + \frac{v^2}{c^2}}} \]

\[ \vec{a} = \frac{d\vec{v}}{dt} = \frac{d^2\vec{r}}{dt^2} = \frac{d^2(\vec{r}(t))}{dt^2} = \left[ \frac{d^2r}{dt^2} - r \left( \frac{d\hat{r}}{dt} \right)^2 \right] \hat{r} + \left( 2 \frac{dr}{dt} \frac{d\hat{r}}{dt} + r \frac{d^2\hat{r}}{dt^2} \right) \hat{\theta} \]

\[ \vec{a} = \frac{d\vec{v}}{dt} = \frac{d\vec{r}}{d\tau} = \frac{d^2\vec{r}}{d\tau^2} = \left( \frac{d^2r}{d\tau^2} - r \left( \frac{d\hat{r}}{d\tau} \right)^2 \right) \hat{r} + \left( 2 \frac{dr}{d\tau} \frac{d\hat{r}}{d\tau} + r \frac{d^2\hat{r}}{d\tau^2} \right) \hat{\theta} \]

Both speeds and accelerations are positive.

\[ \frac{d\vec{v}}{d\tau} = \frac{dt}{dt'} \frac{d}{dt} \left( \frac{\vec{v}}{\sqrt{1 - \frac{v^2}{c^2}}} \right) \]
\[ \mathbf{\tilde{F}}' = \frac{m_0 \mathbf{\tilde{\nu}}}{\sqrt{\frac{1}{c^2} \mathbf{v}^2 + c^2}} = \mathbf{\tilde{F}} = \frac{m_o}{(1 - \mathbf{v}^2/c^2)^{\frac{3}{2}}} \left[ (1 - \frac{\mathbf{v}^2}{c^2}) \frac{d\mathbf{v}}{dt} + \mathbf{v} \frac{d\mathbf{v}}{dt} \right] \]

\[ E_k = \int \mathbf{\tilde{F}}'. d\mathbf{\tilde{r}} = \int \mathbf{\tilde{F}}. d\mathbf{\tilde{r}} = \int -\frac{k}{r^2} \mathbf{\tilde{r}}. d\mathbf{\tilde{r}} \]

\[ E_k = \int \frac{m_0 \mathbf{v} d\mathbf{v}}{\sqrt{\frac{1}{c^2} \mathbf{v}^2 + c^2}} = \int -\frac{k}{r} \mathbf{v} \mathbf{d}r \quad dE_k = \tilde{\mathbf{F}}. d\mathbf{\tilde{r}} = \frac{m_0 \mathbf{v} d\mathbf{v}}{\sqrt{\frac{1}{c^2} \mathbf{v}^2 + c^2}} = -\frac{k}{r} \mathbf{v} \mathbf{d}r \]

\[ E_k = m_o c^2 \sqrt{1 + \frac{\mathbf{v}^2}{c^2}} = \frac{k}{r} + \text{constante} \quad E_R = m_o c^2 \sqrt{1 + \frac{\mathbf{v}^2}{c^2}} - k = m_o c^2 \]

\[ E_R = \frac{m_o c^2}{\sqrt{1 - \frac{\mathbf{v}^2}{c^2}}} - \frac{k}{r} = m_o c^2 \quad \frac{1}{\sqrt{1 - \frac{\mathbf{v}^2}{c^2}}} = \left( 1 + \frac{k}{m_o c^2} \right)^{\frac{1}{2}} = \left( 1 + A \right)^{\frac{1}{2}} \]

In this first variant relativistic kinetic energy is greater than inertial energy \( \frac{m_o c^2}{\sqrt{1 - \frac{\mathbf{v}^2}{c^2}}} > m_o c^2 \). This causes Mercury's perihelion to recede. The planet seems heavier due to the movement.

\[ \frac{d\mathbf{\tilde{r}}}{dt} = \tilde{\mathbf{F}}. \mathbf{\tilde{r}} = \frac{m_0 \mathbf{\tilde{v}} \mathbf{\tilde{v}}}{\sqrt{\frac{1}{c^2} \mathbf{v}^2 + c^2}} = -\frac{k}{r^2} \mathbf{\tilde{r}} \]

\[ \frac{d\mathbf{\tilde{v}}}{dt} = \tilde{\mathbf{F}}. \mathbf{\tilde{v}} = -\frac{k}{r^2} \mathbf{\tilde{r}} \]

\[ \frac{d\mathbf{\tilde{r}}}{dt} = \tilde{\mathbf{F}}. \mathbf{\tilde{r}} = \frac{m_0 \mathbf{\tilde{v}} \mathbf{\tilde{v}}}{\sqrt{\frac{1}{c^2} \mathbf{v}^2 + c^2}} = \frac{m_0 \mathbf{\tilde{v}} \mathbf{\tilde{v}}}{\sqrt{\frac{1}{c^2} \mathbf{v}^2 + c^2}} - \frac{k}{r^2} \mathbf{\tilde{r}}. \mathbf{\tilde{v}} \]

\[ \frac{d\mathbf{\tilde{r}}}{dt} = \tilde{\mathbf{F}}. \mathbf{\tilde{v}} = \frac{m_0 \mathbf{\tilde{v}} \mathbf{\tilde{v}}}{\sqrt{\frac{1}{c^2} \mathbf{v}^2 + c^2}} = \frac{m_0 \mathbf{\tilde{v}} \mathbf{\tilde{v}}}{\sqrt{\frac{1}{c^2} \mathbf{v}^2 + c^2}} - \frac{k}{r^2} \mathbf{\tilde{r}}. \mathbf{\tilde{v}} \]

\[ \mathbf{\tilde{F}} = \frac{m_0}{\sqrt{\frac{1}{c^2} \mathbf{v}^2 + c^2}} \left( \begin{vmatrix} r \left( \frac{d\mathbf{\tilde{v}}}{dt} \right)^2 \mathbf{\tilde{r}} + \left( 2 \frac{dr}{dt} \frac{d\mathbf{\tilde{w}}}{dt} + r \left( \frac{d\mathbf{\tilde{w}}}{dt} \right)^2 \mathbf{\tilde{w}} \right) \end{vmatrix} - \frac{k}{r^2} \mathbf{\tilde{r}} \right) \]

\[ \mathbf{\tilde{F}} \cdot \mathbf{\tilde{v}} = \frac{m_0}{\sqrt{\frac{1}{c^2} \mathbf{v}^2 + c^2}} \left( \begin{vmatrix} \frac{d^2 r}{dt^2} - r \left( \frac{d\mathbf{\tilde{v}}}{dt} \right)^2 \mathbf{\tilde{r}} \end{vmatrix} - \frac{k}{r^2} \mathbf{\tilde{r}} \right) \]

\[ \frac{m_0}{\sqrt{\frac{1}{c^2} \mathbf{v}^2 + c^2}} \left( \begin{vmatrix} \frac{d^2 r}{dt^2} - r \left( \frac{d\mathbf{\tilde{v}}}{dt} \right)^2 \mathbf{\tilde{r}} \end{vmatrix} - \frac{k}{r^2} \mathbf{\tilde{r}} \right) \]

\[ \frac{d\mathbf{\tilde{r}}}{dt} = \frac{\mathbf{\tilde{v}}}{r^2} \quad \frac{dr}{dt} = -\frac{d\mathbf{\tilde{w}}}{dt} \quad \frac{d^2 r}{dt^2} = \frac{-L^2 d^2 w}{r^2} \]

\[ \frac{1}{\sqrt{\frac{1}{c^2} \mathbf{v}^2 + c^2}} \left[ \begin{vmatrix} \frac{-L^2 d^2 w}{r^2} + r \left( \frac{L}{r^2} \right)^2 \mathbf{\tilde{r}} \end{vmatrix} - \frac{k}{m_o r^2} \right] \]

\[ \frac{1}{\sqrt{\frac{1}{c^2} \mathbf{v}^2 + c^2}} \left[ \begin{vmatrix} \frac{L^2 d^2 w}{r^2} + r \left( \frac{L}{r^2} \right)^2 \mathbf{\tilde{r}} \end{vmatrix} = \frac{k}{m_o r^2} \right] \]

\[ \frac{1}{\sqrt{\frac{1}{c^2} \mathbf{v}^2 + c^2}} \left( \begin{vmatrix} \frac{L^2 d^2 w}{r^2} + \frac{1}{r} \mathbf{\tilde{r}} \end{vmatrix} = \frac{k}{m_o L^2} \right] \]

\[ A = \frac{k}{m_o c^2} \quad B = \frac{k}{m_o L^2} \]
\[
\frac{1}{(1 - \frac{v^2}{c^2})^2} = (1 + A \frac{1}{c^2})^3 = 1 + 3A \frac{1}{c^2} + 3A^2 \frac{1}{c^4} + A^3 \frac{1}{c^6} \approx 1 + 3A \frac{1}{c^2} \quad 3A^2 \frac{1}{c^4} + A^3 \frac{1}{c^6} \approx \text{zero}
\]

\[
(1 + 3A \frac{1}{c^2}) \left( \frac{d^2w}{dt^2} + \frac{1}{c^2} \right) = B
\]

\[
(1 + 3A \frac{1}{c^2}) \frac{d^2w}{dt^2} + (1 + 3A \frac{1}{c^2}) \frac{1}{c^2} - B = \text{zero}
\]

\[
\frac{d^2w}{dt^2} + 3A \frac{d^2w}{dt^2} \frac{1}{c^2} + \frac{1}{c^2} + 3A \frac{1}{c^2} - B = \text{zero}
\]

\[
\frac{d^2w}{dt^2} + 3A \frac{d^2w}{dt^2} + w + 3Aw^2 - B = \text{zero}
\]

\[
\frac{d^2w}{dt^2} + w + 3A \frac{d^2w}{dt^2} + w + 3Aw^2 - B = \text{zero}
\]

\[
w = \frac{1}{r} \frac{d}{dr} \left[ 1 + \cos(\theta) \right] \quad \frac{dw}{dt} = -\frac{Q\sin(\theta)}{D} \quad \frac{d^2w}{dt^2} = -\frac{Q\cos(\theta)}{D}
\]

\[
-\frac{Q^2 \cos(\theta)}{D} + \frac{1}{\epsilon D} \left[ 1 + \cos(\theta) \right] + 3A \frac{Q^2 \cos(\theta)}{D} \frac{1}{\epsilon D} \left[ 1 + \cos(\theta) \right] + 3A \left\{ \frac{1}{\epsilon D} \left[ 1 + \cos(\theta) \right] \right\}^2 - B = \text{zero}
\]

\[
-\frac{Q^2 \cos(\theta)}{D} + \frac{1}{\epsilon D} \frac{\cos(\theta)}{D} - \frac{3A^2 \frac{1}{\epsilon D} \frac{\cos(\theta)}{D} - 3AQ^2 \frac{\cos(\theta)}{D} + \frac{3A}{3\epsilon D} + \frac{6A}{3\epsilon D} \frac{\cos(\theta)}{D} + \frac{3A}{3\epsilon D} - \frac{\epsilon DB}{\epsilon \epsilon D A} = \text{zero}
\]

\[
eDB = \frac{\epsilon D M_2 M_3}{m_2 GM_1 G} = 1
\]

\[
(1 - Q^2) \frac{\cos^2(\theta)}{D^2} + \left( \frac{1}{3A} \frac{Q^2}{\epsilon D} + \frac{2}{\epsilon D} \frac{\cos(\theta)}{D} \right) + \frac{1}{\epsilon^2 D^2} - \frac{1}{\epsilon \epsilon D A} = \text{zero}
\]

\[
(1 - Q^2) \frac{\cos^2(\theta)}{D^2} + \left( \frac{1}{3A} \frac{Q^2}{\epsilon D} + \frac{2}{\epsilon D} \frac{\cos(\theta)}{D} \right) + \frac{1}{\epsilon^2 D^2} = \text{zero} \quad Q^2 = \frac{1 + 12A}{1 + 6A}
\]

\[
\left[ 1 - \left( \frac{1}{1 + \frac{12A}{6A}} \right) \frac{\cos^2(\theta)}{D^2} + \frac{1}{\epsilon D} \left[ \frac{1}{3A} + \frac{1}{\frac{12A}{6A}} - \frac{1}{\epsilon D} \left( \frac{1 + 12A}{1 + 6A} \right) \right] + \frac{2}{\epsilon D} \frac{\cos(\theta)}{D} + \frac{1}{\epsilon^2 D^2} = \text{zero}
\]

\[
\left[ 1 + \frac{6A}{\epsilon D} - 1 - \frac{12A}{\epsilon D} \frac{\cos^2(\theta)}{D^2} + \left[ \frac{1}{3A} \left( 1 + \frac{6A}{\epsilon D} \right) - \frac{1}{\epsilon D} \left( 1 + 12A \right) \right] + \frac{2}{\epsilon D} \frac{\cos(\theta)}{D} + \frac{1}{\epsilon^2 D^2} = \text{zero}
\]

\[
\frac{1}{\epsilon D} \left( 1 + \frac{6A}{\epsilon D} \right) = \text{zero}
\]

\[
- \frac{6A \cos^2(\theta)}{\epsilon D} \frac{1}{\epsilon D} \left( \frac{1}{3A} + 1 - \frac{12A}{\epsilon D} \right) + \frac{1}{\epsilon D} \frac{\cos(\theta)}{D} + \frac{1}{\epsilon^2 D^2} = \text{zero}
\]

\[
- \frac{6A \cos^2(\theta)}{\epsilon D} \frac{1}{\epsilon D} \left( \frac{1}{3A} \right) + \frac{1}{\epsilon D} \frac{\cos(\theta)}{D} + \frac{1}{\epsilon^2 D^2} = \text{zero}
\]

\[
\frac{6A \cos^2(\theta)}{\epsilon D} + \frac{\cos(\theta)}{D} - \frac{1}{\epsilon D} \frac{6A}{\epsilon^2 D^2} = \text{zero}
\]

\[
\frac{\cos(\theta)}{D} = -\frac{1}{2.6A} \frac{-1 - 6A \left( \frac{1}{\epsilon^2 D^2} + \frac{6A}{\epsilon^2 D^2} \right)}{1 - 4.6A \left( \frac{1}{\epsilon^2 D^2} + \frac{6A}{\epsilon^2 D^2} \right)}
\]

\[
\frac{\cos(\theta)}{D} = -\frac{1}{2.6A} \frac{1 + 24A \epsilon^2 D^2 + 6A \epsilon^2 D^2}{1 + 24A \epsilon^2 D^2 + 6A \epsilon^2 D^2}
\]

\[
\frac{\cos(\theta)}{D} = -\frac{1}{12} \left( \frac{1 + 12A \epsilon^2 D^2}{12A} \right)
\]

