Mathematical-physical approach to prove that the Navier-Stokes equations provide a correct description of fluid dynamics

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ABSTRACT

This publication takes a mathematical approach to a general solution to the Navier Stokes equations. The basic idea is an mathematical analysis of the unipolar induction according to Faraday. The vector analysis enables both physical-mathematical formulations to be related to one another, since both formulations are mathematically equivalent. Since the unipolar induction has been proven in practice, it can be used as a reference for the significance of the Navier-Stokes equations.

1. INTRODUCTION

The Navier-Stokes equations are a description of the motion of fluids and gases. The problem with the set of equations is that the proof for a solution in 3-dimensional space has not yet been provided. In addition, the math behind the equations is difficult to understand and has not yet been explained plausibly.

One of the so-called Millennium Problems is to prove that the equations have general validity. This paper deals with this last problem and also resolves the first two problems. Since vector calculus was not yet introduced in the lifetime of Claude Louis Marie Henri Navier (1785-1836) and vector calculus was still in its infancy during the lifetime of George Gabriel Stokes (1819-1903) (it was introduced in 1844), this treatise deals with it with formulating a proposal with which the Navier-Stokes equations can be derived from vector calculation and so to solve the problems listed above. In addition, a mathematical connection to the "Maxwell equations" is established in order to prove that the Navier-Stockes equations are also valid in 3-dimensional space. The aim of this paper is not to derive the mathematical foundations of vector calculus that are already known and / or recognized. Reference is only made to these here. This paper compares the Navier-Stokes equations for incompressible Newtonian liquids, at constant pressure, with the equation for unipolar induction, according to Faraday.

2. IDEAS AND METHODS

2.1 IDEA BEHIND THE SOLUTION

The idea is to transfer the vectorial description of the unipolar induction to the Navier-Stokes equations and thus to describe a general validity for the physical behavior, with regard to movement and forces, of substances of all kinds.

Unipolar induction:

$$\vec{E} = \vec{v} \times \vec{B} \qquad (2.1.1)$$

$\vec{E}$ = electric field strength

$\vec{v}$ = velocity

$\vec{B}$ = magnetic flux density

According to the rules of vector analysis, this equation can also be described as follows,

$$\vec{E} = (\vec{v} \times \vec{B}) = (\mathrm{grad}\,\vec{v})\vec{B} - (\mathrm{grad}\,\vec{B})\vec{v} + \vec{v}(\mathrm{div}\,\vec{B}) - \vec{B}(\mathrm{div}\,\vec{v}) \quad . \qquad (2.1.3)$$

The gist of this formula is that a magnetic field is created when an object moves through an electric field.

The material constant μ is given by the relationship $\vec{B} = \mu \vec{H}$.

If $\vec{E} = \vec{\Phi}_1$, $\vec{H} = \vec{\Phi}_2$ and $\mu = a$ are now abstracted, the following equation arises,

$$\vec{\Phi}_1 = (\vec{v} \times a\vec{\Phi}_2) = (\mathrm{grad}\,\vec{v})a\vec{\Phi}_2 - (\mathrm{grad}\,a\vec{\Phi}_2)\vec{v} + \vec{v}(\mathrm{div}\,a\vec{\Phi}_2) - a\vec{\Phi}_2(\mathrm{div}\,\vec{v}) \quad . \qquad (2.1.4)$$

If the terms of the equation are now mathematically reformulated, a new overall expression is created which has an analogy to the Navier-Stokes-equations (2.1.5).

$$\vec{\Phi}_1 = a(\mathrm{grad}\ \vec{v})\vec{v} - a\frac{\delta\vec{\Phi}_2}{\delta t} + \vec{v}(\mathrm{div}(a\vec{\Phi}_2)) - (a\vec{\Phi}_2)(\mathrm{div}\,\vec{v}) \qquad (2.1.5)$$

If the equation (2.1.5) is now multiplied by -1, the result is equation (2.1.6).

$$-\vec{\Phi}_1 = -a(\text{grad } \vec{v})\vec{v} + a\frac{\delta\vec{\Phi}_2}{\delta t} - \vec{v}(\text{div}(a\vec{\Phi}_2)) + (a\vec{\Phi}_2)(\text{div}\vec{v}) \quad (2.1.6)$$

In direct comparison, the following are the Navier-Stockes equations.

$$f = \rho\frac{\delta u}{\delta t} + \rho(\text{grad } u)u - \text{div } \sigma_{(u,p)} + 0 \quad (2.1.7)$$

and

$$\text{div } u = 0 \quad . \quad (2.1.8)$$

Here and also in the following explanations u is equated with the expression of the velocity $\vec{v}$.
Since Φ_2 must be based on a field which contains sources and sinks, i.e. in which density distributions play a role, and which occurs in n-dimensional space, we can assume that the Navier-Stockes equations also have the effect in Map n-dimensional space. The reason for this is that the "Maxwell-equations", which can also be derived from the unipolar-induction, have proven to be a consistent description of electromagnetic fields to this day.

