A Non-linear Generalisation of Quantum Mechanics

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Abstract

A unified perspective to quantum mechanics is presented by which one can (I) Derive Schrödinger equation from Newton Second Law and de Broglie relation, (II) Arrive at a non-linear equation which reduces to Schrödinger equation in a certain approximation, and (III) See that the guiding equation of the de Broglie-Bohm theory is not complete.

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1 Introduction

It is the received wisdom that Planck-Einstein-de Broglie law[1] $p^\nu = \hbar k^\nu$ belongs to the era of 'old quantum mechanics' and that in the realm of quantum mechanics[2] the right (and more fundamental) perspective is to solve the Schrödinger equation for any case at hand. Although from an instrumentalist point of view this perspective has been quite successful, in this paper we advocate another perspective which will prove to be more fruitful with regard to the foundational questions of quantum mechanics. Our perspective is that *quantum mechanics is basically all about* $p^\nu = \hbar k^\nu$. To adopt such perspective we need to first scrutinise our understanding of

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[1]The metric signature $(+,-,-,-)$ is used everywhere in this paper.

[2]In this paper by 'quantum mechanics' we mean '*non-relativistic* quantum mechanics'.

its essential ingredient k^ν. By any rigorous mathematical definition[3] it is required that a wave be defined as a field[4] on spacetime which satisfies a certain equation, without any explicit reference to its four-wavevector. On the other hand, according to our perspective $p^\nu = \hbar k^\nu$ is a fundamental law of nature and appearance of k^ν in such a law suggests that we must enforce all waves to acquire a mathematically well-defined four-wavevector. Consequently we must find a definition for the four-wavevector of a wave ψ in terms of the ψ itself, a process we shall call *harmonisation.*

2 Harmonisation

Considering the simplest case of a complex harmonic wave[5],

$$\psi(x^\mu) = e^{-ik_\mu x^\mu}$$

if we apply the gradient operator to both sides we have,

$$\partial_\mu \psi = -ik_\mu \psi$$

we realise that there are two possibilites for harmonisation:

1. *Operatorial* approach, $\hat{k}_\mu := i\partial_\mu$
2. *Logarithmic* approach[6], $k_\mu := i\frac{\partial_\mu \psi}{\psi} = i\partial_\mu(\log \psi)$

The operatorial approach is familiar for it is the basis of orthodox quantum mechanics. Being thoroughly investigated there is not much one can add to the operatorial approach except a derivation of the Schrödinger equation from applying the de Broglie relation to the *Convective* Newton Second Law.

The logarithmic approach however will be proved to yield a non-linear generalisation which reduces to the Schrödinger equation only if $\nabla \cdot \mathbf{k} = 0$. Finally we will see that the de Broglie-Bohm theory is a yet more special case of the logarithmic approach when in addition to $\nabla \cdot \mathbf{k} = 0$, one assumes that the amplitude of the wavefunction is constant. Therefore

$$\text{Non-linear theory} \supset \text{Orthodox quantum mechanics},$$

and

$$\text{Non-linear theory} \supset \text{de Broglie-Bohm theory}.$$

[3]For example, an entity ϕ which satisfies the wave equation

$$\frac{\partial^2 \phi}{\partial t^2} = v^2 \nabla^2 \phi,$$

where v is the speed of propagation of the wave. Or, an entity ϕ which satisfies the Schrödinger equation.

[4]We only consider scalar fields in this paper. Sufficient conditions of smoothness are also assumed implicitly.

[5]Note that this is not the most general harmonic wave one can write. Moreover notice that in these definitions only forward-in-time waves are considered. It is not clear whether this preference of time direction affects the theory in a decisive manner.

[6]By log the principal values of the complex logarithm function is meant. Equivalently $\psi \neq 0$.

