Eliminate the Irrelevant to the Subject and Prove Equations and Inequalities related to Beal's Conjecture

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Abstract

The subject of this article is exactly to analyze Beal's conjecture and prove it. This proof involves certain basic knowledge of algebra, number theory, and the symmetry of odd points on the number axis.

First, we classify mathematical expressions which consist of A^X, B^Y and C^Z according to the parity of A, B and C, so get rid of two combinations of A^X, B^Y and C^Z, for they have nothing to do with the conjecture.

For the remaining two cases, we first prove the equation $A^X+B^Y=C^Z$ by examples where A, B, and C have at least one common prime factor.

Then prove each kind of $A^X+B^Y\neq C^Z$ by arithmetic fundamental theorem, the mathematical induction, the binomial theorem, the reduction to absurdity, and the interrelation between a sum of two odd numbers and an even number that served as the center of symmetry, where A, B, and C have not a common prime factor, and that each proof takes what it needs.

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1. Introduction

Beal's conjecture states that if $A^X+B^Y=C^Z$, where A, B, C, X, Y and Z are positive integers, and X, Y and Z are all greater than 2, then A, B and C must have a common prime factor.

The conjecture was discovered by Andrew Beal in 1993. Later, it was announced in December 1997 issue of the Notices of the American Mathematical Society; [1]. However, it remains a conjecture that has neither been proved nor disproved.

The conjecture indicates that whoever wants to prove it, must both exemplify $A^X+B^Y=C^Z$ where A, B and C have a common prime factor, and must prove $A^X+B^Y\neq C^Z$ where A, B and C have no a common prime factor.

First of all, we consider the scopes of values of A, B, C, X, Y and Z in $A^X+B^Y=C^Z$ as the necessary constraints, and allow that a part or all of the necessary constraints is replaced by specific values within the scopes.

2. On mathematical expressions which consist of A^X, B^Y and C^Z

We classify mathematical expressions which consist of A^X, B^Y and C^Z according to the parity of A, B and C, and from this get rid of following two kinds of $A^X+B^Y\neq C^Z$ under the necessary constraints.

1) A, B and C are all positive odd numbers;

2) A, B and C are two positive even numbers and one positive odd number.

Then, we continue to have following two kinds which contain $A^X+B^Y=C^Z$ and $A^X+B^Y\neq C^Z$ under the necessary constraints:

1) A, B and C are all positive even numbers；

2) A, B and C are two positive odd numbers and one positive even number.

3. Exemplify $A^X+B^Y=C^Z$ under the necessary constraints

For two retained indefinite equations at above, each of them has many sets of solution as positive integers actually, as shown in the following examples.

When A, B and C are all positive even numbers, let A=B=C=2, X=Y ≥3 and Z=X+1, then $A^X+B^Y=C^Z$ is changed to $2^X+2^X=2^{X+1}$, so $A^X+B^Y=C^Z$ after the assignment of these values has one set of solution with A, B and C as 2, 2 and 2, also A, B and C have one common prime factor 2.

In addition to the above example, let A=B=162, C=54, X=Y=3 and Z=4, then $A^X+B^Y=C^Z$ is changed to $162^3+162^3=54^4$, so $A^X+B^Y=C^Z$ after the assignment of these values has one set of solution with A, B and C as 162, 162 and 54, also A, B and C have two common prime factors 2 and 3.

When A, B and C are two positive odd numbers and one positive even number, let A=C=3, B=6, X=Y=3 and Z=5, then $A^X+B^Y=C^Z$ is changed to $3^3+6^3= 3^5$, so $A^X+B^Y=C^Z$ after the assignment of these values has one set of solution with A, B and C as 3, 6 and 3, also A, B and C have one common prime factor 3.

In addition to the above example, let A=B=7, C=98, X=6, Y=7 and Z=3, then $A^X+B^Y=C^Z$ is changed to $7^6+7^7=98^3$, so $A^X+B^Y=C^Z$ after the assignment of these values has one set of solution with A, B and C as 7, 7 and 98, also A, B and C have one common prime factor 7.

