Simplest Integrals for the Zeta Function and its Generalizations Valid in All $\mathbb{C}$

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Abstract

Using a different approach, we derive integral representations for the Riemann zeta function and its generalizations (the Hurwitz zeta, $\zeta(-k,b)$, the polylogarithm, $\mathrm{Li}_{-k}(e^m)$, and the Lerch transcendent, $\Phi(e^m,-k,b)$), that coincide with their Abel-Plana expressions. A slight variation of the approach leads to different formulae. We also present the relations between each of these functions and their partial sums. It allows one to figure, for example, the Taylor series expansion of $H_{-k}(n)$ about $n=0$ (when k is a positive integer, we obtain a finite Taylor series, which is nothing but the Faulhaber formula). The method used requires evaluating the limit of $\Phi\left(e^{2\pi \boldsymbol{i}\,x},-2k+1,n+1\right)+\pi\boldsymbol{i}\,x\,\Phi\left(e^{2\pi\boldsymbol{i}\,x},-2k,n+1\right)/k$ when x goes to 0, which in itself already makes for an interesting problem.

Summary

1 Introduction

The formula derivations that follow next are based on two main ideas that were introduced in two previous papers, [2] and [3], respectively. In this paper, we explore those ideas further to see what new results they can give.

The Faulhaber formula is a closed expression for the harmonic numbers, $H_{-k}(n)$, when k is a positive integer. It's the sum of positive integer powers of consecutive integers starting at one, and makes use of the Bernoulli numbers, the numbers that appear in the Taylor series expansion of $x/(e^x - 1)$. For positive odd powers, this expression is:

$$H_{-2k+1}(n) = \sum_{j=1}^{n} j^{2k-1} = \frac{n^{2k-1}}{2} + (2k-1)! \sum_{j=0}^{k-1} \frac{B_{2j} n^{2k-2j}}{(2j)!(2k-2j)!}$$

As everyone may know, the limit of $H_{-k}(n)$ as n approaches infinite is the Riemann zeta function when $\Re(k) < 1$, so these two functions are closely interconnected.

The first aforementioned idea is to use the analytic continuation of the Bernoulli numbers, achievable through the zeta function, to extend the Faulhaber formula. Since,

$$\frac{B_{2j}}{(2j)!} = -2(-1)^j (2\pi)^{-2j} \zeta(2j),$$

we obtain the below modified form:

$$H_{-2k+1}(n) = \frac{n^{2k}}{2k} + \frac{n^{2k-1}}{2} + 2(2k-1)!(2\pi\boldsymbol{i})^{-2k}\zeta(2k) - 2(2k-1)!\, n^{2k} \sum_{j=1}^{k} \frac{(2\pi\boldsymbol{i}\, n)^{-2j}\zeta(2j)}{(2k-2j)!}$$

The whole focus is then on how one can find ways to obtain a closed-form for the key sum below:

$$\sum_{j=1}^{k} \frac{(2\pi\boldsymbol{i}\, n)^{-2j}\zeta(2j)}{(2k-2j)!} \tag{1}$$

The second idea, already explained in [3], is the following straightforward identity, which only works for the analytic continuation of the Lerch Φ function at the negative integers:

$$\Phi(e^b, -k, u+v) = k! \sum_{j=0}^{k} \frac{\Phi\left(e^b, -j, v\right) u^{k-j}}{j!(k-j)!} \tag{2}$$

A special case of this identity is the sum of polylogs, when $v = 1$:

$$e^m \, \Phi(e^m, -k, b+1) = k! \sum_{j=0}^{k} \frac{\mathrm{Li}_{-j}\,(e^m)\, b^{k-j}}{j!(k-j)!} \tag{3}$$

Since $e^b = 1$ leads to singularities, the analogous expression for the Hurwitz zeta function at the negative integers was made possible through some relations available in the literature, as explained in [3], and is slightly different:

$$\zeta(-k, u+v) = -\frac{u^{k+1}}{k+1} + k! \sum_{j=0}^{k} \frac{\zeta(-j, v)\, u^{k-j}}{j!(k-j)!}, \tag{4}$$

2 Riemann zeta function

Leveraging the two ideas discussed, we start with an optimal Riemann zeta function and then from it we obtain the Hurwitz zeta, $\zeta(-k, b)$, and the polylogarithm, $\mathrm{Li}_{-k}(e^m)$. And then from the polylogarithm, we obtain the most convoluted one, the Lerch Φ function, $\Phi(e^m, -k, b)$.