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\[
\cos(Q) = \frac{-1 + \frac{(1 + 12A)}{DT}}{12A}
\]

\[
\cos(Q) = \frac{-1 + \frac{12A}{DT}}{12} = \frac{1}{eD}
\]

\[
\cos(Q) = \frac{1}{eD}
\]

\[
\varepsilon - \frac{1}{\cos(Q)} = \text{zero}
\]

For hyperbolic eccentricity \( (\varepsilon) \) is defined as \( \varepsilon = \frac{1}{\cos(Q)} \) where \( (\emptyset) \) is the angle of the asymptote.

**Advance of the Periellium** \( Q < 1 \)

\[
dt = dt'\sqrt{1 + \frac{v^2}{c^2}}
\]

\[
dt' = dt\sqrt{1 - \frac{v^2}{c^2}}
\]

\[
v = \frac{v}{\sqrt{1 + \frac{v^2}{c^2}}}
\]

\[
v' = \frac{v}{\sqrt{1 - \frac{v^2}{c^2}}}
\]

\[
\begin{align*}
\ddot{r} &= r\ddot{\theta} \\
\ddot{v} &= \frac{d^2\theta}{dt^2} + r \dot{\theta} \dot{\theta} \\
\ddot{v}' &= \frac{d^2\theta}{dt'^2} + r \dot{\theta} \dot{\theta} \\
\ddot{\theta} &= \frac{d^2\theta}{dt^2} \\
\ddot{\theta}' &= \frac{d^2\theta}{dt'^2} \\
\end{align*}
\]

\[
\ddot{F} = \frac{m_o \ddot{a}}{\sqrt{1 - \frac{v^2}{c^2}}} = \ddot{F}' = \frac{m_o}{\left(1 + \frac{v^2}{c^2}\right)^{\frac{1}{2}}} \left(1 + \frac{v^2}{c^2}\right) \frac{d^2\theta}{dt^2} - \frac{v^2}{c^2} \frac{dv}{dt} \frac{d\theta}{dt}
\]

\[
E_k = \int \ddot{F}.d\vec{r} = \int \ddot{F}'.d\vec{r}' = \int -\frac{k}{r^2}.d\vec{r} = \int \frac{m_o \ddot{a}}{\sqrt{1 - \frac{v^2}{c^2}}} \\
E_k = \int \frac{m_o \ddot{a}}{\sqrt{1 - \frac{v^2}{c^2}}} = \int m_o \ddot{a} \frac{dv}{dt} = \int \frac{m_o \ddot{a} \frac{dv}{dt}}{\sqrt{1 - \frac{v^2}{c^2}}} = \int \frac{m_o \ddot{a} \frac{dv}{dt}}{\sqrt{1 - \frac{v^2}{c^2}}} = \frac{-m_o^2 c^2}{\sqrt{1 - \frac{v^2}{c^2}}} = \frac{k}{r} \text{ constante}
\]

\[
E_k = -m_o c^2 \sqrt{1 - \frac{v^2}{c^2}} = \frac{k}{r} = \text{ constante} \quad E_R = -m_o c^2 \sqrt{1 - \frac{v^2}{c^2}} - \frac{k}{r} = -m_o c^2
\]

\[
E_R = -m_o c^2 \frac{k}{r} = -m_o c^2 \frac{1}{\left(1 + \frac{v^2}{c^2}\right)^{\frac{1}{2}}} = \left(1 - \frac{k}{m_o c^2} \right)^3 = \left(1 - A \right)^3
\]

In this second variant relativistic kinetic energy is smaller than inertial energy \( \frac{m_o c^2}{\sqrt{1 + \frac{v^2}{c^2}}} \). This causes the advance of Mercury’s perihelion. The planet really is lighter due to movement.
\[
\frac{d^2k}{dt^2} = \tilde{F}_r = \frac{m_o \tilde{v}}{(1 + \frac{v^2}{c^2})^{\frac{3}{2}}} = \frac{m_o \tilde{v} \dot{r}}{(1 + \frac{v^2}{c^2})^{\frac{3}{2}}} = - \frac{k}{r^2} \tilde{r} - \frac{k}{r^2} \tilde{r} \frac{d\tilde{r}}{dt}
\]

\[
\frac{d^2k}{dt^2} = \tilde{F}_r = \frac{m_o \tilde{v}}{(1 + \frac{v^2}{c^2})^{\frac{3}{2}}} = \frac{m_o \tilde{v} \dot{r}}{(1 + \frac{v^2}{c^2})^{\frac{3}{2}}} = - \frac{k}{r^2} \tilde{r}, \tilde{v}
\]

\[
\frac{d^2k}{dt^2} = \tilde{F}_r = \frac{m_o \tilde{v} \ddot{r}}{(1 + \frac{v^2}{c^2})^{\frac{3}{2}}} = \frac{m_o \tilde{v} \ddot{r}}{(1 + \frac{v^2}{c^2})^{\frac{3}{2}}} = - \frac{k}{r^2} \tilde{r}, \tilde{v}
\]

\[
\tilde{F}' = \frac{m_o}{(1 + \frac{v^2}{c^2})^{\frac{3}{2}}} \left[ \left( \frac{d^2r}{dt^2} - r \left( \frac{d\tilde{v}}{dt} \right)^2 \right) \tilde{r} + \left( \frac{2}{r} \frac{dr}{dt} \frac{d\tilde{v}}{dt} + r \frac{d^2\tilde{v}}{dt^2} \right) \tilde{v} \right] = - \frac{k}{r^2} \tilde{r}
\]

\[
\tilde{F}' = \left( \frac{2}{r} \frac{dr}{dt} \frac{d\tilde{v}}{dt} + r \frac{d^2\tilde{v}}{dt^2} \right) \tilde{v} = \text{zero}
\]

\[
\tilde{F}' \tilde{v} = \frac{m_o}{(1 + \frac{v^2}{c^2})^{\frac{3}{2}}} \left[ \left( \frac{d^2r}{dt^2} - r \left( \frac{d\tilde{v}}{dt} \right)^2 \right) \tilde{r} \right] = - \frac{k}{r^2}
\]

\[
\frac{d\tilde{v}}{dt} = \frac{L \tilde{r}}{r^2}, \quad \frac{dr}{dt} = - \frac{L \tilde{r}}{r^2} \quad \frac{d^2r}{dt^2} = - \frac{L^2 \tilde{r}^2}{r^2} \frac{d\tilde{v}}{dt}
\]

\[
\frac{1}{(1 + \frac{v^2}{c^2})^{\frac{3}{2}}} \left[ \frac{-L^2 \tilde{r}^2}{r^2} \frac{d\tilde{v}}{dt} - r \left( \frac{L \tilde{r}}{r^2} \right)^2 \right] = - \frac{k}{m_o} \tilde{v}
\]

\[
\frac{1}{(1 + \frac{v^2}{c^2})^{\frac{3}{2}}} \left[ L \tilde{r}^2 \frac{d\tilde{v}}{dt^2} + r \left( \frac{L \tilde{r}}{r^2} \right)^2 \right] = \frac{k}{m_o} \tilde{v}
\]

\[
\frac{1}{(1 + \frac{v^2}{c^2})^{\frac{3}{2}}} \left( \frac{d^2w}{dt^2} + \frac{d^2w}{dr^2} \right) = \frac{k}{m_o} \tilde{v}
\]

\[
A = \frac{k}{m_o c^2}, \quad B = \frac{k}{m_o l^2}
\]

\[
\frac{1}{(1 + \frac{v^2}{c^2})^{\frac{3}{2}}} = \left( 1 - \frac{A}{r} \right)^{\frac{3}{2}} = 1 - 3A \frac{1}{r} + 3A^2 \frac{1}{r^2} - A^3 \frac{1}{r^3} \equiv 1 - 3A \frac{1}{r}
\]

\[
3A^2 \frac{1}{r^2} - A^3 \frac{1}{r^3} \equiv \text{zero}
\]

\[
\left( 1 - 3A \frac{1}{r} \right) \frac{d^2w}{dt^2} \frac{1}{r} = B
\]

\[
\left( 1 - 3A \frac{1}{r} \right) \frac{d^2w}{dt^2} \frac{1}{r} - B = \text{zero}
\]

\[
\frac{d^2w}{dt^2} - 3A \frac{d^2w}{dt^2} \frac{1}{r} + \frac{1}{r} - 3A \frac{1}{r} - B = \text{zero}
\]

\[
\frac{d^2w}{dt^2} - 3A \frac{d^2w}{dt^2} w + w - 3Aw^2 - B = \text{zero}
\]

\[
\frac{d^2w}{dt^2} + w - 3A \frac{d^2w}{dt^2} w - 3Aw^2 - B = \text{zero}
\]

\[
w = \frac{1}{r} \left[ \frac{1}{ed} [1 + \cos(\Psi Q)] \right] \quad \frac{dw}{dr} = -\frac{Q_s}{B} (\Psi Q) \quad \frac{d^2w}{dt^2} = -\frac{Q^2 \cos(\Psi Q)}{B}
\]

\[
\frac{-Q^2 \cos(\Psi Q)}{B} + \frac{1}{ed} [1 + \cos(\Psi Q)] - 3A \frac{-Q^2 \cos(\Psi Q)}{ed} \frac{1}{ed} [1 + \cos(\Psi Q)] - 3A \left( \frac{1}{ed} [1 + \cos(\Psi Q)] \right)^2 - B = \text{zero}
\]

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\[-Q^2 \cos(\theta Q) \frac{d}{eD} + \frac{1}{eD} + \frac{\cos(\theta Q)}{D} + 3AQ^2 \cos(\theta Q) \frac{1}{eD} + 3AQ^2 \cos^2(\theta Q) \frac{d^2}{eD^2} - \left[ \frac{3A}{e^2D^2} + \frac{6A \cos(\theta Q)}{eD} + \frac{3A \cos^2(\theta Q)}{D^2} \right] - B = \text{zero} \]

\[-Q^2 \cos(\theta Q) \frac{d}{eD} + \frac{1}{eD} + \frac{\cos(\theta Q)}{D} + 3AQ^2 \cos(\theta Q) \frac{1}{eD} + 3AQ^2 \cos^2(\theta Q) \frac{d^2}{eD^2} - \left[ \frac{3A}{e^2D^2} + \frac{6A \cos(\theta Q)}{eD} + \frac{3A \cos^2(\theta Q)}{D^2} \right] - B = \text{zero} \]

\[\left( 3AQ^2 - 3A \right) \cos^2(\theta Q) \frac{d^2}{eD^2} + \left( 1 - Q^2 + 3AQ^2 \frac{1}{eD} - \frac{6A \cos(\theta Q)}{eD} \right) \cos(\theta Q) \frac{d^2}{D} + \frac{1}{eD} - \frac{3A}{e^2D^2} - B = \text{zero} \]

\[\left( 3AQ^2 - 3A \right) \cos^2(\theta Q) \frac{d^2}{eD^2} + \left( 1 - Q^2 + 3AQ^2 \frac{1}{eD} - \frac{6A \cos(\theta Q)}{eD} \right) \cos(\theta Q) \frac{d^2}{D} + \frac{1}{eD} - \frac{3A}{e^2D^2} - B = \text{zero} \]

\[(3AQ^2 - 3A) \frac{\cos^2(\theta Q)}{d^2} + \left( 1 - Q^2 + 3AQ^2 \frac{1}{eD} - \frac{6A \cos(\theta Q)}{eD} \right) \frac{\cos(\theta Q)}{D} + \frac{1}{eD} - \frac{3A}{e^2D^2} - B = \text{zero} \]