It follows that there are undoubtedly $A^X+B^Y=C^Z$ under the necessary constraints, but then A, B and C must have one common prime factor.

Then again, according to the need that proves the conjecture, if we can further prove $A^X+B^Y\neq C^Z$ under the necessary constraints, where A, B and C have not a common prime factor, then the conjecture is tenable surely.

4. Divide $A^X+B^Y\neq C^Z$ into two kinds and prove one kind thereof

When A, B and C are all positive even numbers, they have at least one common prime factor 2, so A, B and C without common prime factor can only be two positive odd numbers and one positive even number.

If A, B, and C have not a common prime factor, then A^X, B^Y and C^Z have not a common prime factor either.

If A^X, B^Y and C^Z have not a common prime factor, then we can divide $A^X+B^Y\neq C^Z$ into following two kinds, and first prove one kind thereof.

(1) Two terms of $A^X+B^Y\neq C^Z$ have a common prime factor, yet another term has not the common prime factor.

Proof· when any two of A^X, B^Y and C^Z have a common prime factor, we can extract this common prime factor from these two terms to become a prime factor of their sum or difference, yet another term has not this common prime factor, accordingly, it can only lead up to $A^X+B^Y\neq C^Z$ or $C^Z-A^X\neq B^Y$ or $C^Z-B^Y\neq A^X$, according to the fundamental theorem of arithmetic; [2].

For $C^Z-A^X\neq B^Y$ and $C^Z-B^Y\neq A^X$, after you transpose a term of each of them, you get $A^X+B^Y \neq C^Z$ too.

Thus there is only $A^X+B^Y \neq C^Z$ under the necessary constraints where any two of A^X, B^Y and C^Z have a common prime factor, yet another have not it.

(2) No two terms of $A^X+B^Y \neq C^Z$ have a common prime factor.

The proof of $A^X+B^Y \neq C^Z$ in the context that no two terms have a common prime factor is the difficult point of this article, so we will elaborate on every relevant section hereinafter.

5. Divide the unproved kind of $A^X+B^Y \neq C^Z$ into four inequalities

The inequality $A^X+B^Y \neq C^Z$ in the context that no two terms have a common prime factor is able to be divided into following two inequalities:

1) $A^X+B^Y \neq (2W)^Z$, i.e. $A^X+B^Y \neq 2^Z W^Z$;

2) $A^X+(2W)^Y \neq C^Z$, i.e. $A^X+2^Y W^Y \neq C^Z$.

In above-listed two inequalities, newly emerging W is an odd number ≥ 1.

Then, we continue to divide $A^X+B^Y \neq 2^Z W^Z$ into following two inequalities:

(1) $A^X+B^Y \neq 2^Z$;

(2) $A^X+B^Y \neq 2^Z O^Z$, where O is an odd number>1, the same below.

Then again, continue to divide $A^X+2^Y W^Y \neq C^Z$ into following two inequalities:

(3) $A^X+2^Y \neq C^Z$;

(4) $A^X+2^Y O^Y \neq C^Z$.

We regard the case that no two terms of $A^X+B^Y \neq C^Z$ have a common prime factor as a qualification, also regard the qualification plus the necessary constraints defined in the introduction as the known constraints for each inequality concerned. Then the proof for $A^X+B^Y \neq C^Z$ under the qualification is turned to prove above-listed 4 inequalities under the known constraints.

6. Several grounds which prove the first two inequalities

Whether you take an even point on the right side of odd point 3 on the number axis or an even number≥4 in the sequence of natural numbers as a symmetric center, you can draw following 4 conclusions from interrelation between the even number and a sum of two odd numbers; [3] and [4].

Conclusion 1˙ The sum of two bilateral symmetric odd numbers is equal to the double of the even number as the symmetric center.

Conclusion 2˙ The sum of two asymmetric odd numbers is not equal to the double of the even number as the symmetric center.

Conclusion 3˙ If the sum of two odd numbers is equal to the double of an even number, then these two odd numbers are bilateral symmetric with the even number as the symmetric center.