2.1 Integral from the literature

First, let's see how we can derive a different expression for $H_{-k}(n)$ than the one we created previously in [2], using a different integral for the zeta function.

We need to obtain a closed-form for (1). If $\Re(k) > 1$, the zeta function integral representation from the literature is:

$$\zeta(k) = \frac{1}{(k-1)!} \int_0^\infty \frac{x^{k-1}}{e^x - 1}\, dx = \frac{1}{(k-1)!} \int_0^1 \frac{(-\log u)^{k-1}}{1-u}\, du \tag{5}$$

Therefore, using equation (5), we have:

$$\sum_{j=1}^{k} \frac{(2\pi \boldsymbol{i}\, n)^{-2j} \zeta(2j)}{(2k-2j)!} = \int_0^1 \sum_{j=1}^{k} \frac{(2\pi \boldsymbol{i}\, n)^{-2j}}{(2k-2j)!} \frac{(-\log u)^{2j-1}}{(2j-1)!(1-u)}\, du$$

Replacing the sum inside the integral with a closed-form, we obtain its analytic continuation, which this time is slightly simpler than before:

$$\begin{aligned}\sum_{j=1}^{k} \frac{(2\pi \boldsymbol{i}\, n)^{-2j} \zeta(2j)}{(2k-2j)!} &= -\frac{(2\pi \boldsymbol{i}\, n)^{-2k}}{2(2k-1)!} \int_0^1 \frac{(2\pi \boldsymbol{i}\, n + \log u)^{2k-1} + (-2\pi \boldsymbol{i}\, n + \log u)^{2k-1}}{1-u}\, du \\ &= \frac{(2\pi \boldsymbol{i}\, n)^{-2k}}{2(2k-1)!} \int_0^\infty \frac{(2\pi \boldsymbol{i}\, n + x)^{2k-1} + (-2\pi \boldsymbol{i}\, n + x)^{2k-1}}{e^x - 1}\, dx\end{aligned}$$

Now, repeating the same process outlined in [2], and making a change of variables, we obtain:

$$\sum_{j=1}^{n} j^k = \frac{n^{k+1}}{k+1} + \frac{n^k}{2} + \zeta(-k) - n^{k+1} \int_0^{\pi/2} (1 - \coth(\pi\, n \tan v)) \frac{\sin k\, v}{(\cos v)^{k+2}} \, dv, \tag{6}$$

which is a bit simpler than the former formula:

$$\sum_{j=1}^{n} j^k = \frac{n^{k+1}}{k+1} + \frac{n^k}{2} + \zeta(-k) + \frac{\pi\, n^{k+2}}{k+1} \int_0^{\pi/2} (\sec v \operatorname{csch}(\pi\, n \tan v))^2 \left(1 - \frac{\cos(k+1)\, v}{(\cos v)^{k+1}}\right) dv \tag{7}$$

As we can see, these formulae are by no means unique. None of these two forms allows one to calculate the Taylor series expansion of $H_{-k}(n)$ about $n = 0$, if $\Re(k) < 0$. They don't get the analytic continuation of $H_{-k}(n)$ over n right either, if $\Re(n) \le 0$ (but over k they always do).

2.2 A simpler expression for $\zeta(k)$

The integral representations we used for the zeta function in (2.1) tend to make the final formulae a bit complicated, so our objective is to try and see if we can find a simpler expression for (1). But in order to do that, first we need to find a simpler expression for $\zeta(k)$.

