An Elementary Approach to the Riemann Hypothesis

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June 27, 2021

Abstract

The Riemann hypothesis is probably true. In this paper I present an approach for it in a very short and condensed way, making use of one of its equivalent problems. But as Carl Sagan once famously said, extraordinary claims require extraordinary evidence. The evidence here is the newly discovered inversion formula for Dirichlet series.

1 Introduction

The Riemann hypothesis is the second of the 7 Millennium Problems proposed by the Clay Mathematics Institute back in year 2000, and it has a 1 million dollar prize for whoever solves it.

It was formulated by Bernhard Riemann in 1859, and has been unsolved for more than 150 years now. The Riemann hypothesis is about the zeta function, $\zeta(s)$, a special case of Dirichlet series:

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}, \text{ if } \Re(s) > 1 \tag{1}$$

A Dirichlet series is any infinite sum of the type,

$$F_a(s) = \sum_{n=1}^{\infty} \frac{a(n)}{n^s}$$

where $a(n)$ is an arithmetic function and $F_a(s)$ is its generating function (hence, the zeta function has $a(n) = 1$).

The analytic continuation of the zeta function to $0 < \Re(s) < 1$ (the so-called critical strip) can be attained by means of the alternating zeta function (also known as the Dirichlet eta function). The zeta function can also be continued to $\Re(s) < 0$, by means of the Riemann functional equation, which pairs $\zeta(s)$ up with $\zeta(1-s)$, therefore enabling points with $\Re(s) > 1$ to complete the function for points with $\Re(s) < 0$. Finally, the Riemann hypothesis is the statement that all the non-trivial zeros of the function thus obtained have $\Re(s) = 1/2$.

Despite its very sound arguments, this paper is not meant to be a definitive proof of the Riemann hypothesis, unless vetted or improved upon by an expert, but it definitely adds novelties to the picture, besides being a really interesting exposition. At worst, the new facts presented here represent evidence in favor of the Riemann hypothesis, even if some details may be missing.

In this article, we try and make a distinction between the zeta function Dirichlet series and the function itself.

2 The background

It stems from the Euler product that when $\Re(s) > 1$, $1/\sum 1/n^s$ is a Dirichlet series that has $a(n) = \mu(n)$, the so-called Möbius function. The Riemann hypothesis is then equivalent to a statement that can be made about this new Dirichlet series and the reciprocal of the zeta function in the upper half portion of the critical strip, $\Re(s) > 1/2$. But let us go over some of the details first.

2.1 The Euler product

In 1737, German mathematician Euler discovered an interesting relationship between the zeta function and the primes known as Euler product, valid when $\Re(s) > 1$:

$$\zeta(s) = \prod_{n=1}^{\infty} \frac{1}{1 - p_n^{-s}} = \sum_{n=1}^{\infty} \frac{1}{n^s} \tag{2}$$

When (2) is inverted, it reveals a relationship between the reciprocal of the zeta function and the square-free numbers:

$$\frac{1}{\zeta(s)} = \prod_{n=1}^{\infty} \left(1 - \frac{1}{p_n^s}\right) = \left(1 - \frac{1}{2^s}\right)\left(1 - \frac{1}{3^s}\right)\left(1 - \frac{1}{5^s}\right)\cdots$$

The observation of this relation allows us to write $1/\zeta(s)$ as function of $\mu(n)$, a function introduced in 1832 by another German mathematician, August F. Möbius, which involves the concept of square-free numbers. That is, provided that $\Re(s) > 1$.

2.2 The Möbius function

A square-free number is a number that can not be divided by any squared prime. In other words, if n is square-free, $p_1 p_2 \cdots p_k$ is its unique prime decomposition. Hence, we can define a function $\mu(n)$ such that:

$$\mu(n) = \begin{cases} 1, & \text{if } n\text{=1} \\ (-1)^k, & \text{if } n \text{ is square-free with } k \text{ prime factors} \\ 0, & \text{if } n \text{ is not square-free} \end{cases}$$

This is the Möbius function from the previous section.

With it, it is easy to write the reciprocal of the zeta function Dirichlet series, which also happens to be a Dirichlet series. By alternating the signs of the terms in the zeta series (and eliminating circa 40% of them, the square-full), the result seems to converge for $\Re(s) > 0$, based on the rationale provided previously,

$$1/\left(\sum_{n=1}^{\infty}\frac{1}{n^s}\right) = \sum_{n=1}^{\infty}\frac{\mu(n)}{n^s} \tag{3}$$

At this point, this equation is still only valid if $\Re(s) > 1$. It stems from the Euler product, and the new series on the right is referred to throughout this text as $F(s)$.