Conclusion 4˙ If the sum of two odd numbers is not equal to the double of an even number, then these two odd numbers are not bilateral symmetric with the even number as the symmetric center.

After this, we will cite these conclusions to prove the first two inequalities.

On the whole, if regard 2^{Z-1} or $2^{Z-1}O^Z$ as a symmetric center, a sum of two asymmetric odd numbers A^X and B^y is not equal to 2^Z or 2^ZO^Z, whether or not these two odd numbers have a common prime factor, according to the conclusion 2 aforementioned. Thus when we continue to prove the first two inequalities, it is necessary to refer, compare and utilize the sum of two symmetric odd numbers on two sides of the symmetric center.

7. Prove $A^X+B^Y \neq 2^Z$ under the known constraints

Let us regard 2^{Z-1} as a symmetric center to prove $A^X+B^Y \neq 2^Z$ under the known constraints by the mathematical induction; [5], *ut infra*.

(1) When Z-1=2, 3, 4, 5 and 6, symmetric odd numbers on two sides of each symmetric center are successively listed below.

1^6, 3, (2^2), 5, 7, (2^3), 3^2, 11, 13, 15, (2^4), 17, 19, 21, 23, 5^2, 3^3, 29, 31, (2^5), 33, 35, 37, 39, 41, 43, 45, 47, 7^2, 51, 53, 55, 57, 59, 61, 63, (2^6), 65, 67, 69, 71, 73, 75, 77, 79, 3^4, 83, 85, 87, 89, 91, 93, 95, 97, 99, 101, 103, 105, 107, 109, 111, 113, 115, 117, 119, 11^2, 123, 5^3, 127

As listed above, it can be seen that there are no two of O^V with $V \geq 3$ on two places of each pair of bilateral symmetric odd numbers with 2^{Z-1} as a symmetric center, where Z-1=2, 3, 4, 5 and 6.

So, there are $A^X+B^Y \neq 2^3$, $A^X+B^Y \neq 2^4$, $A^X+B^Y \neq 2^5$, $A^X+B^Y \neq 2^6$ and $A^X+B^Y \neq 2^7$ under the known constraints, according to the conclusion 2 in section 6, and that any two terms of each inequality have not a common prime factor.

In addition to above results, we also found that there is no B^2 on each symmetric place of A^X, and there is no A^2 on each symmetric place of B^Y, and vice versa.

(2) When Z-1=K with K≥6, you can suppose that there is only $A^X+B^Y \neq 2^{K+1}$ under the known constraints.

(3) When Z-1=K+1, let us prove that there is only $A^X+B^Y \neq 2^{K+2}$ under the known constraints.

Proof· Since there is only $A^X+B^Y \neq 2^{K+1}$ under the known constraints, according to second step of the mathematical induction.

So there is $A^X+(A^X+2B^Y) \neq 2^{K+2}$ under the known constraints.

Next, let $A^X+2B^Y=O_2^M$, then there is $A^X+O_2^M \neq 2^{K+2}$, where O_2 express positive odd numbers, M is the exponent, and similarly hereinafter.

Since there is $A^X+B^Y=2^{K+1}$ under the known constraints except for Y, and Y=1, such as $3^3+37^1=2^6$ and $5^3+131^1=2^8$.

So there is $A^X+(A^X+2B^Y)=2^{K+2}$ under the known constraints except for Y, and Y=1.

Next, let $A^X+2B^1=O_1^L$, then there is $A^X+O_1^L=2^{K+2}$ where O_1 express positive odd numbers, L is the exponent, and similarly hereinafter.

In $A^X+2B^Y=O_2^M$ and $A^X+2B^1=O_1^L$, since two A^X are one and the same, two B are one and the same, and Y≥3, so there is $A^X+2B^Y>A^X+2B^1$, i.e. $O_2^M>O_1^L$.

Since there is $A^X+O_1^L=2^{K+2}$, this indicates that A^X and O_1^L are two bilateral symmetric odd numbers with 2^{K+1} as the symmetric center, according to the conclusion 3 in the section 6.

