A Generalization of Vajda's and D'Ocagne's Identities

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Abstract
In this paper, we present an identity which generalizes Vajda's and D'ocagne's identities involving Fibonacci sequence. Vajda's and D'ocagne's identities are known to contain 3 and 2 variables respectively, but the identity to be presented contains 7 variables. Binet's formula for generating the nth term of Fibonacci sequence will be used in proving the identity.

Keywords: Fibonacci sequence, Binet's formula, Vajda's and D'Ocagne's identities.

1 Introduction

It is widely known that Fibonacci sequence is one of the most commonly used sequences for establishing recurrence relations and many identities which relate to Fibonacci sequence have been discovered over time. For instance, in 1680, Cassini discovered without proof that

$$F_{n-1}F_{n+1} - F_n^2 = (-1)^n$$

which was later independently proven by Robert Simson in 1753.
In 1879, Eugene Charles Catalan discovered and proved that

$$F_n^2 - F_{n-r}F_{n+r} = (-1)^{n-r}F_r^2$$

which generalizes cassini's identity. Another identity which generalizes Cassini's identity was discovered by Philbert Maurice D'Ocagne and the identity states that

$$F_mF_{n+1} - F_{m+1}F_n = (-1)^n F_{m-n} \quad (1)$$

In 1989, Steven Vajda published a book on fibonacci numbers book which contains the identity

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$$F_{n+i}F_{n+j} - F_nF_{n+i+j} = (-1)^n F_iF_j \tag{2}$$

which carries his name and thus generalizes both Cassini's and Catalan's Identities.
Fibonacci sequence is generated by employing the formula:

$$F_k = F_{k-1} + F_{k-2}, k \geq 2, F_0 = 0, F_1 = 1.$$

2 The Identity

The identity be proven states that

if

$$\alpha = F_{n+i+x-z}F_{n+j+y+z}$$
$$\beta = F_{n+x+y-k}F_{n+i+j+k}$$
$$\gamma = (-1)^{n+x+y-k}F_{i+k-y-z}F_{j+k+z-x}$$

then

$$\alpha - \beta = \gamma \tag{3}$$

From (3), we can see that setting $x = y = z = k = 0, a = \alpha, b = \beta, c = \gamma$, we get

$$a = F_{n+i+x-z}F_{n+j+y+z}$$
$$a = F_{n+i+0-0}F_{n+j+0+0}$$
$$a = F_{n+i}F_{n+j} \tag{4}$$

$$b = F_{n+x+y-k}F_{n+i+j+k}$$
$$b = F_{n+0+0-0}F_{n+i+j+0}$$
$$b = F_nF_{n+i+j} \tag{5}$$

$$c = (-1)^{n+x+y-k}F_{i+k-y-z}F_{j+k+z-x}$$
$$c = (-1)^{n+0-0-0}F_{i+0+0-0}F_{j+0+0-0}$$
$$c = (-1)^n F_iF_j \tag{6}$$

Now, we can see that the relationship between (4), (5), and (6) (i.e. $a - b = c$) yields (2)

Also, we can see that setting $a_1 = a, b_1 = b, c_1 = c, i = 1, j = m - n$ in (4), (5) and (6), we get

$$a_1 = F_{n+i}F_{n+j}$$
$$a_1 = F_{n+1}F_{n+m-n}$$
$$a_1 = F_{n+1}F_m \tag{7}$$

$$b_1 = F_nF_{n+i+j}$$
$$b_1 = F_nF_{n+1+m-n}$$
$$b_1 = F_nF_{m+1} \tag{8}$$

$$c_1 = (-1)^nF_iF_j$$
$$c_1 = (-1)^nF_1F_{m-n}$$
$$c_1 = (-1)^nF_{m-n} \tag{9}$$

Now, we can see that the relationship between (7), (8), and (9) (i.e. $a_1 - b_1 = c_1$) yields (1)

3 Prove of the identity

We know from Binet's formula for generating nth Fibonacci number that if

$$\phi = \frac{1+\sqrt{5}}{2}, \varphi = \frac{1-\sqrt{5}}{2}$$

then

$$F_n = \frac{\phi^n - \varphi^n}{\sqrt{5}}$$

where F_n is the nth Fibonacci number

let

$$P_1 = \frac{\phi^{2n+i+x+j+y}}{5}, P_2 = \frac{\varphi^{2n+i+x+j+y}}{5},$$
$$P_3 = \frac{\phi^{n+i+x-z}\varphi^{n+j+y+z}}{5}, P_4 = \frac{\phi^{n+j+y+z}\varphi^{n+i+x-z}}{5},$$
$$P_5 = \frac{\phi^{n+x+y-k}\varphi^{n+i+j+k}}{5}, P_6 = \frac{\phi^{n+i+j+k}\varphi^{n+x+y-k}}{5}$$
$$P_7 = \phi^{i+k-y-z}\varphi^{j+z+k-x}, P_8 = \phi^{j+z+k-x}\varphi^{i+k-y-z}$$
$$P_9 = \phi^{2k+i+j-x-y}, P_{10} = \varphi^{2k+i+j-x-y}$$
$$P_{11} = F_{i+k-y-z}F_{j+k+z-x}$$

From (3), we see that

$$\alpha = F_{n+i+x-z}F_{n+j+y+z}$$
$$\alpha = (\frac{\phi^{n+i+x-z} - \varphi^{n+i+x-z}}{\sqrt{5}})(\frac{\phi^{n+j+y+z} - \varphi^{n+j+y+z}}{\sqrt{5}})$$
$$\alpha = P_1 - P_3 - P_4 + P_2 \tag{10}$$