\[eDB = eDk = eDG_Md_{0} = m_{0} GM_{e} eD = 1 \]

\[(1 - Q^2) \frac{\cos^2(\theta Q)}{D^2} + \left( - \frac{1}{3A} + Q^2 \frac{1}{eD} - \frac{2}{eD} \right) \cos(\theta Q) \frac{d^2}{D} + \frac{1}{3Ae} - \frac{1}{e^2D^2} + \frac{1}{e^2D^2} = \text{zero} \]

\[1 - \left( \frac{1}{3A} + Q^2 \frac{1}{eD} - \frac{2}{eD} \right) \cos(\theta Q) \frac{d^2}{D} + \left[ - \frac{1}{3A} \left( 1 - \frac{6A}{eD} \right) + \frac{1}{3A} \left( 1 - \frac{12A}{eD} \right) + \frac{1}{eD} \left( 1 - \frac{6A}{eD} \right) \right] \cos(\theta Q) \frac{d^2}{D} + \frac{1}{e^2D^2} = \text{zero} \]

\[\left( 1 - \frac{6A}{eD} - 1 + \frac{12A}{eD} \cos^2(\theta Q) \frac{d^2}{D^2} + \left[ - \frac{1}{3A} \left( 1 - \frac{6A}{eD} \right) + \frac{1}{3A} \left( 1 - \frac{12A}{eD} \right) + \frac{1}{eD} \left( 1 - \frac{6A}{eD} \right) \right] \cos(\theta Q) \frac{d^2}{D} + \frac{1}{e^2D^2} \right) \frac{d^2}{D^2} - 1 + \frac{12A}{eD} = \text{zero} \]

\[\frac{6A \cos^2(\theta Q)}{eD} + \left( - \frac{1}{3A} + \frac{2}{eD} + \frac{1}{3A} \frac{4}{eD} - \frac{1}{eD} + \frac{12A}{e^2D^2} + \frac{2}{eD} \frac{12A}{e^2D^2} \right) \cos(\theta Q) \frac{d^2}{D} + \frac{1}{e^2D^2} = \text{zero} \]

\[\frac{6A \cos^2(\theta Q)}{eD} - \frac{\cos(\theta Q)}{D} + \frac{1}{eD} - \frac{6A}{e^2D^2} = \text{zero} \]

\[\cos(\theta Q) \frac{d}{D} = \frac{1}{\sqrt{1 - 6A \left( \frac{1}{eD} \frac{6A}{e^2D^2} \right)}} \]

\[\cos(\theta Q) \frac{d}{D} = \frac{1}{\sqrt{1 - 6A \left( \frac{1}{eD} \frac{6A}{e^2D^2} \right)}} \]

\[\cos(\theta Q) \frac{d}{D} = \frac{1}{\sqrt{\frac{24A}{eD} \frac{144A^2}{e^2D^2}}} \]

\[\cos(\theta Q) \frac{d}{D} = \frac{1}{\sqrt{1 - \frac{12A}{eD}}} \]

\[\cos(\theta Q) \frac{d}{D} = \frac{1 - 12A}{12A} \]

\[\cos(\theta Q) \frac{d}{D} = \frac{1}{eD} \]

\[\varepsilon \frac{1}{\cos(\theta Q)} = \text{zero} \]

For hyperbola eccentricity ($\varepsilon$) is defined as $\varepsilon = \frac{1}{\cos(\theta Q)}$ where ($\theta$) is the angle of the asymptote.
The movements of the ellipses will focus \( F' \) (left) on the origin of the frame.

All ellipses are described by the equation \( r = r(t) = \frac{eD}{1+e\cos(tQ)} = \frac{a(1-e^2)}{1+e\cos(tQ)} = \frac{5(1-0.8^2)}{1+0.8\cos(tQ)} \). In these the angle vector radius \( tQ \), indicates the position of the planet Mercury in all ellipses, the movement of Mercury in the ellipses is counterclockwise, with the value of \( Q \) being the cause of perihelion advancement or retraction.

The first ellipse in blue represents retrogression of the perihelion, where we have \( Q = 1.1 \).

The second red ellipse represents the advancement of the perihelion, in this we have \( Q = 0.9 \). In this ellipse the perihelion and aphelion advance in the trigonometric sense, that is, counterclockwise which is the same direction as the planet's movement in the ellipse.

The fifth ellipse in green represents a stationary ellipse \( Q = 1 \).
§ 29 Yukawa Potential Energy

Newton's gravitational potential energy $E_{PN}$

$$\vec{F} = -\frac{k}{r^2}\hat{r}$$

$$F = |\vec{F}| = \sqrt{\vec{F} \cdot \vec{F}} = \sqrt{\left(-\frac{k}{r^2}\hat{r}\right) \cdot \left(-\frac{k}{r^2}\hat{r}\right)} = \sqrt{\left(\frac{k}{r^2}\right)^2} = \frac{k}{r^2}$$

$$\vec{F} = -\vec{F}$$

$$E_{PN} = -\frac{k}{r}$$

$$F = \frac{dE_p}{dr} = \frac{k}{r^2} \quad \text{for } k > 0$$

Yukawa potential energy $E_{PY}$

$$E_{PY} = -ke^{-ar}/r = -kr^{-1}e^{-ar} \quad \text{for } k > 0 \quad a \geq 0$$

Potential Core Energy $E_N$

Breaking apart $E_{PY} = -ke^{-ar}/r$ we get:

$$E_N = -k\frac{e^{-ar}}{r} \left( -\frac{k}{r} \right) = E_{PN}C_Y$$

$$C_Y = \frac{1}{e^{ar}} \quad \text{for } k > 0 \quad a \geq 0$$

$$a = 0 \rightarrow C_Y = 1 \rightarrow E_N = E_{PN} \quad r = \infty \rightarrow E_N = 0$$

$$\frac{dE_p}{dr} = \frac{d}{dr} (-ke^{-ar}/r) = -k \left\{ (-1)r^{-1-1}\frac{dr}{dr}e^{-a} + (r^{-1})e^{-ar} \left( -\frac{dr}{dr} \right) \right\}$$

$$\frac{dE_p}{dr} = \frac{d}{dr} (-ke^{-ar}/r) = -k(-r^{-2}e^{-ar} - ar^{-1}e^{-ar}) = \frac{ke^{-ar}}{r^2} + ak \frac{e^{-ar}}{r}$$

$$\frac{dE_p}{dr} = \frac{d}{dr} \left( -\frac{k}{r^2} + ak \frac{e^{-ar}}{r} \right)$$

$$E_{PY} = \int dE_p = \int d \left( -\frac{ke^{-ar}}{r} \right) = \int \left( \frac{ke^{-ar}}{r^2} + ak \frac{e^{-ar}}{r} \right) dr = -\frac{ke^{-ar}}{r} + \text{constante}$$

$$\vec{F} = -\frac{dE_p}{dr}\hat{r} = -k \left( \frac{e^{-ar}}{r^2} + a \frac{e^{-ar}}{r} \right)\hat{r} \quad \text{Attractive force}$$

$$\vec{F} = ma^2 = \frac{m_o\bar{a}}{\sqrt{1 - \frac{v^2}{c^2}}} = \frac{m_o}{\sqrt{1 - \frac{v^2}{c^2}}} \frac{dv}{dt}$$

$$\vec{F} = ma^2 = \frac{m_o\bar{a}}{\sqrt{1 + \frac{v^2}{c^2}}} = \frac{m_o}{\sqrt{1 + \frac{v^2}{c^2}}} \frac{dv}{dr}$$

First variant.

$$\vec{F} = -\frac{m_o}{\sqrt{1 - \frac{v^2}{c^2}}} \frac{dv}{dt} = -k \left( \frac{e^{-ar}}{r^2} + a \frac{e^{-ar}}{r} \right)\hat{r}$$

$$E_k = \int \vec{F} \cdot d\vec{r} = \int \frac{m_o}{\sqrt{1 - \frac{v^2}{c^2}}} \frac{dv}{dt} \cdot d\vec{r} = -k \left( \frac{e^{-ar}}{r^2} + a \frac{e^{-ar}}{r} \right) \hat{r} \cdot d\vec{r}$$

$$E_k = \int \vec{F} \cdot d\vec{r} = \int \frac{m_o}{\sqrt{1 - \frac{v^2}{c^2}}} \frac{dv}{dt} = -k \left( \frac{e^{-ar}}{r^2} + a \frac{e^{-ar}}{r} \right) dr$$

$$E_k = \int \vec{F} \cdot d\vec{r} = \int \frac{m_o}{\sqrt{1 - \frac{v^2}{c^2}}} \cdot \vec{v} = -f \left( \frac{e^{-ar}}{r^2} + ak \frac{e^{-ar}}{r} \right) dr$$

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\[ E_k = \int \vec{F} \cdot d\vec{r} = \int \frac{m_o \nu d\nu}{\sqrt{1 - \frac{v^2}{c^2}}} = - \int \left( k \frac{e^{-ar}}{r^2} + ak \frac{e^{-ar}}{r} \right) dr \]

\[ dE_k = \vec{F} \cdot d\vec{r} = \frac{m_o \nu d\nu}{\sqrt{1 - \frac{v^2}{c^2}}} = -k \left( \frac{e^{-ar}}{r^2} + a \frac{e^{-ar}}{r} \right) dr \]

\[ E_k = -m_o c^2 \sqrt{1 - \frac{v^2}{c^2}} = -k \left( \frac{e^{-ar}}{r} \right) + \text{constante} \]

\[ E_k = -m_o c^2 \sqrt{1 - \frac{v^2}{c^2}} = k \frac{e^{-ar}}{r} + \text{constante} \]

\[ E_R = -m_o c^2 \sqrt{1 - \frac{v^2}{c^2}} = -k \frac{e^{-ar}}{r} = -m_o c^2 \]

\[ \sqrt{1 - \frac{v^2}{c^2}} = \frac{m_o c^2}{m_0 c^2} = -k \frac{e^{-ar}}{r} \]

\[ A = \frac{k}{m_0 c^2} \]

\[ \sqrt{1 - \frac{v^2}{c^2}} = 1 - A \frac{e^{-ar}}{r} \]

\[ dE_k = \vec{F} \cdot d\vec{r} = \frac{m_o \nu d\nu}{\sqrt{1 - \frac{v^2}{c^2}}} = -k \left( \frac{e^{-ar}}{r^2} + a \frac{e^{-ar}}{r} \right) dr \]

\[ \frac{dE_k}{dt} = \vec{F} \cdot \frac{d\vec{r}}{dt} = \frac{m_o \nu \frac{d\nu}{dt}}{\sqrt{1 - \frac{v^2}{c^2}}} = -k \left( \frac{e^{-ar}}{r^2} + a \frac{e^{-ar}}{r} \right) \frac{d\vec{r}}{dt} \]

\[ \frac{dE_k}{dt} = \vec{F} \cdot \frac{d\vec{r}}{dt} = \frac{m_o \nu \frac{d\nu}{dt}}{\sqrt{1 - \frac{v^2}{c^2}}} = -k \left( \frac{e^{-ar}}{r^2} + a \frac{e^{-ar}}{r} \right) \frac{d\vec{r}}{dt} \]

\[ \vec{F} = \frac{m_o \vec{a}}{\sqrt{1 - \frac{v^2}{c^2}}} = -k \left( \frac{e^{-ar}}{r^2} + a \frac{e^{-ar}}{r} \right) \vec{r} \]

\[ \vec{F} = \frac{m_o \vec{a}}{\sqrt{1 - \frac{v^2}{c^2}}} \left[ d^2 \frac{r \frac{d\theta}{dt}}{dt} \right] - \left( \frac{r \frac{d\theta}{dt}}{dt} \right)^2 \right] \vec{r} = -k \left( \frac{e^{-ar}}{r^2} + a \frac{e^{-ar}}{r} \right) \vec{r} \]

\[ \int \left[ d^2 \frac{r \frac{d\theta}{dt}}{dt} \right] - \left( \frac{r \frac{d\theta}{dt}}{dt} \right)^2 \right] \vec{r} = -k \left( \frac{e^{-ar}}{r^2} + a \frac{e^{-ar}}{r} \right) \vec{r} \]

\[ \frac{d\theta}{dt} = \frac{L}{r^2} \]