That is to say, let us take 2^{K+1} as the symmetric center, then O_1^L lies on the symmetric place of A^X, undoubtedly the number on the symmetric place of A^X is unique, namely O_1^L on the symmetric place of A^X is unique.

Therefore, when the base number of the number on the symmetric place of A^X takes O_1, the exponent L of O_1 can only be equal to 1, according to the examples we have already shown that $A^X+B^Y=2^{K+1}$ under the known constraints except for Y, and Y=1.

Or to put it bluntly, there is $A^X+O_1^L=2^{K+2}$ under the known constraints except for L, and L can only be equal to 1.

Whereas, due to $O_2^M>O_1^L$, when take 2^{K+1} as the symmetric center, O_2^M does not lie on the symmetric place of A^X, therefore, there is $A^X+O_2^M\neq 2^{K+2}$, according to the conclusion 2 in the section 6.

Based on the result we have already gotten, it is exactly L=1 in O_1^L.

Furthermore, let us analyze $O_2^M>O_1^L$ to confirm different limits of values of the exponent M of O_2^M , in order to complete the proof of $A^X+O_2^M\neq 2^{K+2}$ under the known constraints.

There are five cases of $O_2^M>O_1^L$ only, as listed below.

(1) $O_2>O_1$ and M>L;

(2) $O_2>O_1$ and M=L;

(3) $O_2>O_1$ and M<L;

(4) $O_2=O_1$ and M>L;

(5) $O_2<O_1$ and M>L.

Since three cases of five cases at the above contain M>L, so for three such cases, in the case L=1, there is $M\geq 2$, then there are $A^X+O_2^M\neq 2^{K+2}$ and $A^X+O_2^2\neq 2^{K+2}$ under the known constraints.

For $A^X+O_2^M\neq 2^{K+2}$ under the known constraints, let us substitute Y for M in it because $Y\geq 3$ and $M\geq 3$, then there is $A^X+O_2^Y\neq 2^{K+2}$ under the known constraints.

Since O_2 in $A^X+O_2^Y\neq 2^{K+2}$ and B can express same odd numbers, therefore, after substitute B for O_2, we get $A^X+B^Y\neq 2^{K+2}$ under the known constraints.

For other two cases, when L=1: from M<L to get M=0, such that $O_2^0=1$ and get $A^X+1\neq 2^{K+2}$ under the known constraints; also from M=L to get M=1, such that O_2^1 is a prime or a product of distinct prime factors, and from this get $A^X+O_2^1\neq 2^{K+2}$ under the known constraints.

For inequalities $A^X+O_2^2\neq 2^{K+2}$, $A^X+1\neq 2^{K+2}$ and $A^X+O_2^1\neq 2^{K+2}$ under the known constraints, they are not what the subject asks us to prove, although they are correct too.

Apply the preceding way of doing thing, we can continue to prove that when Z-1=K+2, K+3…up to every integer ≥ K+2, there are likewise $A^X+B^Y\neq 2^{K+3}$, $A^X+B^Y\neq 2^{K+4}$ … up to general $A^X+B^Y\neq 2^Z$ under the known constraints.

8. Prove $A^X+B^Y\neq 2^ZO^Z$ under the known constraints

For the proof of $A^X+B^Y\neq 2^ZO^Z$ under the known constraints, let us do it with one proof and one explanation.

Firstly, take $2^{Z-1}O^Z$ as a symmetric center of A^X and B^Y to prove that O in $A^X+B^Y\neq 2^ZO^Z$ under the known constraints expresses every positive odd number by the mathematical induction, *ut infra*.

(1) When O=1, $2^{Z-1}O^Z$ i.e. 2^{Z-1}, as has been proved, there is only $A^X+B^Y\neq 2^Z$ under the known constraints, in the section 7.

(2) When O=J and J is an odd number ≥1, $2^{Z-1}O^Z$ i.e. $2^{Z-1}J^Z$, and you can suppose that there is only $A^X+B^Y\neq 2^ZJ^Z$ under the known constraints.