From the literature, we know a Taylor series expansion for $x \cot x$, whose k-th derivatives give the Bernoulli numbers, and hence also the zeta function at the even integers:

$$x \cot x = \sum_{k=0}^{\infty} \frac{(-1)^k B_{2k} (2x)^{2k}}{(2k)!}, \text{ if } |x| < \pi$$

But we don't need to use this function, a better option is the generating function of the zeta function at the even integers:

$$\sum_{k=0}^{\infty} \zeta(2k) x^{2k} = -\frac{\pi x \cot \pi x}{2} \tag{8}$$

2.3 Derivatives of trigonometric functions

In [3], we created an expression for the k-th derivatives of the cotangent, so it comes in handy now. Below $\delta_{0\,k} = 1$ iff $k = 0$:

$$\frac{\mathrm{d}^k (\cot a\, x)}{\mathrm{d}\, x^k} = -\boldsymbol{i}\, \delta_{0\,k} - 2\, \boldsymbol{i} (2\, \boldsymbol{i}\, a)^k \operatorname{Li}_{-k}\left(e^{2\, \boldsymbol{i}\, a\, x}\right) \tag{9}$$

For the record, the expressions for the derivatives of the other trigonometric functions are (the translated arc formulae allow one type to converted into another):

$$\frac{\mathrm{d}^k}{\mathrm{d}\, x^k} \left(\frac{1}{\sin a\, x}\right) = -2\, \boldsymbol{i} (2\, \boldsymbol{i}\, a)^k e^{\boldsymbol{i}\, a\, x}\, \Phi\left(e^{2\, \boldsymbol{i}\, a\, x}, -k, \frac{1}{2}\right)$$

$$\frac{\mathrm{d}^k\left(\tan\left(a\,x+b\right)\right)}{\mathrm{d}\,x^k} = \boldsymbol{i}\,\delta_{0\,k} + 2\,\boldsymbol{i}(2\,\boldsymbol{i}\,a)^k\,\mathrm{Li}_{-k}\left(-e^{2\,\boldsymbol{i}(a\,x+b)}\right)$$

$$\frac{\mathrm{d}^k}{\mathrm{d}\,x^k}\left(\frac{1}{\cos\left(a\,x+b\right)}\right) = 2\,\boldsymbol{i}(2\,\boldsymbol{i}\,a)^k\,e^{\boldsymbol{i}(a\,x+b)}\,\Phi\left(-e^{2\,\boldsymbol{i}(a\,x+b)},-k,\frac{1}{2}\right)$$

The problem with equation (9) is the fact it's improper at $x=0$, but from (8) we know the limits exist, so we need to differentiate $x\cot\pi x$, using the Leibniz rule, and take the limit as x goes to 0.

The Leibniz rule tells that the k-th derivative of $xf(x)$ is $kf^{(k-1)}(x)+f^{(k)}(x)$, therefore, for k a non-negative integer, one can write the zeta function at the even integers as:

$$\lim_{x\to 0} -\frac{\pi}{2\,k!}\left(k\left(-\boldsymbol{i}\,\delta_{0\,k-1} - 2\,\boldsymbol{i}(2\,\pi\boldsymbol{i})^{k-1}\,\mathrm{Li}_{-k+1}\left(e^{2\,\pi\boldsymbol{i}\,x}\right)\right) + x\left(-\boldsymbol{i}\,\delta_{0\,k} - 2\,\boldsymbol{i}(2\,\pi\boldsymbol{i})^k\,\mathrm{Li}_{-k}\left(e^{2\,\pi\boldsymbol{i}\,x}\right)\right)\right) = \begin{cases}\zeta(k), & \text{if } k \text{ is even}\\ 0, & \text{if } k \text{ is odd}\end{cases}$$