\[ \frac{d}{dt} \]
\[ \frac{d^2w}{dr^2} + \frac{1}{r} = B r^2 \left( e^{ar} + a e^{-ar} \right) \left( 1 - \frac{A e^{-ar}}{r} \right) \]
\[ B = \frac{k}{\text{mol} L^2} \]
\[ \frac{d^2w}{dr^2} + \frac{1}{r} = B e^{-ar} \left( 1 - A \frac{e^{-ar}}{r} \right) + a B e^{-ar} \left( 1 - A \frac{e^{-ar}}{r} \right) \]
\[ \frac{d^2w}{dr^2} + \frac{1}{r} = B e^{-ar} - A B e^{-ar} \frac{e^{-ar}}{r} + a B e^{-ar} - A B e^{-ar} \frac{e^{-ar}}{r} \]
\[ \frac{d^2w}{dr^2} + \frac{1}{r} = B e^{-ar} - A e^{-2ar} \frac{e^{-ar}}{r} + a B e^{-ar} - A B e^{-2ar} \frac{e^{-ar}}{r} \]
\[ w = \frac{1}{r} \quad r = w^{-1} \]
\[ w = e^{-aw^{-1}} - A B e^{-2a} - 1 + a B w^{-1} e^{-aw^{-1}} - a A B e^{-2aw^{-1}} \]
\[ w = e^{-aw^{-1}} + a B w^{-1} e^{-aw^{-1}} - A B e^{-2aw^{-1}} - a A B e^{-2aw^{-1}} \]
\[ w = (1 + aw^{-1}) B e^{-aw^{-1}} - (w + a) A B e^{-aw^{-1}} \]
\[ w = \left[ (1 + aw^{-1}) - (w + a) A e^{-aw^{-1}} \right] B e^{-aw^{-1}} \]
\[ r = \frac{1}{Q \cos(\theta)} \quad w = \frac{1}{r} = Q \cos(\theta) \quad \frac{dw}{d\theta} = -Q \sin(\theta) \quad \frac{d^2w}{d\theta^2} = -Q \cos(\theta) \]
\[ -Q \cos(\theta) + Q \cos(\theta) = \left[ (1 + aw^{-1}) - (w + a) A e^{-aw^{-1}} \right] B e^{-aw^{-1}} \]
\[ \text{zero} = \left[ (1 + aw^{-1}) - (w + a) A e^{-aw^{-1}} \right] B e^{-aw^{-1}} \]
\[ (1 + aw^{-1}) - (w + a) A e^{-aw^{-1}} = \text{zero} \]
\[ w = \frac{1}{r} = Q \cos(\theta) \quad r = w^{-1} = \frac{1}{Q \cos(\theta)} \]
\[ \left[ 1 + \frac{a}{Q \cos(\theta)} \right] - [Q \cos(\theta) + a] A e^{-aw^{-1}} = \text{zero} \]
\[ Q \cos(\theta) \left( 1 + \frac{a}{Q \cos(\theta)} \right) - Q \cos(\theta) [Q \cos(\theta) + a] A e^{-aw^{-1}} = \text{zero} \]
\[ Q \cos(\theta) + a - Q^2 \cos^2(\theta) A e^{-aw^{-1}} - Q \cos(\theta) a A e^{-aw^{-1}} = \text{zero} \]
\[ -Q \cos(\theta) - a + Q^2 \cos^2(\theta) A e^{-aw^{-1}} + Q \cos(\theta) a A e^{-aw^{-1}} = \text{zero} \]
\[ Q^2 \cos^2(\theta) A e^{-aw^{-1}} - Q \cos(\theta) + Q \cos(\theta) a A e^{-aw^{-1}} - a = \text{zero} \]
\[ Q^2 \cos^2(\theta) A e^{-aw^{-1}} - Q \cos(\theta) (1 - a A e^{-aw^{-1}}) - a = \text{zero} \]
\[ Q^2 \cos^2(\theta) A e^{-aw^{-1}} - Q \cos(\theta) (1 - a A e^{-aw^{-1}}) - a = \text{zero} \]
\[ Q \cos(\theta) = \frac{1 - a A e^{-aw^{-1}} \pm \sqrt{(1 - a A e^{-aw^{-1}})^2 - 4 A e^{-aw^{-1}} (-a)}}{2 A e^{-aw^{-1}}} \]
\[ Q \cos(\theta) = \frac{1 - a A e^{-aw^{-1}} \pm \sqrt{1 - 2 a A e^{-aw^{-1}} + a^2 A^2 e^{-2aw^{-1}} + 4 A A e^{-aw^{-1}}}}{2 A e^{-aw^{-1}}} \]
\[ Q \cos(\theta) = \frac{1 - a A e^{-aw^{-1}} \pm \sqrt{1 + 2 a A e^{-aw^{-1}} + a^2 A^2 e^{-2aw^{-1}}}}{2 A e^{-aw^{-1}}} \]
\[ Q \cos(\theta) = \frac{1 - a A e^{-aw^{-1}} \pm \sqrt{(1 + a A e^{-aw^{-1}})^2}}{2 A e^{-aw^{-1}}} \]
\[ Q \cos(\theta) = \frac{1 - a A e^{-aw^{-1}} \pm (1 + a A e^{-aw^{-1}})}{2 A e^{-aw^{-1}}} \]
\[ Q \cos(\theta) = \frac{1 - aAe^{-aw^{-1}} - 1 - aAe^{-aw^{-1}}}{2Ae^{-aw^{-1}}} \]

\[ Q \cos(\theta) = \frac{-aAe^{-aw^{-1}} - aAe^{-aw^{-1}}}{2Ae^{-aw^{-1}}} \]

\[ w = \frac{1}{r} Q \cos(\theta) = \frac{-a - a}{2} = \frac{-2a}{r} = -a \]

\[ r = \frac{-1}{a} \quad E_p = \text{constante} \]

\[ \frac{dE_p}{dr} = \frac{d}{dr} \left( -k \frac{e^{-ar}}{r} \right) = \left( k \frac{e^{-ar}}{r^2} + ak \frac{e^{-ar}}{r} \right) = \text{zero} \]

\[ k \frac{e^{-ar}}{r^2} + ak \frac{e^{-ar}}{r} = \text{zero} \]

\[ \frac{1}{r} = -a \quad r = \frac{-1}{a} \quad E_p = \text{constante} \]

\[ E_{pY} = -k \frac{e^{-a}}{r} = -kr^{-1}e^{-a} = -(a) e^{-a}e^{-1} = ake \]

\[ Q \cos(\theta) = \frac{1 - aAe^{-aw^{-1}} \pm (1 + aAe^{-aw^{-1}})}{2Ae^{-aw^{-1}}} \]

\[ Q \cos(\theta) = \frac{1 - aAe^{-aw^{-1}} + 1 + aAe^{-aw^{-1}}}{2Ae^{-aw^{-1}}} \]

\[ w = \frac{1}{r} Q \cos(\theta) = \frac{2}{2Ae^{-aw^{-1}}} = \frac{1}{Ae^{-aw^{-1}}} \]

\[ \text{Second variant.} \]

\[ \vec{F} = \frac{m_0}{\sqrt{1 + \frac{v^2}{c^2}}} \frac{dv}{dt} = -k \left( \frac{e^{-a}}{r^2} + a \frac{e^{-ar}}{r} \right) \vec{F} \]

\[ E_k = \int \vec{F} \cdot d\vec{r} = \int \frac{m_0}{\sqrt{1 + \frac{v^2}{c^2}}} \frac{dv}{dt} \cdot d\vec{r} = \int -k \left( \frac{e^{-a}}{r^2} + a \frac{e^{-ar}}{r} \right) \vec{F} \cdot d\vec{r} \]

\[ E_k = \int \vec{F} \cdot d\vec{r} = \int \frac{m_0}{\sqrt{1 + \frac{v^2}{c^2}}} \frac{dv}{dt} \frac{d\vec{r}}{dr} = \int -k \left( \frac{e^{-ar}}{r^2} + a \frac{e^{-ar}}{r} \right) dr \]

\[ E_k = \int \vec{F} \cdot d\vec{r} = \int \frac{m_0}{\sqrt{1 + \frac{v^2}{c^2}}} \frac{dv}{dt} \cdot \vec{r} = -\int \left( k \frac{e^{-ar}}{r^2} + ak \frac{e^{-ar}}{r} \right) dr \]

\[ E_k = \int \vec{F} \cdot d\vec{r} = \int \frac{m_0}{\sqrt{1 + \frac{v^2}{c^2}}} \frac{dv}{dt} = -\int \left( k \frac{e^{-ar}}{r^2} + ak \frac{e^{-ar}}{r} \right) dr \]

\[ E_k = m_0 c^2 \sqrt{1 + \frac{v^2}{c^2}} = -\left( kr^2 + \text{constante} \right) \]

\[ E_k = m_0 c^2 \sqrt{1 + \frac{v^2}{c^2}} = k \frac{e^{-ar}}{r} + \text{constante} \]

\[ E_k = m_0 c^2 \sqrt{1 + \frac{v^2}{c^2}} = k \frac{e^{-ar}}{r} + m_0 c^2 \]

\[ E_R = m_0 c^2 \sqrt{1 + \frac{v^2}{c^2}} - k \frac{e^{-ar}}{r} = m_0 c^2 \]

\[ E_R = m_0 c^2 \sqrt{1 + \frac{(zero)^2}{c^2}} = k \frac{e^{-ar}}{r} = m_0 c^2 \]

\[ \sqrt{1 + \frac{v^2}{c^2}} = \frac{m_0 c^2}{m_0 c^2} + \frac{k e^{-ar}}{r} \]

\[ A = \frac{k}{m_0 c^2} \]

\[ \sqrt{1 + \frac{v^2}{c^2}} = 1 + A \frac{e^{-ar}}{r} \]

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\[ \text{d}E_k = \vec{F} \cdot \text{d}\vec{r} = \frac{m_o \dot{v} v'}{\sqrt{1 + v'^2/c^2}} = -k \left( \frac{e^{-ar}}{r^2} + a \frac{e^{-ar}}{r} \right) \text{d}r \]

\[ \frac{\text{d}E_k}{\text{d}t'} = \vec{F}' \cdot \text{d}\vec{r}' = \frac{m_o \dot{v} v'}{\sqrt{1 + v'^2/c^2}} = -k \left( \frac{e^{-ar}}{r^2} + a \frac{e^{-ar}}{r} \right) \dot{r} \text{d}t' \]

\[ \frac{\text{d}E_k}{\text{d}t'} = \vec{F} \cdot \text{d}\vec{r}' = \frac{m_o \dot{v} v'}{\sqrt{1 + v'^2/c^2}} = -k \left( \frac{e^{-ar}}{r^2} + a \frac{e^{-ar}}{r} \right) \dot{r} \text{d}t' \]

\[ \vec{F}' = \frac{m_o \ddot{a}}{\sqrt{1 + v'^2/c^2}} = -k \left( \frac{e^{-ar}}{r^2} + a \frac{e^{-ar}}{r} \right) \ddot{r} \]

\[ \ddot{r} = \frac{m_o}{\sqrt{1 + v'^2/c^2}} \left[ \left( \frac{d^2 r}{dt'^2} \right)^2 - \left( \frac{d\phi}{dt'} \right)^2 \right] + \left( \frac{2 \ddot{r} \frac{dr}{dt'} + r \frac{d^2 \phi}{dt'^2} \dot{\phi} \right) = -k \left( \frac{e^{-ar}}{r^2} + a \frac{e^{-ar}}{r} \right) \ddot{r} \]

\[ \ddot{a} = \left( \frac{d^2 \phi}{dt'^2} \frac{dr}{dt'} + r \frac{d^2 \phi}{dt'^2} \right) \ddot{\phi} = \text{zero} \quad \frac{dl}{dt'} = \frac{d}{dt} \left( \frac{r^2 \frac{d\phi}{dt'} \frac{dr}{dt'} + r^2 \frac{d^2 \phi}{dt'^2} \right) = \text{zero} \]

\[ \dot{F}_{\phi} = \left( \frac{d}{dt} \frac{d\phi}{dt'} \right) \ddot{\phi} = \text{zero} \]

\[ \ddot{E}_k = \frac{d^2}{dt'^2} \left( r \left( \frac{d\phi}{dt'} \right)^2 \right) \ddot{\phi} = -k \left( \frac{e^{-ar}}{r^2} + a \frac{e^{-ar}}{r} \right) \ddot{\phi} \]

\[ \frac{d\phi}{dt'} = \frac{L}{r} \quad \frac{dr}{dt'} = -L' \frac{dw}{d\phi} \quad \frac{d^2 r}{dt'^2} = -\frac{L^2}{r^2} \frac{d^2 w}{d\phi^2} \]

\[ \frac{d^2 \phi}{dt'^2} \frac{dw}{d\phi} - \frac{1}{r} \left( \frac{d\phi}{dt'} \right)^2 \right] = -k \left( \frac{e^{-ar}}{r^2} + a \frac{e^{-ar}}{r} \right) \left( 1 + A \frac{e^{-ar}}{r} \right) \]

\[ \frac{d^2 w}{d\phi^2} + \frac{1}{r} = \frac{m_o L^2 r^2}{k} \left( \frac{e^{-ar}}{r^2} + a \frac{e^{-ar}}{r} \right) \left( 1 + A \frac{e^{-ar}}{r} \right) \]

\[ \frac{d^2 w}{d\phi^2} + \frac{1}{r} = Br^2 \left( \frac{e^{-ar}}{r^2} + a \frac{e^{-ar}}{r} \right) \left( 1 + A \frac{e^{-ar}}{r} \right) \]
\[
\frac{d^2w}{dr^2} + w = \left[1 + aw^{-1}\right] + (w + a)Ae^{-aw^{-1}}\]

\[
r = \frac{1}{\cos(\theta)} \quad w = \frac{1}{r} = \cos(\theta) \quad \frac{dw}{d\theta} = -\sin(\theta) \quad \frac{d^2w}{d\theta^2} = -\cos(\theta)
\]