2.2 BASICS OF VECTOR-CALCULATION

In order to be able to derive the set of equations of the Navier-Stokes equations from vector calculation, this chapter describes the fundamentals of vector calculation used to solve the problems described in the introduction.
First of all, 3 meta-vectors $\vec{a}$, $\vec{b}$ and $\vec{c}$ are introduced at this point. The three meta-vectors will be used in the following basic mathematical description.

$$\vec{c}=\vec{a}\times\vec{b} \quad (2.2.1)$$

In the next step, the red operator is used on both sides of the equation.

$$rot\,\vec{c} = rot(\vec{a}\times\vec{b}) \quad (2.2.2)$$

Now the right side of the equation is rearranged according to the calculation rules of vector calculation.

$$\text{rot}\,\vec{c} = \text{rot}(\vec{a}\times\vec{b}) = (\text{grad}\,\vec{a})\ \vec{b} - (\text{grad}\,\vec{b})\ \vec{a} + \vec{a}\ \text{div}\,\vec{b} - \vec{b}\ \text{div}\,\vec{a} \qquad (2.2.3)$$

Wird die Gleichung 2.2.3 mit -1 multipliziert entsteht folgender Ausdruck,

$$-\text{rot}\,\vec{c} = -(\text{grad}\,\vec{a})\ \vec{b} + (\text{grad}\,\vec{b})\ \vec{a} - \vec{a}\ \text{div}\,\vec{b} + \vec{b}\ \text{div}\,\vec{a} \quad . \qquad (2.2.4)$$

2.3 SUBSTITUTING THE PHYSICAL COMPONENTS OF THE NAVIER-STOKES-EQUATIONS

In the next step, the metavector $\vec{a}$ in equation 2.2.4 is replaced by the velocity vector $\vec{v}$. The metavector $\vec{b}$ is replaced by the mass occupancy multiplied by the velocity $(\rho \cdot \vec{v})$. The result is the following equation (2.3.1).

$$-(\text{grad}\,\vec{v})\ (\rho\vec{v}) + (\text{grad}\,(\rho\vec{v}))\ \vec{v} - \vec{v}\ \text{div}(\rho\vec{v}) + (\rho\vec{v})\ \text{div}\,\vec{v} = -\,\text{rot}(\vec{v}\times(\rho\vec{v})) \qquad (2.3.1)$$

2.4 BASIC DESCRIPTION

2.4.1 NAVIER-STOKES-EQUATIONS

The formulas of the Navier-Stokes equations and the vector calculation to which reference is made in this publication are presented here. Throughout the elaboration, the form of variation of the incompressible Navier-Stokes equations is referred to and used as a reference. The approach can also be used for other forms of variation of the Navier-Stokes equations.

$$\rho\frac{\delta u}{\delta t} + \rho(\text{grad}\ u)u\ - \text{div}\ \sigma_{(u,\,p)} = f \qquad (2.4.1)$$

$$\text{div}\ u = 0 \qquad (2.4.2)$$

$$\sigma(u,\,p)n = h \qquad (2.4.3)$$

The expression u is used here for the expression of the velocity $\vec{v}$. In order to get a better overview of the proposed solution, the following equations are written one above the other.

$$-(\mathrm{rot}(\vec{a}\times\vec{b})) = -(\mathrm{grad}\,\vec{a})\ \vec{b} + (\mathrm{grad}\,\vec{b})\ \vec{a} - \vec{a}\ \mathrm{div}\,\vec{b} + \vec{b}\ \mathrm{div}\,\vec{a} \qquad (2.4.4)$$

$$-(\mathrm{rot}(\vec{v}\times(\rho\vec{v}))) = -(\rho\vec{v})\ (\mathrm{grad}\,\vec{v}) + (\mathrm{grad}(\rho\vec{v}))\ \vec{v} - \vec{v}\ \mathrm{div}\,(\rho\vec{v}) + (\rho\vec{v})\ \mathrm{div}\,\vec{v} \qquad (2.4.5)$$

$$f = \rho(\mathrm{grad}\ u)u + \rho\frac{\delta u}{\delta t} - \mathrm{div}\ \sigma_{(u,p)} + 0 \qquad (2.4.6)$$

2.5 MATHEMATICAL APPROACH

In the following chapters, the mathematical combination of the individual terms from equations 2.4.2, 2.4.5 and 2.4.6 is discussed in more detail.

