Gauss’ Law for Magnetism

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Abstract
Gauss' law for magnetism states that the magnetic flux across any closed surface is zero. In this paper a rigorous proof of the Gauss’ law for magnetism from the first principle has been presented.

Keywords: Magnetic flux, Closed surface.

1 DERIVATION
Let the origin of a cartesian coordinate system be situated at the center of a sphere $S_1$ of radius $\rho$. Consider a charge $q$ moving with velocity $v$ along the $x$ axis and situated at the center of the sphere at the current instant. Magnetic field $B$ at a distance $r$ due to the moving charge $q$ is

$$ B = \frac{\mu q v \times r}{4 \pi r^3} $$

[Law of Magnetism]

where

$$ C = \left( \frac{\mu q}{4 \pi} \right) $$

and

$$ F = \frac{v \times r}{r^3} = \frac{v i \times (x i + y j + z k)}{r^3} = \frac{v (y k - z j)}{r^3} $$

let $n$ be the unit vector normal to the sphere $S_1$, then

$$ F \cdot n = \frac{v (y k - z j) \cdot (x i + y j + z k)}{\rho^4} $$

[ $r = \rho$ ]

$$ = \frac{v (-zy + zy)}{\rho^4} = 0 $$

$$ \Rightarrow \oint_{S_1} B \cdot dA = \oint_{S_1} C F \cdot n \ dA = 0 \quad (i) $$

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let
\[ \mathbf{F} = f_x \mathbf{i} + f_y \mathbf{j} + f_z \mathbf{k} \]
\[ = -\frac{\mathbf{vz}}{r^3} \mathbf{j} + \frac{\mathbf{vy}}{r^3} \mathbf{k} \]

now
\[ \frac{\partial r}{\partial x} = \frac{x}{r}; \quad \frac{\partial r}{\partial y} = \frac{y}{r}; \quad \frac{\partial r}{\partial z} = \frac{z}{r} \]
so
\[ \frac{\partial f_x}{\partial x} = 0 \]
\[ \frac{\partial f_y}{\partial y} = -v \left( \frac{1}{r^3} \frac{\partial z}{\partial y} - \frac{3yz}{r^5} \right) = v \frac{3yz}{r^5} \]
\[ \frac{\partial f_z}{\partial z} = v \left( \frac{1}{r^3} \frac{\partial y}{\partial z} - \frac{3yz}{r^5} \right) = -v \frac{3yz}{r^5} \]

now
\[ \nabla \cdot \mathbf{F} = \frac{\partial f_x}{\partial x} + \frac{\partial f_y}{\partial y} + \frac{\partial f_z}{\partial z} \]
\[ = \frac{3yz(v - v)}{r^5} = 0 \]
\[ \Rightarrow \nabla \cdot \mathbf{B} = C \nabla \cdot \mathbf{F} = 0 \]

now consider an arbitrary shaped closed surface \( S_2 \) that lies inside the sphere \( S_1 \) and encloses the origin, then
\[ \int_{S_2} \left( \nabla \cdot \mathbf{B} \right) dV = 0 \]
where \( V \) is the volume between the sphere \( S_1 \) and closed surface \( S_2 \)

\[ \Rightarrow \oint_{S_1} \mathbf{B} \cdot d\mathbf{A} + \oint_{S_2} \mathbf{B} \cdot d\mathbf{A} = 0 \]  \hspace{1cm} [ using Divergence theorem ]

\[ \Rightarrow 0 + \oint_{S_2} \mathbf{B} \cdot d\mathbf{A} = 0 \]  \hspace{1cm} [ using (i) ]

\[ \Rightarrow \oint_{S_2} \mathbf{B} \cdot d\mathbf{A} = 0 \]

2 CONCLUSION

Magnetic flux for arbitrary closed surface \( S_2 \) due to a moving charge is zero. Since all magnetic sources consist of moving charges therefore by Superposition principle, Gauss’ law holds for all magnetic sources.
References