\[
-\cos(\theta) + \cos(\theta) = \left[1 + aw^{-1}\right] + (w + a)Ae^{-aw^{-1}}\]

zero = \left[1 + aw^{-1}\right] + (w + a)Ae^{-aw^{-1}}\]

\[
(1 + aw^{-1}) + (w + a)Ae^{-aw^{-1}} = zero \quad w = \frac{1}{r} = \cos(\theta) \quad r = w^{-1} = \frac{1}{\cos(\theta)}
\]

\[
\left[1 + \frac{a}{\cos(\theta)}\right] + [\cos(\theta) + a]Ae^{-aw^{-1}} = zero
\]

\[
\cos(\theta)\left(1 + \frac{a}{\cos(\theta)}\right) + \cos(\theta)[\cos(\theta) + a]Ae^{-aw^{-1}} = zero
\]

\[
\cos(\theta) + a + q^2\cos^2(\theta)Ae^{-aw^{-1}} + \cos(\theta)aAe^{-aw^{-1}} = zero
\]

\[
q^2\cos^2(\theta)Ae^{-aw^{-1}} + \cos(\theta) + \cos(\theta)aAe^{-aw^{-1}} + a = zero
\]

\[
q^2\cos^2(\theta)Ae^{-aw^{-1}} + \cos(\theta)(1 + aAe^{-aw^{-1}}) + a = zero
\]

\[
\cos(\theta) = \frac{-(1 + aAe^{-aw^{-1}}) \pm \sqrt{(1 + aAe^{-aw^{-1}})^2 - 4Ae^{-aw^{-1}}}}{2Ae^{-aw^{-1}}}
\]

\[
\cos(\theta) = \frac{-(1 + aAe^{-aw^{-1}}) \pm \sqrt{1 + 2aAe^{-aw^{-1}} + a^2A^2e^{-2aw^{-1}} - 4aAe^{-aw^{-1}}}}{2Ae^{-aw^{-1}}}
\]

\[
\cos(\theta) = \frac{-(1 - aAe^{-aw^{-1}}) \pm \sqrt{1 - 2aAe^{-aw^{-1}} + a^2A^2e^{-2aw^{-1}}}}{2Ae^{-aw^{-1}}}
\]

\[
\cos(\theta) = \frac{-(1 - aAe^{-aw^{-1}}) \pm \sqrt{(1 - aAe^{-aw^{-1}})^2}}{2Ae^{-aw^{-1}}}
\]

\[
\cos(\theta) = \frac{-(1 - aAe^{-aw^{-1}}) \pm (1 - aAe^{-aw^{-1}})}{2Ae^{-aw^{-1}}}
\]

\[
w = \frac{1}{r} = \cos(\theta) = \frac{-1 + aAe^{-aw^{-1}} + 1 - aAe^{-aw^{-1}}}{2Ae^{-aw^{-1}}} = zero
\]

\[
\cos(\theta) = \frac{-(1 - aAe^{-aw^{-1}}) \pm (1 - aAe^{-aw^{-1}})}{2Ae^{-aw^{-1}}}
\]

\[
\cos(\theta) = \frac{-1 + aAe^{-aw^{-1}} - 1 + aAe^{-aw^{-1}}}{2Ae^{-aw^{-1}}} \quad w = \frac{1}{r} = \cos(\theta) \quad r = w^{-1} = \frac{1}{\cos(\theta)}
\]

\[
w = \frac{1}{r} = \cos(\theta) = \frac{-2 + 2aAe^{-aw^{-1}}}{2Ae^{-aw^{-1}}} = \frac{-1 + aAe^{-aw^{-1}}}{Ae^{-aw^{-1}}} = a - \frac{1}{Ae^{-aw^{-1}}}
\]

§ 29 Yukawa Potential Energy “Continuation”

Newton's gravitational potential energy \(E_p\)

\[
\mathbf{F} = -\frac{k}{r^2} \mathbf{r} \quad \mathbf{E}_p = \mathbf{F} = \mathbf{F} = \sqrt{\mathbf{F}, \mathbf{F}} = \sqrt{\left(-\frac{k}{r^2}\mathbf{r}\right)\left(-\frac{k}{r^2}\mathbf{r}\right)} = \sqrt{\left(\frac{k}{r^2}\right)^2 \mathbf{r} \mathbf{r}} = \sqrt{\left(\frac{k}{r^2}\right)^2} = \frac{k}{r^2}
\]

\[
\mathbf{F} = -\mathbf{F} \quad \mathbf{F} = -\mathbf{F} \quad \mathbf{F} = \frac{dE_p}{dr} = \frac{k}{r^2} \quad k > zero
\]

161/172
\[ \vec{F} = -\frac{dE_p}{dr}\hat{r} = -\frac{k}{r^2}\hat{r} \]

Yukawa potential energy \( E_{pY} \)

\[
E_{pY} = -k\frac{e^{-ar}}{r} = -kr^{-1}e^{-ar} \quad \text{k > zero} \quad \text{a > zero}
\]

\[
\frac{dE_p}{dr} = \frac{d}{dr}(-kr^{-1}e^{-ar}) = -k\left\{(-1)r^{-1-1}\frac{dr}{dr}e^{-ar} + (r^{-1})e^{-ar}(-a\frac{dr}{dr})\right\}
\]

\[
\frac{dE_p}{dr} = \frac{d}{dr}(-kr^{-1}e^{-ar}) = -k(-r^{-2}e^{-ar} - a^{-1}e^{-ar}) = k(r^{-2}e^{-ar} + a^{-1}e^{-ar}) = k\left(\frac{e^{-ar}}{r^2} + a\frac{e^{-ar}}{r}\right)
\]

\[
\frac{dE_p}{dr} = \frac{d}{dr}(-kr^{-1}e^{-ar}) = k\left(\frac{e^{-a}}{r^2} + a\frac{e^{-ar}}{r}\right)
\]

\[
E_p = \int dE_p = k \int \left(\frac{e^{-ar}}{r^2} + a\frac{e^{-ar}}{r}\right) dr = -k\frac{e^{-ar}}{r}
\]

\[ \vec{F} = -\frac{dE_p}{dr} = -k\left(\frac{e^{-ar}}{r^2} + a\frac{e^{-ar}}{r}\right)\hat{r} \] Attractive force

\[
\vec{F} = \frac{m_0}{\sqrt{1 - \frac{v^2c^2}{c^2}}} = \vec{F}' = \frac{m_0}{\sqrt{1 - \frac{v^2c^2}{c^2}}} \left(1 + \frac{v^2}{c^2}\right) \frac{d\vec{r}}{dt} - \frac{v}{c^2} \frac{d\vec{v}}{dt}\]

28.16

\[
\vec{F} = \frac{m_0}{\sqrt{1 - \frac{v^2c^2}{c^2}}} = \vec{F}' = \frac{m_0}{\sqrt{1 + \frac{v^2c^2}{c^2}}} \left(1 + \frac{v^2}{c^2}\right) \frac{d\vec{r}}{dt} - \frac{v}{c^2} \frac{d\vec{v}}{dt} = -k\left(\frac{e^{-ar}}{r^2} + a\frac{e^{-ar}}{r}\right)\hat{r}
\]

\[
E_k = \int \vec{F} \cdot d\vec{r} = \int \vec{F}' \cdot d\vec{r} = \int -k\left(\frac{e^{-ar}}{r^2} + a\frac{e^{-ar}}{r}\right) \hat{r} \cdot d\vec{r}
\]

\[
E_k = \int \frac{m_0}{\sqrt{1 - \frac{v^2c^2}{c^2}}} \left(1 + \frac{v^2}{c^2}\right) \left(\frac{d\vec{r}}{dt} - \frac{v}{c^2} \frac{d\vec{v}}{dt}\right) dr = \int -k\left(\frac{e^{-ar}}{r^2} + a\frac{e^{-ar}}{r}\right) \hat{r} \cdot d\vec{r}
\]

28.17

\[
E_k = \frac{m_0v^2}{\sqrt{1 - \frac{v^2c^2}{c^2}}} = \frac{m_0v^2}{\sqrt{1 + \frac{v^2c^2}{c^2}}} \left(1 + \frac{v^2}{c^2}\right) \frac{d\vec{r}}{dt} = \int -k\left(\frac{e^{-ar}}{r^2} + a\frac{e^{-ar}}{r}\right) \hat{r} \cdot d\vec{r}
\]

28.18

\[
E_k = -m_0c^2\sqrt{1 - \frac{v^2}{c^2}} = -\frac{m_0c^2}{\sqrt{1 + \frac{v^2c^2}{c^2}}} = -\left(-k\frac{e^{-ar}}{r}\right) + \text{constante}
\]

\[
E_R = -m_0c^2\sqrt{1 - \frac{v^2}{c^2}} - k\frac{e^{-ar}}{r} = -m_0c^2
\]

28.18

\[
E_R = -m_0c^2\sqrt{1 - \frac{v^2}{c^2}} - k\frac{e^{-ar}}{r} = -m_0c^2 \quad \frac{1}{\sqrt{1 + \frac{v^2c^2}{c^2}}} = \left(1 - \frac{k}{m_0c^2} \frac{e^{-ar}}{r}\right)^3 = \left(1 - A\frac{e^{-ar}}{r}\right)^3
\]

28.19

\[
\frac{1}{\sqrt{1 + \frac{v^2c^2}{c^2}}} = \left(1 - A\frac{e^{-ar}}{r}\right)^3 \quad \text{A} = \frac{k}{m_0c^2}
\]

28.19

\[
\frac{dE_k}{dt} = \frac{m_0}{\sqrt{1 - \frac{v^2c^2}{c^2}}} \sqrt{\frac{v^2}{c^2} + \frac{v^2}{c^2}} \frac{d\vec{v}}{dt} - \frac{m_0v}{\sqrt{1 - \frac{v^2c^2}{c^2}}} \sqrt{\frac{v^2}{c^2} + \frac{v^2}{c^2}} \frac{d\vec{v}'}{dt} = -k\left(\frac{e^{-ar}}{r^2} + a\frac{e^{-ar}}{r}\right) \frac{dr}{dt} - k\left(\frac{e^{-ar}}{r^2} + a\frac{e^{-ar}}{r}\right) \hat{r} \cdot \frac{d\vec{r}}{dt}
\]
\[
\frac{d\mathbf{E}}{dt} = \mathbf{F}', \quad \mathbf{V}' = \frac{m_0 v_0^2}{c^2} = \frac{m_0 \mathbf{V}'_0}{c^2} = -k \left( \frac{e^{-ar}}{r^2} + a \frac{e^{-ar}}{r} \right) \mathbf{F}.
\]

\[
\frac{d\mathbf{E}}{dt} = \mathbf{F}', \quad \mathbf{V}' = \frac{m_0 v_0^2}{c^2} = \frac{m_0 \mathbf{V}'_0}{c^2} = -k \left( \frac{e^{-ar}}{r^2} + a \frac{e^{-ar}}{r} \right) \mathbf{F}.
\]

\[
\mathbf{F}' = \frac{m_0 v_0}{(1 + v_0^2/c^2)^2},
\]

\[
\mathbf{F}' = \frac{m_0 v_0}{(1 + v_0^2/c^2)^2} \left\{ \frac{d^2r}{dt^2} - r \left( \frac{d\varphi}{dt} \right)^2 \right\} \mathbf{F} + \left( 2 \frac{dr}{dt} \frac{d\varphi}{dt} \frac{d^2\varphi}{dt^2} + r \frac{d^2\varphi}{dt^2} \right) \mathbf{F} = -k \left( \frac{e^{-ar}}{r^2} + a \frac{e^{-ar}}{r} \right) \mathbf{F}.
\]

\[
\mathbf{F}' = \frac{m_0}{(1 + v_0^2/c^2)^2} \left\{ \frac{d^2r}{dt^2} - r \left( \frac{d\varphi}{dt} \right)^2 \right\} \mathbf{F} = -k \left( \frac{e^{-ar}}{r^2} + a \frac{e^{-ar}}{r} \right) \mathbf{F}.
\]

\[
\mathbf{F}' = \frac{m_0}{(1 + v_0^2/c^2)^2} \left\{ \frac{d^2r}{dt^2} - r \left( \frac{d\varphi}{dt} \right)^2 \right\} \mathbf{F} = -k \left( \frac{e^{-ar}}{r^2} + a \frac{e^{-ar}}{r} \right) \mathbf{F}.
\]

\[
\frac{d\varphi}{dt} = \frac{L}{r^2}, \quad \frac{dr}{dt} = -\frac{L}{r^2} \frac{dw}{dr}, \quad \frac{d^2r}{dt^2} = -\frac{L^2}{r^2} \frac{d^2w}{dr^2}
\]

\[
\frac{1}{(1 + v_0^2/c^2)^2} \left\{ \frac{d^2r}{dt^2} - r \left( \frac{d\varphi}{dt} \right)^2 \right\} = -k \frac{\left( \frac{e^{-ar}}{r^2} + a \frac{e^{-ar}}{r} \right)}{m_0}
\]