2.5.1 TERM 2 FROM EQUATIONS 2.4.5 AND 2.4.6

According to the commutative law of multiplication, the factor ρ can change its position as a factor. Therefore it does not matter where the factor ρ is within a term.

$$(\mathrm{grad}\ \vec{v})\ \rho\vec{v} = \rho(\mathrm{grad}\ u)u \qquad (2.5.1)$$

According to the rules of multiplication, the expression ρ from Eq. 2.5.1 can also be calculated first with the velocity u and only then with the gradient of u . Therefore, for Eq. 2.5.1 that they are included in Eq. 2.5.2 can be rewritten.

$$(\mathrm{grad}\ \vec{v})\ \rho\vec{v} = (\mathrm{grad}\ u)\ \rho u \qquad (2.5.2)$$

As already mentioned in chapter 2.4.1, u in equations 2.4.1, 2.4.2, 2.4.3 and 2.4.6 stands for the velocity $\vec{v}$. Therefore, equation 2.5.2 can be rewritten as follows (2.5.3).

$$(\mathrm{grad}\ \vec{v})\ \rho\vec{v} = (\mathrm{grad}\ \vec{v})\ \rho\vec{v} \qquad (2.5.3)$$

That means the second term from equation 2.4.5 and the second term from the equation 2.4.6 can be equated. However, it must be mentioned at this point that the second term from equation 2.4.5 has a minus sign. Whether and how this minus is relevant has to be discussed.

2.5.2 TERM 3 FROM EQUATIONS 2.4.5 AND 2.4.6

First, the third term, equation 2.4.5 ($(\mathrm{grad}(\rho\vec{v}))\,\vec{v}$), is written in column notation. It should be noted that the gradient of a vector results in a matrix.

$$(\mathrm{grad}\,\rho\vec{v})\cdot(\vec{v}) = \begin{pmatrix} \frac{\delta\rho v_x}{\delta x} & \frac{\delta\rho v_x}{\delta y} & \frac{\delta\rho v_x}{\delta z} \\ \frac{\delta\rho v_y}{\delta x} & \frac{\delta\rho v_y}{\delta y} & \frac{\delta\rho v_y}{\delta z} \\ \frac{\delta\rho v_z}{\delta x} & \frac{\delta\rho v_z}{\delta y} & \frac{\delta\rho v_z}{\delta z} \end{pmatrix} \cdot \begin{pmatrix} v_x \\ v_y \\ v_z \end{pmatrix} \qquad (2.5.4)$$

Now, according to the rules of vector calculation, the resulting gradient ($(\mathrm{grad}(\rho\vec{v}))$) is calculated with the velocity vector as a general solution (for all substances). The result is a new vector $\vec{x}_{(v,\rho)}$.

$$(\mathrm{grad}(\rho\vec{v}))\cdot\vec{v} = \begin{pmatrix} \frac{\delta(\rho v_x)\cdot v_x}{\delta x} + \frac{\delta(\rho v_x)\cdot v_y}{\delta y} + \frac{\delta(\rho v_x)\cdot v_z}{\delta z} \\ \frac{\delta(\rho v_y)\cdot v_x}{\delta x} + \frac{\delta(\rho v_y)\cdot v_y}{\delta y} + \frac{\delta(\rho v_y)\cdot v_z}{\delta z} \\ \frac{\delta(\rho v_z)\cdot v_x}{\delta x} + \frac{\delta(\rho v_z)\cdot v_y}{\delta y} + \frac{\delta(\rho v_z)\cdot v_z}{\delta z} \end{pmatrix} = \vec{x}_{(v,\rho)} \qquad (2.5.5)$$

For substances that are not subject to any deformation and have a homogeneous mass coverage, the following expression (2.5.6).

$$(\mathrm{grad}(\rho\vec{v}))\cdot\vec{v} = \begin{pmatrix} \frac{\delta(\rho v_x)}{\delta x}\cdot v_x + 0 + 0 \\ 0 + \frac{\delta(\rho v_y)}{\delta y}\cdot v_y + 0 \\ 0 + 0 + \frac{\delta(\rho v_z)}{\delta z}\cdot v_z \end{pmatrix} = \vec{x}_{(v,\rho)} \qquad (2.5.6)$$