However, unlike the zeta function series, $F(s)$ should converge even for $1/2 < \Re(s) \leq 1$, and according to the literature, a certain statement about it and the zeta function is equivalent to the Riemann hypothesis.

2.3 A restatement of the Riemann hypothesis

The Riemann hypothesis is equivalent to the statement that the equation,

$$\frac{1}{\zeta(s)} = \sum_{n=1}^{\infty}\frac{\mu(n)}{n^s}, \tag{4}$$

is valid for every s with real part greater than 1/2.

This restatement says that when we invert the zeta function Dirichlet series, we obtain another Dirichlet series, $F(s)$, that is in keeping with the analytically continued values of the zeta function (that is, values such as $1/2 < \Re(s) \leq 1$). This restatement requires that $F(s)$ be defined only for $\Re(s) > 1/2$ for the conjecture to hold, due to the following rationale.

If the Riemann Hypothesis is true, then all non-trivial zeros of $\zeta(s)$ lie on the line $\mathrm{Re}(s) = 1/2$. This means that $\zeta(s)$ has no zeros in the open half-plane $\mathrm{Re}(s) > 1/2$. Since $\zeta(s)$ is non-zero there, we can define $1/\zeta(s)$ as an analytic function in that half-plane.

But the Dirichlet series in (4) is the unique Dirichlet series with that analytic function as sum if it converges. So, if the RH is true, then $1/\zeta(s)$ has an analytic continuation to $\mathrm{Re}(s) > 1/2$, and this continuation must coincide with the Dirichlet series — if it converges. And in fact, if the series converges on $\mathrm{Re}(s) > 1/2$, it must be equal to $1/\zeta(s)$, and thus implies the RH.

To help with the exposition, let us define $F_N(s)$ as the partial sums of $F(s)$,

$$F_N(s) = \sum_{n=1}^{N}\frac{\mu(n)}{n^s} \tag{5}$$

3 The solution

What this proof really establishes is the validity of equation (4) for every s with $\Re(s) > 1/2$, even if $\zeta(s) = 0$ for some such s (which would mean the Riemann hypothesis is false). In that case, however, the equation would still hold in a sense, if $F(s)$ diverged (which seems implausible, since Dirichlet series either always converge or never converge for $\Re(s)$ in a given domain). The Riemann hypothesis therefore implies that $F(s)$ never diverges for $\Re(s) > 1/2$.

Convergence of $F(s)$ for $\Re(s) > 1/2$ is highly likely, as can be verified empirically through its new integral representation (which will be introduced in a subsequent paper),

$$\sum_{n=1}^{\infty} \frac{\mu(n)}{n^s} = \frac{1}{2\pi \boldsymbol{i}\,\Gamma(s)} \int_{c-\boldsymbol{i}\infty}^{c+\boldsymbol{i}\infty} \frac{z^{-x}}{x} \frac{\Gamma(s+x)}{\zeta(s+x)} + \frac{z^{s-x}}{s-x} \frac{\Gamma(x)}{\zeta(x)}\, dx, \text{ if } 0 < c < \Re(s), \tag{6}$$

where $z > 0$ is a degree of freedom and the convergence of the integral very likely mirrors that of $F(s)$. However, since numeric computations of this integral also seem to work for $\Re(s) > 0$, this claim needs to be further investigated.

Intuitively, it seems easy to see why $F(s)$ should always converge when $\Re(s) > 1/2$, the sum of the square-free numbers with an odd number of primes (raised to s) should always be an infinity more or less equal to the sum of those with an even number of primes and together these infinities should cancel out. As a matter of fact, these infinities will be exactly equal when $s = 1$ (which is a pole in the zeta function), causing $F(1)$ to zero out.

3.1 Inversion formula for Dirichlet series

Before laying out the points of the proof, let us review the inversion formula for Dirichlet series, whose proof and ubiquitous evidence for were given in paper [1]. It is a proven theorem that relates a Dirichlet series to its arithmetic function, and can be stated as follows.

