A strengthened form of the strong Goldbach conjecture

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Abstract. This paper presents a proof of a strengthened form of the strong Goldbach conjecture. Whereas the traditional approaches focus on the control over the distribution of the prime numbers by means of circle method and sieve theory, we will show that the solution lies in the constructive properties of the primes, reflecting their multiplicative character within the natural numbers.

Moreover, by complementing this proof we can derive two contradictory statements, thereby constituting an antinomy within classical mathematics.

Notations. Let \( \mathbb{N} \) denote the natural numbers starting from 1, let \( \mathbb{N}_n \) denote the natural numbers starting from \( n > 1 \) and let \( \mathbb{P}_3 \) denote the prime numbers starting from 3.

Theorem (Strengthened strong Goldbach conjecture (SSGB)). Every even integer greater than 6 can be expressed as the sum of two different primes.

Proof. We define the set
\[ S_g := \{ (p^k, m_k, q^k) \mid k, m \in \mathbb{N}; p, q \in \mathbb{P}_3, p < q; m = (p + q) / 2 \} \]

SSGB is equivalent to saying that every integer \( x \geq 4 \) is the arithmetic mean of two different odd primes and so it is equivalent to saying that all integers \( x \geq 4 \) appear as \( m \) in a middle component \( m_k \) of \( S_g \).

Let us assume \( \neg\text{SSGB} \) now. This means that there is at least one \( n \geq 4 \) such that for each fixed \( k \geq 1 \) \( n_k \) is different from all the \( m_k \) generated in \( S_g \).

Let \( S_{g^*} \) be the set such that \( \neg\text{SSGB} \Rightarrow S_g = S_{g^*} \), i.e. \( S_{g^*} \) is the set \( S_g \) under the assumption of the existing \( n \). We will show that \( S_{g^*} \) has the same elements as \( S_g \), i.e. as if we do not assume the existence of \( n \), and that therefore the assumed \( n \) actually does not exist.

The whole range of \( \mathbb{N}_3 \) can be expressed by the triple components of \( S_g \), since every integer \( x \geq 3 \) can be written as some \( p^k \) with \( k = 1 \) when \( x \) is prime, as some \( p^k \) with \( k \neq 1 \) when \( x \) is composite and not a power of 2, or as \( (3 + 5)^k / 2 \) when \( x \) is a power of 2, where \( p \in \mathbb{P}_3, k \in \mathbb{N} \).

According to the above three kinds of expression by \( S_g \) triple components, for any \( n \geq 4 \) given by \( \neg\text{SSGB} \) we have the property
\[ (C): \forall k \in \mathbb{N} \exists (p^k', m_k', q^k') \in S_g \quad n_k = p^k' \lor n_k = m_k' = 4k'. \]

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So, every \( nk \) given by \( -SSGB \) equals a component of some \( S_g \) triple that exists by definition.

Moreover, since all pairs \((p, q)\) of odd primes with \( p < q \) are used in \( S_g \) and so all arithmetic means \( m \) of two odd primes are generated, we have that an \( n \geq 4 \) given by \( -SSGB \) cannot be the arithmetic mean of a pair of odd primes not used in \( S_g \). This results in the property

\[(M): \neg \exists p, q \in \mathbb{P}_3, p < q \quad n = (p + q) / 2.\]

Because the properties (C) and (M) hold for any \( n \) given by \( -SSGB \), \( S_g \) can be written as the union of the following triples, which would otherwise be impossible.

(i) \( S_g \) triples of the form \((pk' = nk, mk', qk')\) with \( k' = k \) in case \( n \) is prime, due to (C)

(ii) \( S_g \) triples of the form \((pk' = nk, mk', qk')\) with \( k' \neq k \) in case \( n \) is composite and not a power of 2, due to (C)

(iii) \( S_g \) triples of the form \((3k', 4k' = nk, 5k')\) in case \( n \) is a power of 2, due to (C)

(iv) all remaining \( S_g \) triples of the form \((pk' = nk, mk', qk')\), \((pk', mk' = nk, qk')\) or \((pk', mk', qk' = nk)\)

and

(v) \( S_g \) triples of the form \((pk' \neq nk, mk' \neq nk, qk' \neq nk)\), i.e. those \( S_g \) triples where none of the \( nk \)'s equals a component.

The triples in (iv) comprise all \( S_g \) triples where \( nk \) occurs as a component redundantly to the occurrences in (i) - (iii). We can split the triples in (iv) as follows.