This expression is simplified to the equation 2.5.7.

$$\left(\text{grad}\left(\rho \vec{v}\right)\right) \cdot \vec{v} = \begin{pmatrix} \frac{\delta\left(\rho v_x\right)}{\delta x} \cdot v_x \\ \frac{\delta\left(\rho v_y\right)}{\delta y} \cdot v_y \\ \frac{\delta\left(\rho v_z\right)}{\delta z} \cdot v_z \end{pmatrix} \tag{2.5.7}$$

For Newtonian liquids with constant pressure, the mass occupancy is constant and is interpreted as density ρ . Therefore it can be excluded as a factor on the right side of the equation. This results in equation 2.5.8.

$$\left(\text{grad}\left(\rho \vec{v}\right)\right) \cdot \vec{v} = \rho \begin{pmatrix} \frac{\delta v_x}{\delta x} \cdot v_x \\ \frac{\delta v_y}{\delta y} \cdot v_y \\ \frac{\delta v_z}{\delta z} \cdot v_z \end{pmatrix} \tag{2.5.8}$$

Now, on the right side of the equation, the velocity is derived from the distance $\frac{\delta \vec{v}}{\delta \vec{s}}$. The following equation (2.5.9) arises.

$$\frac{\delta \vec{v}}{\delta \vec{s}} = \frac{\delta}{\delta t} \tag{2.5.9}$$

This expression from equation 2.5.9 is now inserted into equation 2.5.8 and assuming that the term $\vec{v}$ is equated with the term u equation 2.5.10 is the result.

$$\left(\text{grad}\left(\rho \vec{v}\right)\right) \cdot \vec{v} = \rho \begin{pmatrix} \frac{\delta}{\delta t} \cdot v_x \\ \frac{\delta}{\delta t} \cdot v_y \\ \frac{\delta}{\delta t} \cdot v_z \end{pmatrix} = \rho \begin{pmatrix} \frac{\delta v_x}{\delta t} \\ \frac{\delta v_y}{\delta t} \\ \frac{\delta v_z}{\delta t} \end{pmatrix} = \rho\left(\frac{\delta \vec{v}}{\delta t}\right) = \rho \frac{\delta u}{\delta t} \tag{2.5.10}$$

The result from equation 2.5.10 now corresponds to the third term from equation 2.4.6.

$$(\text{grad}(\rho \vec{v}))\, \vec{v} = \rho \frac{\delta u}{\delta t} \qquad (2.5.11)$$

2.5.3 TERM 4 FROM EQUATIONS 2.4.5 AND 2.4.6

First the fourth term from equation 2.4.6 is written, $\text{div}(\sigma_{(u,p)})$. The term $\sigma_{(u,p)}$ stands for the mechanical normal stress, which here depends on the velocity u and the pressure p. It is defined as the viscous stresstensor (2.5.12).

$$\sigma = \begin{pmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22} & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} \end{pmatrix} \qquad (2.5.12)$$

Applying the divergence to this tensor creates a vector, i.e. a tensor of the first degree (2.5.13).

$$\text{div}\ \sigma = \begin{pmatrix} \frac{\delta \sigma_{11}}{\delta x} + \frac{\delta \sigma_{12}}{\delta y} + \frac{\delta \sigma_{13}}{\delta z} \\ \frac{\delta \sigma_{21}}{\delta x} + \frac{\delta \sigma_{22}}{\delta y} + \frac{\delta \sigma_{23}}{\delta z} \\ \frac{\delta \sigma_{31}}{\delta x} + \frac{\delta \sigma_{32}}{\delta y} + \frac{\delta \sigma_{33}}{\delta z} \end{pmatrix} = \begin{pmatrix} \sigma_{a\,div} \\ \sigma_{b\,div} \\ \sigma_{c\,div} \end{pmatrix} \qquad (2.5.13)$$

The vector resulting from the $\text{div}\ \sigma$ has the physical unit $\frac{g}{\vec{m} \cdot s^2} = \vec{F}$. With this unit, the dependence of σ can be mapped on both the speed u and the pressure p . That is why σ can also be written for $\sigma_{(u,p)}$.

