New integrals with Barnes function

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Abstract
In this paper, I study three particular integrals where I give three conjectural formulas in terms of Barnes G-function. There are a article where there are already several similary logarithmics integrals, this is the paper: rediscovery of Malmsten’s integrals, their evaluation by contour integration methods and some related results (1). And also, there are a interest for these three integrals because actually softwares as Mathematica or Maple didn’t give a correct closed form.

1 Definition
The Barnes function is defined as the following Weierstrass product:

\[ G(1 + z) = (2\pi)^\frac{z}{2} e^{-\frac{z^2}{2}} \prod_{k=1}^{\infty} \left(1 + \frac{z^2}{k}\right)^k e^{-\frac{z^2}{k}} \]  

where gamma is the Euler-Mascheroni constant.

The following properties of G are well-known.

2 Properties

- \[ G(1) = 1 \]  
- \[ G(1 + z) = G(z)\Gamma(z) \]

3 First integral: \[ \int_{0}^{\infty} \log\left(t^2 + z^2\right) \frac{dt}{e^{a\pi t} - 1} \]
Where a and z are a positive integer number or a positive rational fraction.
I have successively (A is the Glaisher-Kinkelin’s constant (5))

\[
\int_0^\infty \frac{t \log(t^2 + z^2)}{e^{\pi t} - 1} \, dt = -4 \log(G(z/2)) - 4 \log(\Gamma(z/2)) + \frac{1}{3} - 4 \log(A)
\]
\[
- \frac{\log(2)}{2} z^2 + \log(2) z + \frac{\log(2)}{3} + \frac{z^2 \log(z)}{2} - \frac{3 z^2}{4} + z \log(\pi)
\]

Then

\[
\int_0^\infty \frac{t \log(t^2 + z^2)}{e^{4\pi t} - 1} \, dt = - \log(G(z)) - \log(\Gamma(z)) + \frac{1}{12} - \log(A)
\]
\[
+ \frac{\log(2) z^2}{2} + \frac{z^2 \log(z)}{2} - \frac{3 z^2}{4} + \frac{z \log(\pi)}{2}
\]

Then

\[
\int_0^\infty \frac{t \log(t^2 + z^2)}{e^{a\pi t} - 1} \, dt = - \frac{4}{9} \log\left(G\left(\frac{3z}{2}\right)\right) - \frac{4}{9} \log\left(\Gamma\left(\frac{3z}{2}\right)\right) + \frac{1}{27} - \frac{4 \log(A)}{9} - \frac{\log(2) z^2}{2} + \frac{\log(2) z}{3} + \frac{\log(2) z}{27} + \frac{z^2 \log(z)}{2} - \frac{3 z^2}{4} + \frac{z \log(\pi)}{3} + \log(3) \left(\frac{z^2}{2} - \frac{1}{27}\right)
\]

Then

\[
\int_0^\infty \frac{t \log(t^2 + z^2)}{e^{4a\pi t} - 1} \, dt = - \frac{\log(G(2z))}{4} - \frac{\log(\Gamma(2z))}{4} + \frac{1}{48} - \frac{\log(A)}{4}
\]
\[
+ \frac{\log(2) z^2}{2} + \frac{\log(2) z}{4} - \frac{\log(2) z}{48} + \frac{z^2 \log(z)}{2} - \frac{3 z^2}{4} + \frac{z \log(\pi)}{4}
\]

And now in general

\[
\int_0^\infty \frac{t \log(t^2 + z^2)}{e^{a\pi t} - 1} \, dt = -4 \frac{\log(G((1/2)az))}{a^2} - 4 \frac{\log(\Gamma((1/2)az))}{a^2} + \frac{1}{3 a^2} - 4 \frac{\log(A)}{a^2}
\]
\[
- \frac{\log(2) z^2}{2} + \frac{\log(2) z}{a} + \frac{\log(2) z}{3 a^2} + \frac{z^2 \log(z)}{2} - \frac{3 z^2}{4} + \frac{z \log(\pi)}{a} + \log(a) \left(\frac{z^2}{2} - \frac{1}{3 a^2}\right)
\]
4 Second integral: \( \int_0^\infty \frac{t \log(t^2 + z^2)}{\sinh(a \pi t)} \, dt \)

Where \( a \) and \( z \) are a positive integer number or a positive rational fraction.