\[
\frac{1}{(1 + v_0^2/c^2)^2} \left\{ \frac{d^2r}{dt^2} + \frac{L^2}{r^2} \frac{d\varphi}{dt} \right\} = \frac{k}{m_0} \left( \frac{e^{-ar}}{r^2} + a \frac{e^{-ar}}{r} \right)
\]

\[
\frac{1}{(1 + v_0^2/c^2)^2} \left\{ \frac{d^2r}{dt^2} + \frac{L^2}{r^2} \frac{d\varphi}{dt} \right\} = \frac{k}{m_0} \left( \frac{e^{-ar}}{r^2} + a \frac{e^{-ar}}{r} \right)
\]

\[
\frac{1}{(1 + v_0^2/c^2)^2} \left\{ \frac{d^2w}{dr^2} + \frac{1}{r} \right\} = \frac{k}{m_0 L^2} r^2 \left( \frac{e^{-ar}}{r^2} + a \frac{e^{-ar}}{r} \right)
\]

\[
\frac{1}{(1 + v_0^2/c^2)^2} \left\{ \frac{d^2w}{dr^2} + \frac{1}{r} \right\} = Br^2 \left( \frac{e^{-ar}}{r^2} + a \frac{e^{-ar}}{r} \right)
\]

\[
\frac{1}{(1 + v_0^2/c^2)^2} \left\{ \frac{d^2w}{dr^2} + \frac{1}{r} \right\} = Br^2 \left( \frac{e^{-ar}}{r^2} + a \frac{e^{-ar}}{r} \right)
\]

\[
\frac{1}{3A^2 - A^3 e^{-3ar}/r^3} \approx 1 - 3A \frac{e^{-ar}}{r} + 3A^2 \frac{e^{-2ar}}{r^2} - A^3 \frac{e^{-3ar}}{r^3} \approx 1 - 3A \frac{e^{-ar}}{r}
\]

\[
3A^2 \frac{e^{-2ar}}{r^2} - A^3 \frac{e^{-3ar}}{r^3} \approx zero
\]

\[
A = \frac{k}{m_0 c^2}
\]

\[
\left( 1 - 3A \frac{e^{-ar}}{r} \right) \left( \frac{d^2w}{dr^2} + \frac{1}{r} \right) = Br^2 \left( \frac{e^{-ar}}{r^2} + a \frac{e^{-ar}}{r} \right)
\]

\[
\left( 1 - 3A \frac{e^{-ar}}{r} \right) \frac{d^2w}{dr^2} + \left( 1 - 3A \frac{e^{-ar}}{r} \right) \frac{1}{r} = Br^2 \left( \frac{e^{-ar}}{r^2} + a \frac{e^{-ar}}{r} \right)
\]

\[
\frac{d^2w}{dr^2} = 3A \frac{d^2w}{dr^2} e^{-ar} + \frac{1}{r} - 3A \frac{e^{-ar}}{r} = Br^2 \left( \frac{e^{-ar}}{r^2} + a \frac{e^{-ar}}{r} \right)
\]

\[
\frac{d^2w}{dr^2} = 3A \frac{d^2w}{dr^2} e^{-ar} + \frac{1}{r} - 3A \frac{e^{-ar}}{r} = Be^{-ar} + r e^{-ar}
\]
\[\frac{d^2 w}{d \phi^2} w + \frac{w^2}{r^2} - 3A \frac{d^2 w}{d \phi^2} e^{-ar} + \frac{1}{r^2} - 3A \frac{e^{-ar}}{r^3} = B e^{-ar} + aBe^{-ar} \]

\[w = \frac{1}{r} \]

\[\frac{d^2 w}{d \phi^2} w - 3A \frac{d^2 w}{d \phi^2} e^{-ar} w^2 + w^2 - 3Ae^{-ar}w^3 = Be^{-ar}w + aBe^{-a} \]

\[\frac{d^2 w}{d \phi^2} w + w^2 = 3A \frac{d^2 w}{d \phi^2} e^{-ar} w^2 + 3Ae^{-ar}w^3 + Be^{-ar}w + aBe^{-ar} \]

\[\frac{d^2 w}{d \phi^2} w + w^2 = e^{-ar} \left( 3A \frac{d^2 w}{d \phi^2} w^2 + 3Aw^3 + Bw + aB \right) \]

28.22

\[w = \frac{1}{r} = xe^{i\phi} + ye^{-i\phi} \]

\[\frac{d w}{d \phi} = ixe^{i\phi} - iye^{-i\phi} \]

\[\frac{d^2 w}{d \phi^2} = -xe^{i\phi} - ye^{-i\phi} \]

\[i = \sqrt{-1} \]

\[\left( -xe^{i\phi} - ye^{-i\phi} \right) \left( xe^{i\phi} + ye^{-i\phi} \right) + \left( xe^{i\phi} + ye^{-i\phi} \right)^2 = e^{-ar} \left( 3A \frac{d^2 w}{d \phi^2} w^2 + 3Aw^3 + Bw + aB \right) \]

\[\left( -xe^{i\phi} - ye^{-i\phi} \right) \left( xe^{i\phi} + ye^{-i\phi} \right) = \left( -xe^{i\phi}(xe^{i\phi}) + (-xe^{i\phi})(ye^{-i\phi}) + (-ye^{-i\phi})(xe^{i\phi}) + (-ye^{-i\phi})(ye^{-i\phi}) \right) \]

\[= \left( -x^2e^{2i\phi} - xy - ye^{-2i\phi} \right) = \left( y^2e^{2i\phi} + 2xy + y^2e^{-2i\phi} \right) = \left( xe^{i\phi} + ye^{-i\phi} \right)^2 \]

\[-\left( xe^{i\phi} + ye^{-i\phi} \right)^2 + \left( xe^{i\phi} + ye^{-i\phi} \right)^2 = e^{-a} \left( 3A \frac{d^2 w}{d \phi^2} w^2 + 3Aw^3 + Bw + aB \right) \]

\[\text{zero} = e^{-ar} \left( 3A \frac{d^2 w}{d \phi^2} w^2 + 3Aw^3 + Bw + aB \right) \]

\[3A \frac{d^2 w}{d \phi^2} w^2 + 3Aw^3 + Bw + aB = \text{zero} \]

\[3A(-xe^{i\phi} - ye^{-i\phi}) (xe^{i\phi} + ye^{-i\phi})^2 + 3A(xe^{i\phi} + ye^{-i\phi})^3 + B(xe^{i\phi} + ye^{-i\phi}) + aB = \text{zero} \]

\[3A(-xe^{i\phi} - ye^{-i\phi})(xe^{i\phi} + ye^{-i\phi})^2 = -3A(xe^{i\phi} + ye^{-i\phi})(xe^{i\phi} + ye^{-i\phi})^2 = -3A(xe^{i\phi} + ye^{-i\phi})^3 \]

\[-3A(xe^{i\phi} + ye^{-i\phi})^3 + 3A(xe^{i\phi} + ye^{-i\phi})^3 + B(xe^{i\phi} + ye^{-i\phi}) + aB = \text{zero} \]

\[B(xe^{i\phi} + ye^{-i\phi}) + aB = \text{zero} \quad xe^{i\phi} + ye^{-i\phi} + a = \text{zero} \]

\[w = \frac{1}{r} = xe^{i\phi} + ye^{-i\phi} = -a \]
§30 Energy

In §28 simplified calculation of the perihelion retraction we obtain:

\[ E_k = \int \frac{m_0 v^2 dv}{\sqrt{1 - \frac{v^2}{c^2}}} = \int \frac{m_0 v dv}{\sqrt{1 - \frac{v^2}{c^2}}} = 0 \quad \text{dE}_k = \dot{F}, \, \text{d}r = \frac{m_0 v^2 dv}{\sqrt{1 + \frac{v^2}{c^2}}} = -\frac{k}{r} \text{dr} \]

28.08

\[ E_k = m_0 c^2 \sqrt{1 + \frac{v^2}{c^2}} = \frac{m_0 c^2}{\sqrt{1 - \frac{v^2}{c^2}}} = k \quad \text{E}_R = m_0 c^2 \sqrt{1 + \frac{v^2}{c^2}} - \frac{k}{r} = m_0 c^2 \]

28.09

\[ E_R = \frac{m_0 c^2}{\sqrt{1 - \frac{v^2}{c^2}}} - \frac{k}{r} = m_0 c^2 \]

\[ r \quad \frac{1}{(1 - \frac{v^2}{c^2})} = (1 + \frac{k}{m_0 c^2})^3 = (1 + \lambda z)^3 \]

28.10

In this first variant relativistic kinetic energy is greater than inertial energy \( m_0 c^2 > m_0 c^2 \). This causes Mercury's perihelion to recede. The planet seems heavier due to the movement.

\[ E_k = \int \frac{m_0 v^2 dv}{\sqrt{1 - \frac{v^2}{c^2}}} = \int \frac{m_0 v dv}{\sqrt{1 - \frac{v^2}{c^2}}} = 0 \quad \text{dE}_k = \dot{F}, \, \text{d}r = \frac{m_0 v^2 dv}{\sqrt{1 + \frac{v^2}{c^2}}} = -\frac{k}{r} \text{dr} \]

28.08

\[ E_k = \int_{r=0}^{r} \frac{m_0 v^2 dv}{\sqrt{1 + \frac{v^2}{c^2}}} = \int_{v=0}^{v} \frac{m_0 v dv}{\sqrt{1 - \frac{v^2}{c^2}}} = \frac{k}{r} \text{dr} \]

E_k = \frac{m_0 c^2}{\sqrt{1 + \frac{v^2}{c^2}}} \left(1 + \frac{v^2}{c^2}\right) = \frac{m_0 c^2}{\sqrt{1 - \frac{v^2}{c^2}}} \left(1 + \frac{v^2}{c^2}\right)

E_k = \frac{m_0 c^2}{\sqrt{1 + \frac{v^2}{c^2}}} \left(1 + \frac{v^2}{c^2}\right) - m_0 c^2 = \frac{k}{r} \quad m_0 c^2 \left(1 + \frac{v^2}{c^2}\right) \geq m_0 c^2 \quad \frac{m_0 c^2}{\sqrt{1 - \frac{v^2}{c^2}}} \geq m_0 c^2

30.1

Defining potential energy as

\[ E_p = -\frac{m_0}{r} \]

30.2

And applying in 1 we have

\[ E_k = m_0 c^2 \left(1 + \frac{v^2}{c^2}\right) - m_0 c^2 = \frac{m_0 c^2}{\sqrt{1 - \frac{v^2}{c^2}}} - m_0 c^2 = -E_p \]

30.3

In 3 we have the energy conservation principle written as \( E_k + E_p = 0 \)

30.4

With 3 the kinetic energy equal to

\[ E_k = m_0 c^2 \left(1 + \frac{v^2}{c^2}\right) - m_0 c^2 = \frac{m_0 c^2}{\sqrt{1 - \frac{v^2}{c^2}}} - m_0 c^2 \]

30.5

n 3 the lowest energy of the system is the inertial energy of rest

\[ E_0 = m_0 c^2 \]

30.6

In 3 the highest energy of the system is

\[ E = m_0 c^2 \left(1 + \frac{v^2}{c^2}\right) = \frac{m_0 c^2}{\sqrt{1 - \frac{v^2}{c^2}}} \]

30.7

Now defining \( p = \frac{m_0 v}{\sqrt{1 - \frac{v^2}{c^2}}} \quad \text{v} = \frac{m_0 v^2}{\sqrt{1 - \frac{v^2}{c^2}}} = m_0 v \)

\[ T' = m_0 c^2 \left(1 \frac{v^2}{c^2}\right) \quad T = -m_0 c^2 \left(1 - \frac{v^2}{c^2}\right) \]

30.8

And knowing that

\[ E = c \sqrt{m_0 c^2 + p^2} = \frac{m_0 c^2}{\sqrt{1 - \frac{v^2}{c^2}}} \quad \frac{m_0 c^2}{\sqrt{1 - \frac{v^2}{c^2}}} + m_0 c^2 \left(1 - \frac{v^2}{c^2}\right) = \text{vp} - T \]

30.9

If in 9 \( m_0 = 0 \quad \text{So} \quad E = \text{cp} \quad \text{30.10} \]
Applying 8 and 9 in 7 we obtain the greatest energy of the written system as:

\[ E = c\sqrt{m_0^2 c^2 + p^2} = T' = vp - T \]

This we have \( vp = T' + T \) 30.7b

With 8 and 9 we can write 3 in the form:

\[ E_k = m_0 c^2 \sqrt{1 + \frac{v^2}{c^2}} - m_0 c^2 = \frac{m_0 v^2}{\sqrt{1 - \frac{v^2}{c^2}}} + m_0 c^2 \sqrt{1 - \frac{v^2}{c^2}} - m_0 c^2 = -E_p \]

30.3b

\[ E_k = T' - E_0 = vp - T - E_0 = -E_p \]

30.3c

From 3c we can define the resting inertial energy \( E_0 = m_0 c^2 \) and \( E_0 = vp - T + E_p = m_0 c^2 \).