3 Schrödinger equation from Newton Second Law

In view of de Broglie's wave-particle duality, all classical variables like momentum which were functions of time only, become fields in quantum mechanics. It turns out that the classical law of motion (Newton second law and related definitions) is easily extended to the quantum case if we treat our classical dynamical variables as functions of trajectory and time, viz. if $\mathcal{V}$ is a classical dynamical variable, then the correct treatment in quantum mechanics should treat it as

$$\mathcal{V} = \mathcal{V}(\mathbf{x}(t), t).$$

Consequently, using chain rule[7]

$$\mathbf{F} = \frac{d\mathbf{p}(\mathbf{x}(t),t)}{dt} = \frac{\partial \mathbf{p}}{\partial t} + \mathbf{v} \cdot \nabla \mathbf{p} = \frac{\partial \mathbf{p}}{\partial t} + \frac{1}{m}\mathbf{p} \cdot \nabla \mathbf{p} \tag{1}$$

which means we are dealing with a convective form of the Newton second law.

To proceed with the derivation, notice that in Euclidean space we have the following identity

$$\mathbf{p} \cdot \nabla \mathbf{p} = \frac{1}{2}\nabla(\mathbf{p} \cdot \mathbf{p}),$$

Which is readily proved if we use the index notation

$$\nabla(\mathbf{p} \cdot \mathbf{p}) = \partial_\mu (p^\rho p_\rho)$$

where $\mu, \rho = 1, 2, 3$. By the product rule of differentiation we have

$$\partial_\mu (p^\rho p_\rho) = p^\rho (\partial_\mu p_\rho) + p_\rho \partial_\mu p^\rho$$

In the first term, lower one ρ and raise the other. Although $p^\alpha = -p_\alpha$, the negative sign from raising is cancelled by the negative sign from lowering. We thus have

$$\partial_\mu (p^\rho p_\rho) = 2p_\rho \partial_\mu p^\rho \Longrightarrow \nabla(\mathbf{p} \cdot \mathbf{p}) = 2\mathbf{p} \cdot \nabla \mathbf{p}$$

Using this identity (1) can be written as

$$\mathbf{F} = \frac{\partial \mathbf{p}}{\partial t} + \frac{1}{2m}\nabla(\mathbf{p} \cdot \mathbf{p})$$

We can now quantise this equation using the quantum mechanical operator $\mathbf{p} = -i\hbar\nabla$. After acting both sides on ψ, we get the *quantised Newton second law*,

$$\mathbf{F}\psi(\mathbf{x}, t) = -\frac{\hbar^2}{2m}\nabla\nabla^2\psi - i\hbar\frac{\partial}{\partial t}\nabla\psi. \tag{2}$$

For a conservative force field $\mathbf{F}(\mathbf{x}) = -\nabla V(\mathbf{x})$, after substitution we have,

$$-\nabla V\psi = -i\hbar\nabla\frac{\partial}{\partial t}\psi - \frac{\hbar^2}{2m}\nabla(\nabla \cdot \nabla)\psi$$

[7] We need not worry here about possible non-commutativities familiar from ordianry quantum mechanics. such non-commutativities happen for canonically conjugate (Fourier dual) variables only. No use of Fourier transform/duals is made in our discussion.

which is

$$i\hbar\nabla\frac{\partial}{\partial t}\psi = \nabla\left(V - \frac{\hbar^2}{2m}\nabla^2\right)\psi \tag{3}$$

i.e. (the gradient of) the Schrödinger equation.

4 Non-linear theory

We apply the de Broglie relation $\mathbf{p} = \hbar\mathbf{k}$ to the logarithmic approach to get

$$\mathbf{p} = -i\hbar\nabla(\log\psi) \quad \text{and} \quad E = i\hbar\frac{\partial}{\partial t}(\log\psi) \tag{4}$$

Substituting (4) in the law of conservation of energy[8]

$$E = \frac{\mathbf{p}\cdot\mathbf{p}}{2m} + V,$$

yields

$$i\hbar\frac{\partial}{\partial t}\log\psi = -\frac{\hbar^2}{2m}(\nabla\log\psi)^2 + V \tag{5}$$