Theorem 1 Suppose that $F_a(s)$ is a Dirichlet series and $a(n)$ is its associated arithmetic function. Then for any positive integer q such that $F_a(2q) < \infty$, $a(n)$ is given by:

$$a(n) = -2 \sum_{i=q}^{\infty} (-1)^i (2\pi n)^{2i} \sum_{j=q}^{i} \frac{(-1)^j (2\pi)^{-2j} F_a(2j)}{(2i+1-2j)!}$$

Proof 1 Although not obvious, this is a very powerful result. The above power series converges for all n and is the analytic continuation of:

$$-\frac{\sin 2\pi n}{\pi n} \sum_{j=q}^{\infty} n^{2j} F_a(2j), \tag{7}$$

since they have the same Taylor series expansion and (7) only converges for $|n| < 1$. In some cases it is possible to find a closed-form for $a(n)$, though it can be challenging.

The proof is short and simple:

$$-2\sum_{i=1}^{\infty}(-1)^i(2\pi n)^{2i}\sum_{j=1}^{i}\frac{(-1)^j(2\pi)^{-2j}F_a(2j)}{(2i+1-2j)!}\Rightarrow$$

$$-2\sum_{i=1}^{\infty}(-1)^i(2\pi n)^{2i}\sum_{j=1}^{i}\frac{(-1)^j(2\pi)^{-2j}}{(2i+1-2j)!}\sum_{k=1}^{\infty}\frac{a(k)}{k^{2j}}\Rightarrow$$

$$\sum_{k=1}^{\infty}a(k)\left(-2\sum_{i=1}^{\infty}(-1)^i(2\pi n)^{2i}\sum_{j=1}^{i}\frac{(-1)^j(2\pi)^{-2j}k^{-2j}}{(2i+1-2j)!}\right)$$

The theorem then follows from the following equation:

$$-2\sum_{i=1}^{\infty}(-1)^i(2\pi n)^{2i}\sum_{j=1}^{i}\frac{(-1)^j(2\pi k)^{-2j}}{(2i+1-2j)!}=\begin{cases}1, & \text{if } n=k\\ 0, & \text{otherwise}\end{cases} \tag{8}$$

And the above equation is justified for being the convolution of $\mu_0(n)$, the unit function, and the arithmetic function of the series k^{-s}, $b(n)$, since from the convolution formula:

$$c(n)=(\mu_0 * b)(n)=\sum_{d|n}\mu_0(d)b\left(\frac{n}{d}\right)=b(n)=\begin{cases}1, & \text{if } n=k\\ 0, & \text{otherwise}\end{cases}$$

For more details on the proof, please refer to [1]. □

This is a real breakthrough and a surprising result. It says that every Dirichlet series has coefficients given by a Taylor series. One of the advantages of this formula is that if you know $F_a(s)$ at the even integers, you know the series expansion of the $a(n)$. Another advantage is that it extends $a(n)$ to the complex numbers. Perhaps this extended function even has some deeper connection to the Dirichlet series, though that is not currently known.

The inversion formula also has a property that is trivial to see. For any complex s:

$$\frac{a(n)}{n^s}=-2\sum_{i=q}^{\infty}(-1)^i(2\pi n)^{2i}\sum_{j=q}^{i}\frac{(-1)^j(2\pi)^{-2j}F_a(s+2j)}{(2i+1-2j)!}$$

Theorem 2 Let $G(s)$ be a function that admits a Dirichlet series representation, $F_a(s)$, in at least part of its domain. Then at the integers the $a(n)$, derived from G, are finite and not all zero.

Proof 2 Self-evident. □

In other words, the inversion formula can be used to check if a function is a Dirichlet series. It says nothing about the convergence domain of $F_a(s)$, though.

3.2 Riemann hypothesis proof

First we note that, even though the zeta function only admits a Dirichlet series representation for $\Re(s) > 1$, its reciprocal can be expressed as a Dirichlet series supposedly for $\Re(s) > 0$.

