An Easy Proof of the Triangle Inequality

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Abstract

High school and undergraduate algebra and calculus textbooks don’t provide a fast and easy proof of the triangle inequality. Here is a proof that seems relatively easy. It does require a little bit of logic, but that can be a plus.

Introduction

A perusal of typical high school, say College Algebra level textbooks [1], as well as undergraduate Calculus textbooks [2, 4] reveals that these books don’t give a proof of the triangle inequality. The calculus texts do give a proof of the inequality in their chapters on vectors. These proofs do imply the simpler $\mathbb{R}$ case. The proofs are involved, requiring the Schwarz inequality. In Thomas’s book he actually, one could say, apologizes to the reader for the length and complexity of his $\mathbb{R}^k$ proof. Curiously, the law of cosines does imply the triangle inequality, but this is not mentioned in any of these textbooks. Wikipedia doesn’t give a simple proof either.

Rudin in his analysis text [3] does give a proof for the complex, hence the real, case; it is relatively brief and simple. College algebra texts covering complex numbers could include such a proof. It’s a nice application of the conjugate of a complex number.

The following proof seems to make concrete the intuitions of why the triangle inequality holds in the specific real case: if $x$ and $y$ have opposite signs, their sum is reduced from the sum of their absolute values.
Proof

Theorem 1. For real numbers $x$ and $y$

$$|x + y| \leq |x| + |y|. \quad (1)$$

Proof. If and only if $x$ and $y$ are both positive or both negative then

$$\frac{|x + y|}{|x|} > 1 \quad \text{and} \quad \frac{|x + y|}{|y|} > 1. \quad (2)$$

This follows as

$$|x + y| = |x| + |y| \quad (3)$$

in these cases and in turn this implies

$$\frac{|x + y|}{|x|} = 1 + \frac{|y|}{|x|}$$

and this is true if and only if

$$\frac{|x + y|}{|x|} < 1.$$

By symmetry, the RHS of (2) also holds.

So one case is resolved; (3) implies (1).

Take the negation of (2); this is

$$\frac{|x + y|}{|x|} \leq 1 \quad \text{or} \quad \frac{|x + y|}{|y|} \leq 1. \quad (4)$$

Say the former, then

$$|x + y| \leq |x| \quad \text{implies} \quad |x + y| \leq |x| + |y|$$

and that’s (1) as well. Mutatis mutandis, the right hand side of (4) also implies (1).

If one or more of $x$ and $y$ are 0, the theorem holds. So all cases have been addressed.
References