To get the Schrödinger equation, notice that (5) is equivalent to

$$\boxed{i\hbar\frac{\partial\psi}{\partial t} = -\frac{\hbar^2}{2m}\frac{|\nabla\psi|^2}{\psi} + V\psi} \tag{6}$$

which differs from the Schrödinger equation only by the term

$$\frac{|\nabla\psi|^2}{\psi},$$

which we now show only in the *special case* where $\mathbf{k}$ is a solenoidal field, is equal to the corresponding term in Schrödinger equation. To avoid loss of generality we consider the relativistic condition of the vanishing of divergence of four-wavevector

$$\partial^\mu k_\mu = 0 \Longrightarrow \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\log\psi - \nabla\cdot\mathbf{k} = 0, \tag{7}$$

in the approximation

$$\frac{\partial^2}{\partial t^2}\log\psi \approx 0,$$

we have

$$\nabla\cdot\mathbf{k} = 0 \tag{8}$$

$$\Longrightarrow \nabla\cdot(\frac{\nabla\psi}{\psi}) = 0$$

$$\Longrightarrow \frac{\psi\nabla^2\psi - |\nabla\psi|^2}{\psi^2} = 0$$

for $\psi \neq 0$,

$$\frac{|\nabla\psi|^2}{\psi} = \nabla^2\psi, \tag{9}$$

[8]Similar to the familiar derivation of the Schrödinger equation using conservation of energy.

In other words, Schrödinger equation is an approximate special case of a non-linear theory. We can now manifestly see how linearity arises from non-linearity, and how an eigenvalue problem which is the representative of quantum *discreteness* is only an approximation to non-linearity. In this light the superposition principle is only an approximate feature of nature and has a limited domain of applicability.
Being a non-linear dispersive equation (6) possesses soliton solutions, rendering collapse of the wavefunction a redundant notion, for the old picture of Schrödinger's wave packets –which were interpreted as particles– is now immune to dispersion.

5 de Broglie-Bohm theory

Thus far it is proved that Schrödinger equation is a special case of (6). We now show that the guiding equation of the de Broglie-Bohm theory holds only for wavefunctions with constant (uniform) amplitude.
To see this we use the polar representation of a complex-valued function $\psi : \mathbb{R}^3 \times \mathbb{R} \to \mathbb{C}$

$$\psi(\mathbf{x}, t) = R(\mathbf{x}, t)e^{iS(\mathbf{x},t)}$$

where $R : \mathbb{R}^3 \times \mathbb{R} \to \mathbb{R}$ and $S : \mathbb{R}^3 \times \mathbb{R} \to \mathbb{R}$.
Applying de Broglie relation

$$\mathbf{p} = \hbar\mathbf{k} \Longrightarrow \mathbf{p} = -i\hbar\frac{\nabla\psi}{\psi} = -i\hbar\frac{1}{\psi}\left(\nabla Re^{iS(\mathbf{x},t)} + iR\nabla Se^{iS(\mathbf{x},t)}\right)$$

yields

$$\boxed{\mathbf{p} = -i\hbar\frac{\nabla R}{R} + \hbar\nabla S} \tag{10}$$

In natural unit $\hbar = 1$,

$$\mathbf{p} = -i\frac{\nabla R}{R} + \nabla S$$

while in the context of de Broglie-Bohm theory[1]

$$\mathbf{p} = \nabla S,$$

so there is a missing term:

$$-i\frac{\nabla R}{R}.$$

In the literature one usually uses the non-relativistic momentum $\mathbf{p} = m\mathbf{v}$ to find the velocity field of particles as

$$\mathbf{v} = \frac{\nabla S}{m},$$

the complete relation, however, is

$$\mathbf{v} = -i\frac{\nabla R}{mR} + \frac{\nabla S}{m} \tag{11}$$

As classical velocity cannot be a complex number the only way a particle can have classical trajectory is when

$$\nabla R = 0, \quad R \neq 0$$

which gives

$$\psi(\mathbf{x}, t) = Ce^{iS(\mathbf{x},t)}$$

where C is the constant of integration. We therefore conclude that the concept of a classical path makes sense only for wavefunctions with constant amplitudes.

References

[1] Dürr, Detlef - Teufel, Stefan (2009). 'Bohmian Mechanics'. Springer. Page 145.