The analytic continuation of the zeta function to $0 < \Re(s) < 1$ is achieved through the Dirichlet eta function, $\eta(s)$, that is, the below equation holds for $\Re(s) > 0$:

$$\zeta(s) = \frac{\eta(s)}{1-2^{1-s}} = \frac{1}{1-2^{1-s}} \sum_{j=1}^{\infty} \frac{(-1)^{j+1}}{j^s},$$

With everything that has been said, we are now in a position to produce a short proof. Suppose that for $\Re(s) > 1/2$,

$$\frac{1}{\zeta(s)} = \frac{1-2^{1-s}}{\eta(s)} = \sum_{n=1}^{\infty} \frac{a(n)}{n^s} \tag{9}$$

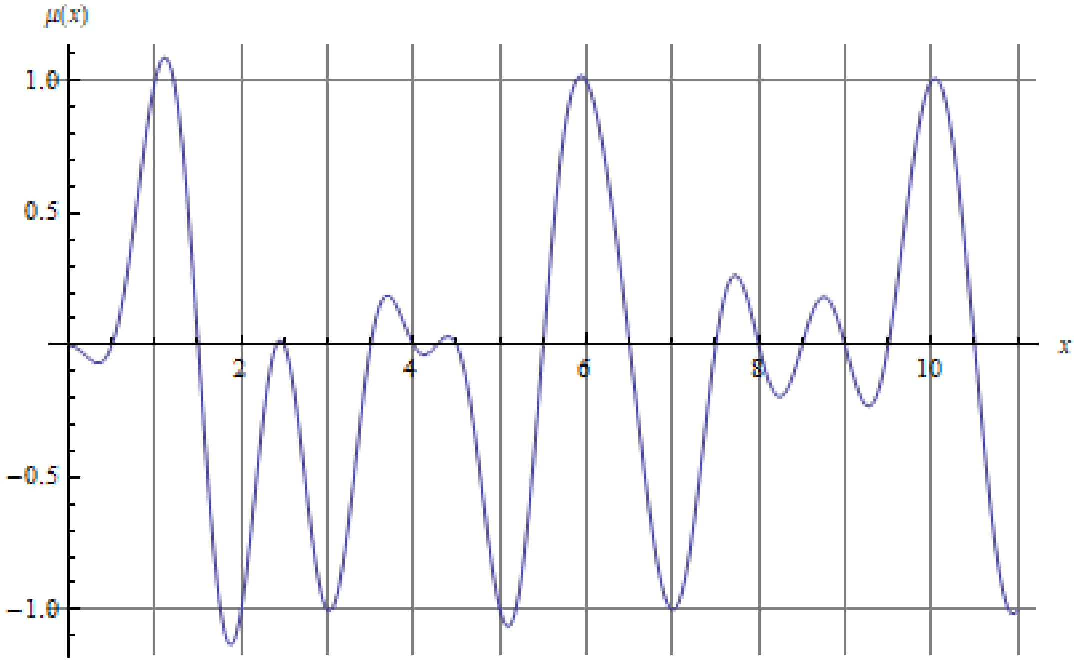
What is $a(n)$?

The inversion formula tells us that $a(n)$ is given by the below power series:

$$a(n) = -2 \sum_{i=1}^{\infty} (-1)^i (2\pi n)^{2i} \sum_{j=1}^{i} \frac{(-1)^j (2\pi)^{-2j} \zeta(2j)^{-1}}{(2i+1-2j)!} \tag{10}$$

Despite the fact that this formula yields the coefficients of the Dirichlet series $1/\zeta(s)$ independently of its convergence domain, it has no known closed-form and hence in principle we do not know what values it assumes at the positive integers. However, since for $\Re(s) > 1$ we know from the Euler product that $a(n) = \mu(n)$, and (10) does not depend on s, we conclude that $a(n) = \mu(n)$ for every positive integer n.

Below the graph of $\mu(n)$ was plotted in the $(0, 11)$ interval for some insight into its shape and local minima and maxima (it crosses the x-axis at the square-full and half-integers):



Prior to this discovery it was not even known that the Möbius function could be continuous or analytic, or what its shape could look like. Therefore, there is no question that equation (4) holds whenever $F(s)$ converges ($F(s)$ is the Dirichlet series representation of $(1-2^{1-s})/\eta(s)$).

4 Conclusion

The equivalent problem shown in this paper is a result from the literature, some mathematician has found a way to prove that it implies the Riemann hypothesis. The missing piece of the puzzle was to come up with an alternative way to prove that,

$$\frac{1}{\zeta(s)} = \sum_{n=1}^{\infty} \frac{a(n)}{n^s} \Rightarrow a(n) = \mu(n),$$

without the requirement that $\Re(s) > 1$, which the inversion formula for Dirichlet series does. At the very least, this paper has shown that this equation must hold whenever $F(s)$ converges.

References

[1] Risomar Sousa, Jose *An exact formula for the prime counting function, eprint arXiv:1905.09818,* 2019.