30.11

Defining Lagrangean as

\[ L = T - E_p = -m_0 c^2 \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}} + \frac{k}{r} \]

30.12

This Lagrangian meets \( \frac{d}{dt} \left( \frac{\partial L}{\partial \dot{r}} \right) = \frac{\partial L}{\partial r} \) according to §24.

So from 11 temos:

\[ E'_0 = T' + E_p = m_0 c^2 \quad \text{and} \quad E_0 = vp - L = m_0 c^2 \]

30.13

In §28 simplified calculation of perihelion advance we obtain:

\[
E_k = \int m_0 v dv = \int m_0 v dv' = \int -\frac{k}{r^2} dr \quad \text{de}E_k = \ddot{r}' \dot{r} = \frac{m_0 v dv}{\sqrt{1 - \frac{v^2}{c^2}}} = -\frac{k}{r^2} dr
\]

28.17

\[ E_k = -m_0 c^2 \sqrt{1 - \frac{v^2}{c^2}} = \frac{m_0 c^2}{\sqrt{1 + \frac{v^2}{c^2}}} = \frac{k}{r} \quad \text{constant} \quad E_R = -m_0 c^2 \sqrt{1 - \frac{v^2}{c^2}} - \frac{k}{r} = -m_0 c^2 \]

28.18

\[ E_R = -m_0 c^2 \frac{1}{\sqrt{1 + \frac{v^2}{c^2}}} = \frac{k}{r} = -m_0 c^2 \frac{1}{\frac{1}{1 + \frac{v^2}{c^2}}} \left( 1 \right) = \left( 1 - \frac{k}{m_0 c^2 r} \right)^3 \]

28.19

In this second variant relativistic kinetic energy is smaller than inertial energy \( \frac{m_0 c^2}{\sqrt{1 + \frac{v^2}{c^2}}} \). This causes the advance of Mercury's perihelion. The planet really is lighter due to movement.

\[ E_k = \int m_0 v dv = \int m_0 v dv' = \int -\frac{k}{r^2} dr \]

28.17

\[ E_k = \int_{v=0}^{v} m_0 v dv = \int_{v'=0}^{v'} m_0 v dv = \int_{r=0}^{r} -\frac{k}{r^2} dr \]

\[ E_k = -m_0 c^2 \sqrt{1 - \frac{v^2}{c^2}} = -m_0 c^2 \frac{1}{\sqrt{1 + \frac{v^2}{c^2}}} \]

\[ E_k = -m_0 c^2 \sqrt{1 - \frac{v^2}{c^2}} = -m_0 c^2 \frac{1}{\sqrt{1 + \frac{v^2}{c^2}}} \]

\[ E_k = -m_0 c^2 \sqrt{1 - \frac{v^2}{c^2}} = -m_0 c^2 \frac{1}{\sqrt{1 + \frac{v^2}{c^2}}} \]

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\[ E_k = -m_0 c^2 \sqrt{1 - \frac{v^2}{c^2}} = -m_0 c^2 \frac{1}{\sqrt{1 + \frac{v^2}{c^2}}} \]

\[ E_k = -m_0 c^2 \sqrt{1 - \frac{v^2}{c^2}} = -m_0 c^2 \frac{1}{\sqrt{1 + \frac{v^2}{c^2}}} \]
$$E_k = -m_o c^2 \sqrt{1 - \frac{v^2}{c^2}} + m_o c^2 = -\frac{m_o c^2}{\sqrt{1 + \frac{v^2}{c^2}}} + m_o c^2 = \frac{k}{r}$$

$$E_k = m_o c^2 - m_o c^2 \sqrt{1 - \frac{v^2}{c^2}} = m_o c^2 - \frac{m_o c^2}{\sqrt{1 + \frac{v^2}{c^2}}} \geq m_o c^2 \sqrt{1 - \frac{v^2}{c^2}} \geq \frac{m_o c^2}{\sqrt{1 + \frac{v^2}{c^2}}}$$  

Where applying 2 the definition of potential energy  

$$E_p = -\frac{k}{r}.$$  

We get  

$$E_k = m_o c^2 - m_o c^2 \sqrt{1 - \frac{v^2}{c^2}} = m_o c^2 - \frac{m_o c^2}{\sqrt{1 + \frac{v^2}{c^2}}} = -E_p$$  

In 15 we have the principle of conservation of energy written as  

$$E_k + E_p = 0.$$  

Being 15 the kinetic energy equal to:  

$$E_k = m_o c^2 \sqrt{1 - \frac{v^2}{c^2}} = \frac{m_o c^2}{\sqrt{1 + \frac{v^2}{c^2}}}$$  

In 15 the biggest energy of the system is the inertial energy of rest  

$$E_0 = m_o c^2.$$  

At 15 the lowest energy in the system is  

$$E' = m_o c^2 \sqrt{1 - \frac{v^2}{c^2}} = \frac{m_o c^2}{\sqrt{1 + \frac{v^2}{c^2}}}$$  

Now defining  

$$p' = \frac{m_o v'}{\sqrt{1 + \frac{v^2}{c^2}}} = m_o v \quad v' p' = \frac{m_o v'^2}{\sqrt{1 + \frac{v^2}{c^2}}} = m_o v'$$  

And knowing that  

$$E' = c \sqrt{m_o c^2 - p'^2} = \frac{m_o c^2}{\sqrt{1 + \frac{v^2}{c^2}}} = \frac{m_o c^2}{\sqrt{1 + \frac{v^2}{c^2}}} + m_o c^2 \frac{1 + \frac{v^2}{c^2}}{\sqrt{1 + \frac{v^2}{c^2}}} = -v' p' + T'$$  

If in 20  

$$m_o = 0$$  

so  

$$E' = t e p'$$  

Proving 20:  

$$E' = \frac{m_o c^2}{\sqrt{1 + \frac{v^2}{c^2}}} = c \sqrt{m_o c^2 - v'^2} = c \sqrt{m_o c^2 - \left(\frac{m_o v^2}{1 + \frac{v^2}{c^2}}\right)^2} = c \sqrt{m_o c^2 - \frac{m_o c^2}{1 + \frac{v^2}{c^2}}} = m_o c \sqrt{\frac{c^2}{1 + \frac{v^2}{c^2}} - \frac{v^2}{1 + \frac{v^2}{c^2}}}$$  

$$E' = \frac{m_o c^2}{\sqrt{1 + \frac{v^2}{c^2}}} = m_o c \sqrt{c^2 \left(1 + \frac{v^2}{c^2}\right) - v'^2} = \frac{m_o c^2}{\sqrt{1 + \frac{v^2}{c^2}}}$$  

$$E' = m_o c^2 \sqrt{1 - \frac{v^2}{c^2}} = c \sqrt{m_o c^2 - v'^2} = c \sqrt{m_o c^2 - \left(m_o v\right)^2} = c \sqrt{m_o c^2 - m_o v^2} = m_o c \sqrt{c^2 - v'^2} = m_o c^2 \sqrt{1 - \frac{v^2}{c^2}}$$  

Applying 8, 19 and 20 to 18 results in the lowest energy of the written system as:  

$$E' = c \sqrt{m_o c^2 - v'^2} = -T = -v' p' + T'$$  

This we have  

$$v' p' = T' + T$$  

With 19 and 20 we can write 15 in the form:
\[ E_k = m_o c^2 - m_o c^2 \sqrt{1 - \frac{v^2}{c^2}} = m_o c^2 + \frac{m_o v^2}{\sqrt{1 + \frac{v^2}{c^2}}} - m_o c^2 \sqrt{1 + \frac{v^2}{c^2}} = -E_p \]  

30.15b

\[ E_k = E_0 + T = E_0 + v'p' - T' = -E_p \]  

30.15c

From 15c we can define the resting inertial energy \( E'_0 = m_o \) \( E_0 = m_o \) in the form:

\[ E_0 = -T - E_p = m_o c^2 \]  

and \( E'_0 = -v'p' + T' - E_p = m_o c^2. \)  

30.22

Defining Lagrangean as \( L' = T' - E_p = m_o c^2 \sqrt{1 + \frac{v^2}{c^2}} + \frac{k}{r} \)  

30.23

This Lagrangian meets \( \frac{d}{dx} \left( \frac{\partial L'}{\partial \dot{x}} \right) = \frac{\partial L'}{\partial x} \) prove no final.

\[ E_0 = -T - E_p = m_o c^2 \]  

\( E'_0 = -v'p' + L' = m_o c^2 \)  

30.24

Rewriting 11 e 22:

\[ E'_0 = T' + E_p = m_o c^2 \]  

and \( E_0 = vp - T + E_p = m_o c^2. \)  

30.11

\[ E_0 = -T - E_p = m_o c^2 \]  

and \( E'_0 = -v'p' + T' - E_p = m_o c^2. \)  

30.22

Equating \( E_0 \) of 11 with \( E_0 \) of 22 we have:

\[ E_0 = vp - T + E_p = -T - E_p = m_o c^2 \]

This we get \( vp = -2E_p \) \( -E_p = \frac{vp}{2} = \frac{1}{2} \frac{m_o v^2}{\sqrt{1 - \frac{v^2}{c^2}}} \)  

30.25

Matching \( -E_p = \frac{vp}{2} \) the kinetic energy of 3b we have:

\[ E_k = m_o c^2 \sqrt{1 + \frac{v^2}{c^2}} - m_o c^2 = m_o c^2 + m_o c^2 \sqrt{1 - \frac{v^2}{c^2}} - m_o c^2 = -E_p = \frac{vp}{2} = \frac{1}{2} \frac{m_o v^2}{\sqrt{1 - \frac{v^2}{c^2}}} \]  

30.3d

In 3d we should have:

\[ \frac{m_o v^2}{\sqrt{1 - \frac{v^2}{c^2}}} + m_o c^2 \sqrt{1 - \frac{v^2}{c^2}} - m_o c^2 = \frac{1}{2} \frac{m_o v^2}{\sqrt{1 - \frac{v^2}{c^2}}} \]

\[ m_o v^2 + m_o c^2 \left( 1 - \frac{v^2}{c^2} \right) - m_o c^2 \sqrt{1 - \frac{v^2}{c^2}} = \frac{1}{2} m_o v^2 \]

\[ \frac{1}{2} m_o v^2 + m_o c^2 - m_o c^2 \frac{v^2}{c^2} - m_o c^2 \sqrt{1 - \frac{v^2}{c^2}} = 0 \]

\[ \frac{1}{2} m_o v^2 + m_o c^2 - m_o c^2 \frac{v^2}{c^2} - m_o c^2 \sqrt{1 - \frac{v^2}{c^2}} = 0 \]

\[ m_o c^2 - \frac{c^2}{2c^2} m_o c^2 - m_o c^2 \sqrt{1 - \frac{v^2}{c^2}} = 0 \]

\[ m_o c^2 \left( 1 - \frac{v^2}{2c^2} \right) - m_o c^2 \sqrt{1 - \frac{v^2}{c^2}} = 0 \]

The approximation \( \left( 1 - \frac{v^2}{2c^2} \right) \approx \sqrt{1 - \frac{v^2}{c^2}} \) is the cause of Mercury’s perihelion setback.

\[ m_o c^2 \sqrt{1 - \frac{v^2}{c^2}} - m_o c^2 \sqrt{1 - \frac{v^2}{c^2}} = 0 \]  

Result that proves that \( E_k = \frac{1}{2} \frac{m_o v^2}{\sqrt{1 - \frac{v^2}{c^2}}} \).
Equating $E_0$ of 11 with $E_0$ of 22 we have:

$$E_0 = T' + E_p = -v'p' + T' - E_p = m_c c^2$$

This we get:

$$v'p' = -2E_p$$

$$-E_p = \frac{v'p'}{2} = \frac{1}{2} \frac{m_c v'^2}{\sqrt{1 - v'^2/c^2}}$$

Matching $-E_p = \frac{v'p'}{2}$ the kinetic energy of 15b we have:

$$E_k = m_o c^2 - m_o c^2 \sqrt{1 - \frac{v^2}{c^2}} = m_o c^2 + \frac{m_o v^2}{\sqrt{1 + \frac{v^2}{c^2}}} - m_o c^2 \sqrt{1 + \frac{v^2}{c^2}} = -E_p = \frac{v'p'}{2} = \frac{1}{2} \frac{m_o v^2}{\sqrt{1 + \frac{v^2}{c^2}}}$$

In 15d we should have:

$$m_o c^2 \frac{1}{\sqrt{1 + \frac{v'^2}{c^2}}} - m_o c^2 \sqrt{1 + \frac{v^2}{c^2}} = \frac{1}{2} m_o v^2 \frac{1}{\sqrt{1 + \frac{v^2}{c^2}}}$$

$$m_o c^2 + \frac{1}{2} m_o v^2 - m_o c^2 \sqrt{1 + \frac{v^2}{c^2}} = 0$$

$$m_o c^2 \sqrt{1 + \frac{v'^2}{c^2}} + \frac{1}{2} m_o v'^2 - m_o c^2 \left(1 + \frac{v'^2}{c^2}\right) = 0$$

$$m_o c^2 \sqrt{1 + \frac{v'^2}{c^2}} - m_o c^2 = \frac{1}{2} m_o c^2 v'^2 = 0$$

$$m_o c^2 \left(1 + \frac{v'^2}{c^2}\right) - m_o c^2 \left(1 + \frac{v^2}{c^2}\right) = 0$$

The approximation $(1 + \frac{1}{2} \frac{v^2}{c^2}) \approx \sqrt{1 + \frac{v^2}{c^2}}$ is the cause of the advance of Mercury's perihelion.

$$m_o c^2 \sqrt{1 + \frac{v^2}{c^2}} - m_o c^2 \sqrt{1 + \frac{v'^2}{c^2}} = 0$$

Result that proves that $E_k = \frac{1}{2} \frac{m_o v^2}{\sqrt{1 + \frac{v^2}{c^2}}}$.  