We know this identity:

\[
(\sinh(\pi t))^{-1} = 2 \left(e^{\pi t} - 1\right)^{-1} - 2 \left(e^{2\pi t} - 1\right)^{-1}
\]

And so we can write that

\[
(\sinh(a \pi t))^{-1} = 2 \left(e^{a \pi t} - 1\right)^{-1} - 2 \left(e^{2a \pi t} - 1\right)^{-1}
\]

Now using the precedent formula we obtain

\[
\int_0^\infty \frac{t \log(t^2 + z^2)}{\sinh(a \pi t)} \, dt = -8 \log \left(G \left(\frac{1}{2}a z\right)\right) - 8 \log \left(\Gamma \left(\frac{1}{2}a z\right)\right) + 2 \log \left(G \left(a z\right)\right) + 2 \log \left(\Gamma \left(a z\right)\right) + \frac{1}{2a^2} - \frac{6 \log(A)}{a^2} - \frac{2 \log(2)}{3a^2} + \frac{z \log(\pi)}{a} - \frac{\log(a)}{2a^2}
\]

5 Third integral: \( \int_0^\infty \frac{t \log(t^2 + z^2)}{e^{a \pi t} + 1} \, dt \)

Where \( a \) and \( z \) are a positive integer number or a positive rational fraction.

Using the same principale than the first integral and with several closed form of the integral, finally

\[
\int_0^\infty \frac{t \log(t^2 + z^2)}{e^{a \pi t} + 1} \, dt = -4 \log \left(G \left(\frac{1}{2}a z\right)\right) - 4 \log \left(\Gamma \left(\frac{1}{2}a z\right)\right) + 2 \log \left(G \left(a z\right)\right) + 2 \log \left(\Gamma \left(a z\right)\right) + \frac{1}{6a^2} - \frac{2 \log(A)}{a^2} - \frac{4 \log(2)}{2a^2} + \frac{2 \log\left(\frac{z^2}{2a^2}\right)}{2a^2} + \log(\pi) \left(-\frac{z^2}{2} - \frac{1}{6a^2}\right)
\]

6 Examples of applications

First example

Consider the integral

\[
\int_0^\infty \frac{t \log(t^2 + 2^2)}{e^{(3/2)\pi t} - 1} \, dt
\]

So we see \( a=3/2 \) and \( z=2 \). We obtain

\[
-\frac{83}{27} + \frac{8 \log(A)}{9} + \frac{4 \log(2)}{3} + \frac{50 \log(3)}{27}
\]
**Second example**

Consider the integral

\[ \int_0^\infty \frac{t \log(t^2 + 2^2)}{\sinh((1/3)\pi t)} \, dt \]

So we see \(a=1/3\) and \(z=2\). We obtain

\[ -\frac{3}{2} + 18 \log(A) + 29 \log(2) - \frac{9 \log(3)}{4} + 18 \log(\pi) - \frac{5 \sqrt{3}\pi}{3} + \frac{5 \Psi(1,1/3) \sqrt{3}}{2\pi} - 36 \log(\Gamma(1/3)) \]

Where \(\Psi(1,\frac{1}{3})\) is the trigamma function at \(1/3\) (6).

**Third example**

Consider the integral

\[ \int_0^\infty \frac{t \log(t^2 + 3^2)}{e^{(1/2)\pi t} + 1} \, dt \]

So we see \(a=1/2\) and \(z=3\). We obtain

\[ \frac{83}{12} - 2 \log(A) - \frac{35 \log(2)}{3} - \frac{9 \log(3)}{2} - 6 \log(\pi) - 4 \frac{G}{\pi} + 12 \log(\Gamma(1/4)) \]

Where \(G\) is the Catalan’s constant (7).

### 7 References

(1): Iaroslav V. Blagouchine, Rediscovery of Malmsten’s integrals, their evaluation by contour integration methods and some related results (2014)


(3),(4): https://dlmf.nist.gov/5.17


(7): https://mathworld.wolfram.com/CatalansConstant.html