(3) When O=J+2, $2^{Z-1}O^Z$ i.e. $2^{Z-1}(J+2)^Z$, let us prove that there is only $A^X+B^Y\neq 2^Z(J+2)^Z$ under the known constraints.

Proof· Since there is $(J+2)^Z = J^Z + 2C_Z^1 J^{Z-1} + 2^2 C_Z^2 J^{Z-2} + ... + 2^n C_Z^n J^{Z-n} + ... + 2^Z C_Z^Z$, according to the binomial theorem; [6] and [7].

So there is $2^Z(J+2)^Z = 2^Z(J^Z + 2C_Z^1 J^{Z-1} + 2^2 C_Z^2 J^{Z-2} + ... + 2^n C_Z^n J^{Z-n} + ... + 2^Z C_Z^Z)$, i.e. $2^Z(J+2)^Z = 2^Z J^Z + 2^Z(2C_Z^1 J^{Z-1} + 2^2 C_Z^2 J^{Z-2} + ... + 2^n C_Z^n J^{Z-n} + ... + 2^Z C_Z^Z)$, and one part of this equation will be used in the following an equation and an inequality.

Due to in $A^X+B^Y=2^ZJ^Z$, A^X and B^Y are two bilateral symmetric odd numbers with $2^{Z-1}J^Z$ as the symmetric center, according to the conclusion 3 in the section 6.

Next, on the premise of Z≥3, if either of X and Y in $A^X+B^Y=2^ZJ^Z$ is more than or equal to 3, then we can exemplify that another is surely equal to 1, for example $13^3+547^1=2^37^3$ and $19^3+(7\times131)^1=2^5\times3^5$.

Since A^X and B^Y can switch places, so only let Y=1, then there is $A^X+B^Y=2^ZJ^Z$ under the known constraints except for Y, or there is $A^X+B^1=2^ZJ^Z$ under the known constraints.

So there is $A^X + B^1 + 2^Z(2C_Z^1 J^{Z-1} + 2^2 C_Z^2 J^{Z-2} + ... + 2^n C_Z^n J^{Z-n} + ... + 2^Z C_Z^Z) = 2^Z(J+2)^Z$.

Evidently, $B^1+2^Z(2C_Z^1J^{Z-1}+2^2C_Z^2J^{Z-2}+...+2^nC_Z^nJ^{Z-n}+...+2^ZC_Z^Z)$ is an odd number, and let $B^1+2^Z(2C_Z^1J^{Z-1}+2^2C_Z^2J^{Z-2}+...+2^nC_Z^nJ^{Z-n}+...+2^ZC_Z^Z)=O_1^S$.

As a consequence, there is $A^X+O_1^S=2^Z(J+2)^Z$ derived from participation of B^1.

While, there is $A^X+B^Y\neq 2^ZJ^Z$ under the known constraints, according to second step of the mathematical induction.

So there is $A^X+B^Y+2^Z(2C_Z^1J^{Z-1}+2^2C_Z^2J^{Z-2}+...+2^nC_Z^nJ^{Z-n}+...+2^ZC_Z^Z)\neq 2^Z(J+2)^Z$

Likewise $B^Y+2^Z(2C_Z^1J^{Z-1}+2^2C_Z^2J^{Z-2}+...+2^nC_Z^nJ^{Z-n}+...+2^ZC_Z^Z)$ is an odd number, and let $B^Y+2^Z(2C_Z^1J^{Z-1}+2^2C_Z^2J^{Z-2}+...+2^nC_Z^nJ^{Z-n}+...+2^ZC_Z^Z)=O_2^G$.

As another consequence, there is $A^X+O_2^G\neq 2^Z(J+2)^Z$ derived from participation of B^Y with $Y\geq 3$.

