On the Last Numbers of Positive Integers

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Abstract: In this note, we are interested in the last numbers of positive integers; for example, for 20211206, the last number is 6, typically we note that for any positive integer \( a \), the last numbers of \( a^5 \) and \( a \) are the same.

Key Words: The last number of a positive integer, Fermat’s small theorem, series of integers, Yamane’s problem, repeated series.

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1 Introduction

In this note, we are interested in the last numbers of positive integers; for example, for 20211206, the last number is 6, typically we note that for any positive integer \( a \), the last numbers of \( a^5 \) and \( a \) are the same.
2 Main results

For any natural numbers (positive integers) $a$ and $b$ we consider the series

$$\{(a - n, b + n)\}_{n=\infty}^{n=-\infty},$$

as in

$$\ldots, (a - 2, b + 2), (a - 1, b + 1), (a, b), (a + 1, b - 1), (a + 2, b - 2), \ldots.$$

Then, we have

**Theorem 2.1** In the series

$$\{(a - n)^3 + (b + n)^3\}_{n=\infty}^{n=-\infty},$$

the last numbers are the repeated series composing at most 4 numbers.

Indeed, we note the identity

$$S_5 = (a - 5)^3 + (b + 5)^2$$

$$= (a + b) \left( a^2 - ab + b^2 - 15(a - b - 5) \right).$$

Note that all the cases

$$(a + b)15(a - b - 5)$$

is a multiply of 10; that is, the last numbers of $S_0 = a^3 + b^3$ and $S_5$ are the same. Therefore, in general, we have the theorem.

In general, let $L_n$ denote the last number of an integer $n$.

**Corollary 2.1** If $L(a + b) = 0$, then $L(a^3 + b^3) = 0$.

Theorem 2.1 states that the series is repeated with at most 4 numbers. By examining the details, we have

**Corollary 2.2** For

$L(a + b) = 1$,

the series is repeated as

$$1, 7, 9, 7, 1.$$

Similarly,

$$2 : 6, 8, 2, 8, 6.$$
3 An application of the Fermat’s small theorem

In connection with the last numbers of positive integers, we obtain the pleasant theorem:

Theorem 3.1 For any positive integer \(a\), the last numbers of \(a^5\) and \(a\) are the same.

Indeed, in the identity

\[
a^5 - a = a(a^{5-1} - 1),
\]

from the Fermat’s small theorem ([1, 2]), if \(a\) is not a multiply of 5, since

\[
(a^{5-1} - 1) \equiv 0, \quad (\text{mod } 5),
\]

we see that for the both cases of even \(a\) and odd \(a\), \(a^5 - a\) is a multiply of 10 and so, we have the desired result.

If \(a\) is a multiply of 5, then the result is trivial.
4 Open problems

In this note we are interested in the last number of positive integers. How will be such an problem? It seems that this type problem will be a new type one. Therefore, we would like to propose a general problem as the Yamane’s problem:

Yamane’s Problem: Discuss or derive the results about the last numbers of positive integers.

In particular, we can propose

(Y1): What is the general theorem of Theorem 2.1?
(Y2): Could we derive the result of Theorem 3.1 type?

References
