Gamma function, Lambert W-function: “The integral”

Edgar Valdebenito
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Abstract. We give an integral involving the Gamma function and the Lambert W-function.

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Introduction
Recall that
\[
\left( \Gamma \left( \frac{1}{3} \right) \right)^3 = \frac{16 \frac{3}{\sqrt{3}}}{\sqrt{3}} \int_0^\infty \int_0^\infty \frac{\frac{3}{\sqrt{3}} \cosh x}{\cosh (x+y) + \cosh (x-y)} \, dx \, dy
\]
where \( \Gamma(x) \) is the Gamma function.

The (real-valued) Lambert W-functions are solutions of the nonlinear equation
\[
w e^w = y , \ y \in \mathbb{R}
\]
If \( y > 0 \), there is a unique real solution, \( w(y) \), satisfying \( 0 < w(y) < \infty \). If \( -\frac{1}{e} \leq y < 0 \), there are exactly two real solutions, \( w_0(y) \) and \( w_{-1}(y) \), satisfying respectively \( -1 \leq w_0(y) < 0 \) and \( -\infty < w_{-1}(y) \leq -1 \). Clearly, \( w(0 +) = 0, w(0 -) = 0 \), and \( w_{-1}(0 -) = -\infty \), while \( w_0(-1/e) = w_{-1}(-1/e) = -1 \). For \( y < -1/e \), there are no real solutions of \( w e^w = y , \ y \in \mathbb{R} \). For a discussion of the various branches of the Lambert W-functions, also in the complex plane, see [6].

Integral for \( \left( \Gamma \left( 1/3 \right) \right)^3 \)

For \( \alpha = 4 \frac{3}{\sqrt{2}} \ln 2 \), we have
\[
\left( \Gamma \left( \frac{1}{3} \right) \right)^3 = \alpha + \int_\alpha^\infty \left( 1 - \sqrt{1 - 4 \left( \frac{3}{2x} W \left( \frac{2x}{3} \right) \right)^{3/2}} \right) \, dx
\]
where \( W(x) \) is the Lambert W-function.
References