Due to $Y\geq 3$, so $B^Y+2^Z(2C_Z^1J^{Z-1}+2^2C_Z^2J^{Z-2}+...+2^nC_Z^nJ^{Z-n}+...+2^ZC_Z^Z)$ is greater than $B^1+2^Z(2C_Z^1J^{Z-1}+2^2C_Z^2J^{Z-2}+...+2^nC_Z^nJ^{Z-n}+...+2^ZC_Z^Z)$ i.e. there is $O_2^G>O_1^S$.

Due to $A^X+O_1^S=2^Z(J+2)^Z$ as above a consequence, it indicates that A^X and O_1^S are two bilateral symmetric odd numbers with $2^{Z-1}(J+2)^Z$ as the symmetric center, according to the conclusion 3 in section 6.

That is to say, let us take $2^{Z-1}(J+2)^Z$ as the symmetric center, then O_1^S lies on the symmetric place of A^X, undoubtedly the number on the symmetric place of A^X is unique, namely O_1^S on the symmetric place of A^X is unique.

Therefore, when the base number of the number on the symmetric place of A^X takes O_1, the exponent S of O_1 can only be equal to 1, according to the examples we have already shown that $A^X+B^Y=2^ZJ^Z$ under the known constraints except for Y, and Y=1.

Or to put it bluntly, there is $A^X+O_1^S=2^Z(J+2)^Z$ under the known constraints except for S, and S can only be equal to 1.

Whereas, due to $O_2^G>O_1^S$, when take $2^{Z-1}(J+2)^Z$ as the symmetric center, such that O_2^G does not lie on the symmetric place of A^X, so there is $A^X+O_2^G\neq 2^Z(J+2)^Z$, according to the conclusion 2 in the section 6.

Based on the result we have already gotten, it is exactly S=1 in O_1^S. Furthermore, let us analyze $O_2^G>O_1^S$ to confirm different limits of values of the exponent G of O_2^G, in order to complete the proof of $A^X+O_2^G\neq 2^Z(J+2)^Z$ under the known constraints.

There are five cases of $O_2^G>O_1^S$ only, as listed below.

(1) $O_2>O_1$ and G>S;

(2) $O_2=O_1$ and G>S;

(3) $O_2<O_1$ and G>S;

(4) $O_2>O_1$ and G=S;

(5) $O_2>O_1$ and G<S.

Since first three cases of five cases at the above contain G>S, so for the three such cases, in the case S=1, there is G≥2, then there are $A^X+O_2^G\neq 2^Z(J+2)^Z$ and $A^X+O_2^2\neq 2^Z(J+2)^Z$ under the known constraints.

For $A^X+O_2^G\neq 2^Z(J+2)^Z$ under the known constraints, we substitute Y for G in it because Y≥3 and G≥3, then there is $A^X+O_2^Y\neq 2^Z(J+2)^Z$ under the known constraints.

Since O_2 in $A^X+O_2^Y\neq 2^Z(J+2)^Z$ and B can express same odd numbers, thus, after substitute B for O_2, there is $A^X+B^Y\neq2^Z(J+2)^Z$ under the known constraints.

For other two cases, when S=1: from G<S to get G=0, such that $O_2^G=1$ and get $A^X+1\neq2^Z(J+2)^Z$ under the known constraints; also from G=S to get G=1, such that O_2^1 is a prime or a product of distinct prime factors, and from this get $A^X+O_2^1\neq2^Z(J+2)^Z$ under the known constraints.

For $A^X+O_2^G\neq2^Z(J+2)^Z$ under the known constraints except for G, and G=0, 1 and 2, it is not what the subject asks us to prove, although it is correct too.

Apply the preceding way of doing thing, we can continue to prove that when O=J+4, J+6, …up to every odd number≥J+4, there are likewise $A^X+B^Y\neq2^Z(J+4)^Z$, $A^X+B^Y\neq2^Z(J+6)^Z$ … up to general $A^X+B^Y\neq2^ZO^Z$ under the known constraints.

Since O expresses all positive odd numbers at here, so it contains inevitably such odd numbers whose each and A^X or B^Y have at least one common prime factor. For inequalities in this case, we have proven them in the section 4.