From 25 and 26 results

$$v_p = v'p'$$

Applying 25 to E_0 of 22:

$$E_0 = -T - E_p = -T + \frac{v_p}{2} = m_o c^2 \sqrt{1 - \frac{v^2}{c^2}} + \frac{1}{2} m_o v^2 \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}} = m_o c^2 \left(1 - \frac{2v^2}{2c^2} + \frac{v^2}{2c^2}\right)$$

$$E_0 = -T - E_p = -T + \frac{v_p}{2} = m_o c^2 \frac{1}{\sqrt{1 - \frac{v'^2}{c^2}}} \left(1 - \frac{v^2}{2c^2}\right) \approx \frac{m_o c^2}{\sqrt{1 - \frac{v^2}{c^2}}} \sqrt{1 - \frac{v^2}{c^2}} = m_o c^2$$

Applying 25 to $E_0$ of 11:

$$E_0 = v_p - T + E_p = \frac{2v_p}{2} - T - \frac{v_p}{2} = -T + \frac{v_p}{2}$$

Result already obtained on 28.

Applying 26 in $E'_0$ of 11
\[ E' = T' + E_p = T' - \frac{v'p'}{2} = m_o c^2 \left( 1 + \frac{v^2}{c^2} \right) - \frac{1}{2} m_o v^2 = m_o c^2 \sqrt{1 + \frac{v'^2}{2c^2} - \frac{v^2}{c^2}} \]  
\[ 30.29 \]

\[ E'_0 = T' + E_p = T' - \frac{v'p'}{2} = m_o c^2 \left( 1 + \frac{v'^2}{2c^2} \right) \approx m_o c^2 \sqrt{1 + \frac{v'^2}{c^2}} = m_o c^2 \]  
\[ 30.29 \]

The approximation that exists in 28 and 29 is the cause of the advance and setback of Mercury's perihelion.

Applying 26 to \( E'_0 \) of 22

\[ E'_0 = -v'p' + T' - E_p = -\frac{2v'p'}{2} + T' + \frac{v'p'}{2} = T' - \frac{v'p'}{2} \]  
Result already obtained on 29.

Proof that \( \frac{d}{dt} \left( \frac{\partial L}{\partial \dot{\mathbf{x}}} \right) = \frac{\partial L}{\partial \mathbf{x}} \)

\[ \begin{align*}
F'_x &= \frac{d}{dt} \left( \frac{\partial L}{\partial \dot{\mathbf{x}}} \right) = \frac{\partial L}{\partial \dot{\mathbf{x}}} = F_x = \frac{d}{dt} \left[ \frac{\partial}{\partial \mathbf{x}} \left( m_o c^2 \sqrt{1 + \frac{v^2}{c^2}} \right) \right] = \frac{\partial}{\partial \mathbf{x}} \left( \frac{\partial x}{\partial \mathbf{r}} \right) \\
\mathbf{v}' &= \frac{ds}{dt} = \sqrt{\dot{x}^2 + \dot{y}^2 + \dot{z}^2} \quad ds = |ds| = \sqrt{dx^2 + dy^2 + dz^2} \\
\mathbf{F}' &= F'_x \hat{i} + F'_y \hat{j} + F'_z \hat{k} = -k \frac{\dot{y}}{r} \hat{i} - k \frac{\dot{z}}{r} \hat{j} - k \frac{\dot{x}}{r} \hat{k} = -k \frac{\dot{r}}{r} \hat{r} = -k \frac{\dot{r}}{r} \hat{r} \\
p' &= m_o c^2 \sqrt{1 + \frac{v^2}{c^2}} = m_o c^2 \left( 1 + \frac{v'^2}{2c^2} \right) - \frac{v'p'}{2} = m_o c^2 \sqrt{1 + \frac{v'^2}{2c^2} - \frac{v^2}{c^2}} \\
\frac{\partial v}{\partial x} &= \frac{\partial}{\partial x} \left( \sqrt{\dot{x}^2 + \dot{y}^2 + \dot{z}^2} \right) = \frac{1}{2} \left( \dot{x}^2 + \dot{y}^2 + \dot{z}^2 \right)^{-\frac{1}{2}} \dot{x} \frac{\partial x}{\partial \mathbf{r}} = \frac{\dot{x}}{\sqrt{\dot{x}^2 + \dot{y}^2 + \dot{z}^2}} \\
p'_x &= m_o c^2 \sqrt{1 + \frac{v^2}{c^2}} = m_o c^2 \sqrt{1 + \frac{v'^2}{2c^2} - \frac{v^2}{c^2}} \\
\frac{d}{dt} \left( \frac{\partial l}{\partial \mathbf{x}} \right) &= \frac{\partial l}{\partial \mathbf{x}} = F'_x = \frac{d}{dt} \left( m_o c^2 \sqrt{1 + \frac{v^2}{c^2}} \right) = \frac{\partial}{\partial \mathbf{x}} \left( \frac{\partial x}{\partial \mathbf{r}} \right) \\
\frac{\partial v}{\partial x} &= \frac{\partial}{\partial x} \left( \sqrt{\dot{x}^2 + \dot{y}^2 + \dot{z}^2} \right) = \frac{1}{2} \left( \dot{x}^2 + \dot{y}^2 + \dot{z}^2 \right)^{-\frac{1}{2}} \dot{x} \frac{\partial x}{\partial \mathbf{r}} = \frac{\dot{x}}{\sqrt{\dot{x}^2 + \dot{y}^2 + \dot{z}^2}} \\
\frac{d}{dt} \left( m_o c^2 \sqrt{1 + \frac{v^2}{c^2}} \right) &= \frac{d}{dt} \left( m_o c^2 \sqrt{1 + \frac{v'^2}{2c^2} - \frac{v^2}{c^2}} \right) = \frac{d}{dt} \left( m_o c^2 \sqrt{1 + \frac{v'^2}{c^2}} \right) = \frac{d}{dt} \left( m_o c^2 \sqrt{1 + \frac{v'^2}{2c^2} - \frac{v^2}{c^2}} \right) = \frac{d}{dt} \left( m_o c^2 \sqrt{1 + \frac{v'^2}{c^2}} \right) \\
\frac{d}{dt} \left( \sqrt{1 + \frac{v^2}{c^2}} \right) &= \frac{1}{2} \left( 1 + \frac{v^2}{c^2} \right)^{-\frac{1}{2}} \cdot \frac{1}{2} \left( \dot{x}^2 + \dot{y}^2 + \dot{z}^2 \right)^{-\frac{1}{2}} \dot{x} \frac{\partial x}{\partial \mathbf{r}} = \frac{1}{\sqrt{1 + \frac{v^2}{c^2}}} \frac{d}{dt} \left( \sqrt{1 + \frac{v^2}{c^2}} \right) \\
F'_x &= \frac{d}{dt} \left( \frac{m_o c^2}{\sqrt{1 + \frac{v^2}{c^2}}} \right) = \frac{m_o c^2}{\sqrt{1 + \frac{v^2}{c^2}}} \frac{\partial}{\partial \mathbf{x}} \left( \frac{\partial x}{\partial \mathbf{r}} \right) \\
\frac{\partial v}{\partial x} &= \frac{\partial}{\partial x} \left( \sqrt{\dot{x}^2 + \dot{y}^2 + \dot{z}^2} \right) = \frac{1}{2} \left( \dot{x}^2 + \dot{y}^2 + \dot{z}^2 \right)^{-\frac{1}{2}} \dot{x} \frac{\partial x}{\partial \mathbf{r}} = \frac{\dot{x}}{\sqrt{\dot{x}^2 + \dot{y}^2 + \dot{z}^2}} \\
F'_x &= \frac{d}{dt} \left( \frac{m_o c^2}{\sqrt{1 + \frac{v^2}{c^2}}} \right) = \frac{m_o c^2}{\sqrt{1 + \frac{v^2}{c^2}}} \frac{\partial}{\partial \mathbf{x}} \left( \frac{\partial x}{\partial \mathbf{r}} \right) \\
F'_x &= \frac{d}{dt} \left( \frac{m_o c^2}{\sqrt{1 + \frac{v^2}{c^2}}} \right) = \frac{m_o c^2}{\sqrt{1 + \frac{v^2}{c^2}}} \frac{\partial}{\partial \mathbf{x}} \left( \frac{\partial x}{\partial \mathbf{r}} \right) \\
\mathbf{F}' &= F'_x \hat{i} + F'_y \hat{j} + F'_z \hat{k} \\
\mathbf{F}' &= \frac{m_o c^2}{\sqrt{1 + \frac{v^2}{c^2}}} \left[ \hat{i} \left( \dot{x} + \frac{v'^2}{2c^2} \right) + \hat{j} \left( \dot{y} + \frac{v'^2}{2c^2} \right) + \hat{k} \left( \dot{z} + \frac{v'^2}{2c^2} \right) \right] \\
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\[
\vec{F} = \frac{m_0}{(1 + \frac{v^2}{c^2})^2} \left\{ \vec{x}(1 + \frac{v^2}{c^2}) \hat{i} - \vec{x}' \frac{dv}{dt} \frac{d}{dt} \right\} + \vec{y}' \left( 1 + \frac{v^2}{c^2} \right) \hat{j} - \vec{y} \frac{dv}{dt} + \vec{z}' \left( 1 + \frac{v^2}{c^2} \right) \hat{k} - \vec{z} \frac{dv}{dt} \hat{k}
\]

\[
\vec{F} = \frac{m_0}{(1 + \frac{v^2}{c^2})^2} \left\{ (1 + \frac{v^2}{c^2}) (\vec{x}' \hat{i} + \vec{y}' \hat{j} + \vec{z}' \hat{k}) \hat{i} - (\vec{x}' + \vec{y}' \hat{j} + \vec{z}' \hat{k}) \frac{dv}{dt} \right\}
\]

\[
\vec{F} = \frac{m_0}{(1 + \frac{v^2}{c^2})^2} \left\{ (1 + \frac{v^2}{c^2}) \frac{dv}{dt} - \frac{v^2}{c^2} \frac{dv}{dt} \right\}
\]

\[
= 28.16
\]

§30 Energy Continuation

With 3d, 6, and 9 we get:

\[
E_k = E - E_0 = \frac{m_0 c^2}{\sqrt{1 - \frac{v^2}{c^2}}} - m_0 c^2 = \frac{m_0 v^2}{\sqrt{1 - \frac{v^2}{c^2}}} + m_0 c^2 \sqrt{1 - \frac{v^2}{c^2}} = \frac{1}{2} m_0 c^2 = \frac{1}{2} vp
\]

30.30

This we get:

\[
\frac{v^2}{c^2} = \frac{4E_k^2}{c^2 p^2}
\]

30.31

That applied in 8 results:

\[
p = \frac{m_0 v}{\sqrt{1 - \frac{v^2}{c^2}}} \rightarrow E_k = \frac{c}{2} \sqrt{p^2 - m_0^2 v^2}
\]

30.32

If at 32 \( m_0 = 0 \) then:

\[
E_k = \frac{cp}{2} = \frac{1}{2} vp \rightarrow v = c
\]

30.33

For a particle with velocity \( c = \lambda \gamma \) and zero resting mass \( m_0 = 0 \) we have:

\[
E = \hbar \gamma \quad \quad p = \frac{\hbar}{\lambda}
\]

30.34

Applying \( c = \lambda \gamma \) and 34 in 33 results:

\[
E_k = \frac{cp}{2} = \frac{\lambda \hbar}{\lambda} \frac{\hbar}{\lambda} = \frac{\hbar^2}{2} \rightarrow E_k = \frac{\hbar}{2} \gamma
\]

30.35

From 34 and 35 we have:

\[
E = 2E_k
\]

30.36

Applying 36 in 30 we have:

\[
E_k = E - E_0 \rightarrow E_k = 2E_k - E_0 \rightarrow E_0 = E_k = \frac{E}{2}
\]

30.37

Applying \( E_0 = m_0 c^2 \) in 37 we obtain:

\[
E_0 = E_k = \frac{E}{2} = m_0 c^2 \rightarrow m_0 = \frac{E}{2c^2}
\]

30.38

With 33 and 38 we get:

\[
E_k = \frac{cp}{2} = m_0 c^2 \rightarrow p = 2m_0 c
\]

30.39
"Although nobody can return behind and perform a new beginning, any one can begin now and create a new end"
(Chico Xavier)

Bibliography
http://www.wbabin.net/physics/faraj7.htm