The fourth term from Eq. 2.4.5 shows the following relationship, $\vec{v}\ \text{div}(\rho \vec{v})$. In this context, the $\text{div}(\rho \vec{v})$ provides a purely numerical value.

$$\text{div}(\rho \vec{v}) = \frac{\delta(\rho v_x)}{\delta x} + \frac{\delta(\rho v_y)}{\delta y} + \frac{\delta(\rho v_z)}{\delta z} \qquad (2.5.14)$$

If the scalar expression from Eq. 2.5.14, however, multiplied by the velocity, as shown in the fourth term from Eq. 2.4.5 is required, however, a vector results (2.5.15).

$$\vec{v}\ \operatorname{div}(\rho\vec{v}) = \begin{pmatrix} v_x \cdot \left(\frac{\delta(\rho v_x)}{\delta x} + \frac{\delta(\rho v_y)}{\delta y} + \frac{\delta(\rho v_z)}{\delta z}\right) \\ v_y \cdot \left(\frac{\delta(\rho v_x)}{\delta x} + \frac{\delta(\rho v_y)}{\delta y} + \frac{\delta(\rho v_z)}{\delta z}\right) \\ v_z \cdot \left(\frac{\delta(\rho v_x)}{\delta x} + \frac{\delta(\rho v_y)}{\delta y} + \frac{\delta(\rho v_z)}{\delta z}\right) \end{pmatrix} \tag{2.5.15}$$

The resulting vector from Eq. 2.5.15, like the vector resulting from Eq. 2.5.13, the physical unit $\frac{g}{\vec{m} \cdot s^2} = \vec{F}$. In addition, this vector is also dependent on the pressure p and the velocity u . The next thing in common is that both vectors make a statement about the tensions within a substance. For these reasons we can use the term 4 from Eq. 2.4.5 and the term 4 from Eq. 2.4.6 equate. Hence the following expression arises.

$$\vec{v}\ \operatorname{div}(\rho\vec{v}) = \operatorname{div}(\sigma_{(u,p)}) \ . \tag{2.5.16}$$

2.5.4 TERM 5 FROM EQUATIONS 2.4.5 AND 2.4.6

Equation 2.2.2 says that $\operatorname{div}(\vec{v}) = 0$ is. If the divergence $\vec{v}$ is derived from the fifth term, from Eq. 2.4.5, inserted in the fifth term of equation 2.4.6, instead of the 0, both expressions can be used via the relationship from Eq. 2.5.17 are equated.

$$(\rho\vec{v}) \cdot \operatorname{div}(\vec{v}) = (\rho\vec{v}) \cdot 0 = 0 \tag{2.5.17}$$

2.5.5 TERM 1 FROM EQUATIONS 2.4.5 AND 2.4.6

Since the first term of Eq. 2.4.6 (f) is not precisely defined, it can be calculated with the first term from Eq. 2.4.5 ($rot(v\times(\rho\vec{v}))$) are equated. By equating the first term f from equation 2.4.6 and the first term $rot(v\times(\rho\vec{v}))$ from equation 2.4.5, the following expression results (2.5.18),

$$rot(v\times(\rho\vec{v})) = f \quad . \qquad (2.5.18)$$

Here, too, there is a minus sign in the first term of equation 2.4.5. For this term, too, it must be discussed whether and what effects this sign has on equation 2.5.1.8.

3. DISCUSSION

1. It remains to be discussed whether this expression $\mathrm{div}(\vec{v}) = 0$ is valid for all substances, including those that are not subject to Newton's laws. The problem is that the following relationship holds (3.1.1),

$$\mathrm{div}(\vec{v}) = (\mathrm{Sp})\mathrm{grad}(\vec{v}) \quad . \qquad (3.1.1)$$

If the relationship from Eq. 3.1.1 should apply, then, for the statement $\mathrm{div}(\vec{v}) = 0$, $\mathrm{grad}(\vec{v}) = 0$. The question here would be what effect this would have on the two equations 2.4.5 and 2.4.6.

2. What effects would an inhomogeneous density distribution of a substance have on the solution approach?

3. What effects would it have if the mass occupancy were included in the solution as a vector quantity?

4. Is the approach from equation 2.1.4 a fundamental law of nature that is valid for all substances?

5. With reference to the question to 4, which state of aggregation then have physical fields?

6. Term 1 ($-\mathrm{rot}\left(\vec{v}\times(\rho\,\vec{v})\right)$) and term 2 ($-(\rho\,\vec{v})\ (\mathrm{grad}\,\vec{v})$) from equation 2.4.5 have a minus sign. It remains to be discussed whether and what effect this has on equating the two equations 2.4.5 and 2.4.6.

4. CONCLUSIONS

Due to the mathematical connection to the vector calculation and the physical-mathematical connection to the flow law of electrodynamics, the general validity of the Navier-Stokes equations could be adequately described.

5. CONFLICTS OF INTEREST

The author(s) declare(s) that there is no conflict of interest regarding the publication of this paper.

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