When A^X, B^Y and 2^ZO^Z have at least one common prime factor, we need to use each common prime factor to divide each and every term of the inequality, and then proceed as before.

Excepting the above, no two terms of each of remainder inequalities have a common prime factor, and that let us substitute O_δ for O to express an odd number in every such remainder inequality.

As thus, we have proved $A^X+B^Y\neq2^ZO_\delta^Z$.

Secondly, since there are many even numbers between $2^{Z-1}O_\delta^Z$ and $2^ZO_\delta^Z$, each of them can likewise become a symmetric center of A^X and B^Y, and it seem that being left out to use them.

But, in fact, we have already proved $A^X+B^Y\neq 2^{Z-1}O_\delta^Z$ and $A^X+B^Y\neq 2^Z O_\delta^Z$, you do not need to prove $A^X+B^Y\neq(2^{Z-1}+2h)O_\delta^Z$ where $h\geq1$ and $2^Z>2^{Z-1}+2h$, because when 2^{Z-1} rises to $2^{Z-1}+2h$, the exponent of 2 will decrease, and O_δ^Z will have also change accordingly, however, these changes only occur in positive odd numbers or/and their exponents.

9. Prove $A^X+2^Y\neq C^Z$ under the known constraints

In this section, we are going to prove $A^X+2^Y\neq C^Z$ under the known constraints by the reduction to absurdity; [8] and [9].

***Proof*·** Based on exemplified $A^X+B^y=2^{K+1}$ under the known constraints except for Y, and Y=1, in the section 7, so there is $O_1^M+O_2^L=2^Y$ where O_1 and O_2 are positive odd numbers, M and Y≥3, and L=1.

Assume that there is $A^X+2^Y=C^Z$ under the known constraints, then there is $A^X+O_1^M+O_2^L=C^Z$, i.e. $A^X+O_1^M=C^Z-O_2^L$.

Since there is $A^X+O_1^M\neq 2^G$ in the context of X, M and G≥3, according to proven $A^X+B^Y\neq 2^Z$ under the known constraints in the section 7.

So there is $C^Z-O_2^L\neq 2^G$, then after transpose a term of it, we get $O_2^L+2^G\neq C^Z$.

It is obvious that such an inequality $O_2^L+2^G\neq C^Z$ does not hold water, because there is surely $O_2^L+2^G=C^Z$ in the context that O_2 and C are positive odd numbers, G and Z≥3, and L=1 or even 2, such as $87^1+2^8=7^3$ and $7^2+2^5=3^4$.

Now that we deduce a false inequality derived from such an assumption, which means that such an assumption is wrong.

That is to say, $A^X+2^Y=C^Z$ under the known constraints is wrong either.

Therefore, there is only $A^X+2^Y\neq C^Z$ under the known constraints.

10. Prove $A^X+2^YO^Y\neq C^Z$ under the known constraints

We apply also the reduction to absurdity to prove $A^X+2^YO^Y\neq C^Z$ under the known constraints, in this section.

Proof· Based on exemplified $A^X+B^Y=2^ZJ^Z$ under the known constraints except for Y, and Y=1, in the section 8, so there is $O_3^M+O_4^L=2^YO^Y$ where O_3, O_4 and O are positive odd numbers, M and Y≥3, and L=1.

Assume that there is $A^X+2^YO^Y=C^Z$ under the known constraints, then there is $A^X+O_3^M+O_4^L=C^Z$, i.e. $C^Z-O_4^L=A^X+O_3^M$.

Since there is $C^Z-O_4^L=2^GO_n^G$ in the context that C, O_4 and O_n are positive odd numbers, Z and G≥3, and L=1 or even 2, such as $11^3-35^1=2^4\times3^4$; $3^4-7^2=2^5\times1^5$ and $1419857^5-1747866711689283^2=2^3\times6975757441^3$.

So there is $A^X+O_3^M=2^GO_n^G$.