Also from (3), we see that

$$\beta = F_{n+x+y-k}F_{n+i+j+k}$$
$$\beta = (\frac{\phi^{n+x+y-k} - \varphi^{n+x+y-k}}{\sqrt{5}})(\frac{\phi^{n+i+j+k} - \varphi^{n+i+j+k}}{\sqrt{5}})$$
$$\beta = P_1 - P_5 - P_6 + P_2 \tag{11}$$

Deducting (11) from (10) gives

$$\alpha - \beta = (P_5 + P_6) - (P_3 + P_4) \tag{12}$$

From (12), let

$$V_1 = -(P_3 + P_4)$$
$$V_2 = (P_5 + P_6)$$

then

$$V_1 = -(\frac{\phi^{n+i+x-z}\varphi^{n+j+y+z}}{5} + \frac{\phi^{n+j+y+z}\varphi^{n+i+x-z}}{5})$$
$$V_1 = -\frac{(\phi\varphi)^{n+x+y-k}}{(\phi\varphi)^{n+x+y-k}}(\frac{\phi^{n+i+x-z}\varphi^{n+j+y+z}}{5} + \frac{\phi^{n+j+y+z}\varphi^{n+i+x-z}}{5})$$
$$V_1 = -\frac{1}{5}((\phi\varphi)^{n+x+y-k})(\frac{\phi^{n+i+x-z}}{\phi^{n+x+y-k}}\frac{\varphi^{n+j+y+z}}{\varphi^{n+x+y-k}} + \frac{\phi^{n+j+y+z}}{\phi^{n+x+y-k}}\frac{\varphi^{n+i+x-z}}{\varphi^{n+x+y-k}})$$

$$V_1 = -\frac{1}{5}((\phi\varphi)^{n+x+y-k})(\phi^{i+k-y-z}\varphi^{j+z+k-x} + \phi^{j+z+k-x}\varphi^{i+k-y-z})$$

Note that

$$\phi\varphi = -1$$
$$V_1 = -\frac{1}{5}((-1)^{n+x+y-k})(P_7 + P_8)$$

Also,

$$V_2 = \frac{(\phi^{n+x+y-k}\varphi^{n+i+j+k}}{5} + \frac{\phi^{n+i+j+k}\varphi^{n+x+y-k}}{5})$$

$$V_2 = \frac{(\phi\varphi)^{n+x+y-k}}{(\phi\varphi)^{n+x+y-k}}(\frac{\phi^{n+x+y-k}\varphi^{n+i+j+k}}{5} + \frac{\phi^{n+i+j+k}\varphi^{n+x+y-k}}{5})$$

$$V_2 = \frac{1}{5}((\phi\varphi)^{n+x+y-k})(\frac{\phi^{n+x+y-k}}{\phi^{n+x+y-k}}\frac{\varphi^{n+i+j+k}}{\varphi^{n+x+y-k}} + \frac{\phi^{n+i+j+k}}{\phi^{n+x+y-k}}\frac{\varphi^{n+x+y-k}}{\varphi^{n+x+y-k}})$$

$$V_2 = \frac{1}{5}((\phi\varphi)^{n+x+y-k})(\phi^0\varphi^{2k+i+j-x-y} + \phi^{2k+i+j-x-y}\varphi^0)$$

$$V_2 = \frac{1}{5}((\phi\varphi)^{n+x+y-k})(\phi^{2k+i+j-x-y} + \varphi^{2k+i+j-x-y})$$

Note that

$$\phi\varphi = -1$$

$$V_2 = \frac{1}{5}((-1)^{n+x+y-k})(P_9 + P_{10})$$

Now, we see from (12) that

$$\alpha - \beta = (P_5 + P_6) - (P_3 + P_4)$$

$$\alpha - \beta = V_1 + V_2$$

$$\alpha - \beta = \frac{1}{5}((-1)^{n+x+y-k})(P_9 - P_7 - P_8 + P_{10}) \tag{13}$$

From (3), we see that

$$\gamma = (-1)^{n+x+y-k}F_{i+k-y-z}F_{j+k+z-x}$$

$$\gamma = (-1)^{n+x+y-k}(P_{11}) \tag{14}$$

But

$$P_{11} = F_{i+k-y-z}F_{j+k+z-x}$$

$$P_{11} = (\frac{\phi^{i+k-y-z} - \varphi^{i+k-y-z}}{\sqrt{5}})(\frac{\phi^{j+k+z-x} - \varphi^{j+k+z-x}}{\sqrt{5}})$$

$$P_{11} = \frac{1}{5}(\phi^{i+k-y-z} - \varphi^{i+k-y-z})(\phi^{j+k+z-x} - \varphi^{j+k+z-x}) \tag{15}$$

We can see that the expansion of (15) gives

$$P_{11} = \frac{1}{5}(P_9 - P_7 - P_8 + P_{10}) \tag{16}$$

So, putting (16) in (14) gives

$$\gamma = \frac{1}{5}(-1)^{n+x+y-k}(P_9 - P_7 - P_8 + P_{10}) \tag{17}$$

Since (13) equals (17) then, (3) is true which completes the proof.

4 Conclusion

In this paper, we have stated and proven an identity which generalizes both Vajda's and D'Ocagne's identities (involving Fibonacci sequence) using Binet's formula for generating nth Fibonacci number.

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