(iv, a) \( S_g \) triples of the form \((pk', mk', qk' = nk)\) with \( k' = k \) in case \( n \) is prime

(iv, b) \( S_g \) triples of the form \((pk', mk' = nk, qk')\) with \( k' = k \) in case \( n \) is prime

(iv, c) \( S_g \) triples of the form \((pk', mk', qk' = nk)\) with \( k' \neq k \) in case \( n \) is composite and not a power of 2

(iv, d) \( S_g \) triples of the form \((pk', mk' = nk, qk')\) with \( k' \neq k \) in case \( n \) is composite and not a power of 2

(iv, e) \( S_g \) triples of the form \((pk', mk' = nk, qk')\) with \( k' = k \) in case \( n \) is composite

(iv, f) \( S_g \) triples of the form \((pk' = nk, mk', qk')\) in case \( n \) is a power of 2

(iv, g) \( S_g \) triples of the form \((pk', mk', qk' = nk)\) in case \( n \) is a power of 2

(iv, h) \( S_g \) triples of the form \((pk', mk' = nk, qk')\) with \( m \neq 4 \) in case \( n \) is a power of 2.
The types (iv, a) - (iv, h) are of merely informative character. For the sake of completeness also the triples of type (iv, b) and (iv, e) are listed. Of course, they cannot exist due to (M). Also, depending on n and k the triples of some other types may not exist.

Let $S_a$ denote the union of the triples of types (i) to (iv), i.e. all $S_g$ triples where one of the nk’s equals one of the three components (affected triples), and let $S_{na}$ denote the union of the triples of type (v), i.e. all those $S_g$ triples where none of the nk’s equals a component (not affected triples).

Then, as $S_g$ consists of all triples of the types (i) to (v) we have $S_g = S_a \cup S_{na}$. On the other hand, since every triple of $S_g$ either belongs to $S_a$ or to $S_{na}$, $S_a$ and $S_{na}$ are complementary subsets of $S_g$ and we have $S_g = S_a \cup S_{na}$.

Thus, we conclude that $S_g$ has the same elements as $S_g$, i.e. as if we do not assume the existence of n. So, in fact there is no such n and we obtain

$\neg SSGB \Rightarrow SSGB$. This proves the theorem. $\square$

**Note.** The above splitting of all the $S_g$ triples into two complementary subsets $S_a$ and $S_{na}$ is independent of our information about $S_g$ and it is also independent of the property behind n. The splitting works solely on the basis of the existence of n.

**Theorem.** The theory based on ZFC is inconsistent, i.e. there is a statement $P$ such that both $P$ and its negation $\neg P$ can be deduced in ZFC.

**Proof.** We use the set

$$S_g = \{ (pk, mk, qk) | k, m \in \mathbb{N}; p, q \in \mathbb{P}_3, p < q; m = (p + q) / 2 \}$$

from the proof above and we recall

$SSGB$: $\forall x \in \mathbb{N}_4$ $\exists (pk, mk, qk) \in S_g$ $x = m$.

$\neg SSGB$: $\exists x \in \mathbb{N}_4$ $\forall (pk, mk, qk) \in S_g$ $x \neq m$.

For each fixed $k \geq 1$, we set $M(k) := \{ mk | (pk, mk, qk) \in S_g \}$. Then, by definition

(I) $SSGB \iff M(k) = k\mathbb{N}_4$ for every $k \geq 1$.

(II) $\neg SSGB \iff M(k) \neq k\mathbb{N}_4$ for every $k \geq 1$.

$\neg SSGB$ means that there is at least one $n \geq 4$ such that for each fixed $k \geq 1$ nk is different from all the mk generated in $S_g$. Correspondingly, $SSGB$ means that there is no such n.
In analogy to the set \( S_g \) in the proof above, let \( S_g^+ \) be the set such that \( SSGB \Rightarrow S_g = S_g^+ \), i.e. \( S_g^+ \) is the set \( S_g \) under the assumption of the non-existence of \( n \). Since an \( n \) that does not exist has no effect on the set \( S_g \) and so all triples of \( S_g \) belong to \( S_g^+ \), the set \( S_g^+ \) trivially equals \( S_g \).

So, we have that \( S_g^+ \) equals \( S_g \) as it is defined, i.e. \( S_g \) without any assumption, and that also \( S_g^- \) equals \( S_g \) as it is defined, i.e. \( S_g \) without any assumption:

\[
\exists \text{ sets } S, S' \text{ such that } S = S' \text{ and } (SSGB \Rightarrow S_g = S) \text{ and } (\neg SSGB \Rightarrow S_g = S') .
\]

As this formula holds for the set \( S_g \) it particularly holds for the set \( M(1) \):

\[
\exists \text{ sets } S, S' \text{ such that } S = S' \text{ and } (SSGB \Rightarrow M(1) = S) \text{ and } (\neg SSGB \Rightarrow M(1) = S') .
\]

By using (II), this implies

\( (III) \) \( SSGB \Rightarrow M(1) \neq \mathbb{N}_4 \).

Since in the proof above we have shown \( SSGB \), additionally by “ex falso quodlibet” we have

\( (IV) \) \( \neg SSGB \Rightarrow M(1) = \mathbb{N}_4 \).

All in all, the statements (I) - (IV) yield

\( (SSGB \Rightarrow (M(1) = \mathbb{N}_4 \text{ and } M(1) \neq \mathbb{N}_4)) \text{ and } (\neg SSGB \Rightarrow (M(1) = \mathbb{N}_4 \text{ and } M(1) \neq \mathbb{N}_4)) \)

\( \iff \)

\( (V) \) \( M(1) = \mathbb{N}_4 \text{ and } M(1) \neq \mathbb{N}_4 \).

\( \square \)