It is obvious that the equality $A^X+O_3^M=2^GO_n^G$ in the context of X, M and G≥3 does not hold water, according to proven $A^X+B^Y\neq2^ZO^Z$ under the known constraints in the section 8.

Now that we deduce a false equality derived from such an assumption, which means that such an assumption is wrong.

That is to say, $A^X+2^YO^Y=C^Z$ under the known constraints is wrong either.

Therefore, there is only $A^X+2^YO^Y\neq C^Z$ under the known constraints.

11. Make a Summary and reach the conclusion

To sum up, on the one hand, we give examples to have exemplified $A^X+B^Y=C^Z$ under the necessary constraints, where A, B and C have at least one common prime factor, in the section 3.

On the other hand, we have already proved every kind of $A^X+B^Y\neq C^Z$ under the necessary constraints, where A, B and C have not a common prime factor, in these sections 4, 7, 8, 9 and 10.

Now that we have already exemplified each kind of equations and have proved each kind of inequalities relating to the conjecture, then we continue to make a comparison between $A^X+B^Y=C^Z$ and $A^X+B^Y\neq C^Z$ under the necessary constraints, therefore reaches undoubtedly such a conclusion that an indispensable prerequisite of $A^X+B^Y=C^Z$ under the necessary constraints is exactly that A, B and C must have a common prime factor.

The proof was thus brought to a close. As a consequence, the Beal's conjecture is tenable.

P.S. Prove Fermat's last theorem from proven Beal's conjecture

Fermat's last theorem is a special case of Beal's conjecture, whereas Beal's conjecture is a generalization of Fermat's last theorem; [10].

In this case, you let X=Y=Z, then $A^X+B^Y=C^Z$ is changed to $A^X+B^X=C^X$.

After the Beal's conjecture is proved to be true, we divide each term of $A^X+B^X=C^X$ by the greatest common divisor of three terms of this equation itself, and get a set of solution with A, B and C as positive integers without common prime factor.

It is obvious that such a result is in contradiction with proven Beal's conjecture. As thus, we have proved Fermat's last theorem by the reduction to absurdity, as easy as pie.

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References

[1] R.Daniel Mauldin, "A Generalization of Fermat's Last Theorem:The Beal Conjecture and Prize Problem", NOTICES OF THE AMS, VOLUME 44 NUMBER 11, pp.1436-1437, 1997.

[2] Hardy, G. H. and Wright, E. M. "Statement of the Fundamental Theorem of Arithmetic," "Proof of the Fundamental Theorem of Arithmetic," and "Another Proof of the Fundamental Theorem of Arithmetic." § 1.3, 2.10 and 2.11 in An Introduction to the Theory of Numbers, 5th ed. Oxford, England: Clarendon Press, pp. 3 and 21, 1979.

[3] H. Weyl, "Symmetry" , Princeton Univ. Press (1952) (Translated from German).

[4] Sloane, N. J. A. Sequences A000217/M2535, A103307, and A103308 in "The On-Line Encyclopedia of Integer Sequences."

[5] Courant, R. and Robbins, H. "The Principle of Mathematical Induction" and "Further Remarks on Mathematical Induction." §1.2.1 and 1.7 in What Is Mathematics?: An Elementary Approach to Ideas and Methods, 2nd ed. Oxford, England: Oxford University Press, pp. 9-11 and 18-20, 1996.

[6] Coolidge, J. L. "The Story of the Binomial Theorem." Amer. Math. Monthly 56, 147-157, 1949.

[7] Boyer, C. B. and Merzbach, U. C. "The Binomial Theorem." A History of Mathematics, 2nd ed. New York: Wiley, pp. 393-394, 1991.

[8] Coxeter, H. S. M. and Greitzer, S. L. Geometry Revisited. Washington, DC: Math. Assoc. Amer., p. 16, 1967.

[9] Hardy, G. H. A Mathematician's Apology, reprinted with a foreword by C. P. Snow. New York: Cambridge University Press, p. 34, 1993.

[10] William L. Hosch, "Beal's conjecture", The Editors of Encyclopaedia Britannica, Nov 26, 2024.