We therefore have (note $\mathbb{1}_{2|j}=1$ if j is even, 0 otherwise):

$$\sum_{j=0}^{k}\frac{(2\pi\boldsymbol{i}\,n)^{-2j}\zeta(2j)}{(2k-2j)!} = \sum_{j=0}^{2k}\frac{\mathbb{1}_{2|j}(2\pi\boldsymbol{i}\,n)^{-j}\zeta(j)}{(2k-j)!}$$
$$= \lim_{x\to 0} -\frac{\pi}{2}\sum_{j=0}^{2k}\frac{(2\pi\boldsymbol{i}\,n)^{-j}}{(2k-j)!j!}\left(j\left(-\boldsymbol{i}\,\delta_{0\,j-1} - 2\boldsymbol{i}(2\pi\boldsymbol{i})^{j-1}\,\mathrm{Li}_{-j+1}\left(e^{2\pi\boldsymbol{i}\,x}\right)\right) + x\left(-\boldsymbol{i}\,\delta_{0\,j} - 2\boldsymbol{i}(2\pi\boldsymbol{i})^j\,\mathrm{Li}_{-j}\left(e^{2\pi\boldsymbol{i}\,x}\right)\right)\right)$$

Simplifying the calculation (keeping just the right-hand side of the equation, the limit):

$$\lim_{x\to 0}\frac{1}{4n(2k-1)!} + \frac{\pi\boldsymbol{i}\,x}{2(2k)!} + \pi\boldsymbol{i}\sum_{j=0}^{2k}\frac{(2\pi\boldsymbol{i}\,n)^{-j}}{(2k-j)!j!}\left(j\,(2\pi\boldsymbol{i})^{j-1}\,\mathrm{Li}_{-j+1}\left(e^{2\pi\boldsymbol{i}\,x}\right) + x\,(2\pi\boldsymbol{i})^j\,\mathrm{Li}_{-j}\left(e^{2\pi\boldsymbol{i}\,x}\right)\right)$$

Simplifying a little more:

$$\lim_{x\to 0}\frac{1}{4n(2k-1)!} + \frac{\pi\boldsymbol{i}\,x}{2(2k)!} + \frac{1}{2}\sum_{j=1}^{2k}\frac{n^{-j}\,\mathrm{Li}_{-j+1}\left(e^{2\pi\boldsymbol{i}\,x}\right)}{(2k-j)!(j-1)!} + \pi\boldsymbol{i}\,x\sum_{j=0}^{2k}\frac{n^{-j}\,\mathrm{Li}_{-j}\left(e^{2\pi\boldsymbol{i}\,x}\right)}{(2k-j)!j!} \tag{10}$$

2.4 A limit involving the Lerch Φ

Now, applying the identity (3) to (10), one deduces the following:

$$\lim_{x\to 0}\frac{1}{4n(2k-1)!} + \frac{\pi\boldsymbol{i}\,x}{2(2k)!} + \frac{n^{-2k}e^{2\pi\boldsymbol{i}\,x}}{2(2k-1)!}\Phi\left(e^{2\pi\boldsymbol{i}\,x},-2k+1,n+1\right) + \pi\boldsymbol{i}\,x\frac{n^{-2k}e^{2\pi\boldsymbol{i}\,x}}{(2k)!}\Phi\left(e^{2\pi\boldsymbol{i}\,x},-2k,n+1\right)$$

This means that to figure out the sum, we need to figure out the limit:

$$\lim_{x\to 0}\frac{1}{4n(2k-1)!} + \frac{n^{-2k}}{2(2k-1)!}\left(\Phi\left(e^{2\pi\boldsymbol{i}\,x},-2k+1,n+1\right) + \frac{\pi\boldsymbol{i}\,x}{k}\Phi\left(e^{2\pi\boldsymbol{i}\,x},-2k,n+1\right)\right) \tag{11}$$

At this point, we might think it'd help to use the closed-form of the Lerch Φ at the negative integers from reference [3],

$$\Phi(e^{2\pi \boldsymbol{i}\, x}, -k,\, n+1) = -\frac{1}{e^{2\pi \boldsymbol{i}\, x}-1}\sum_{q=0}^{k} q!\left(\frac{e^{2\pi \boldsymbol{i}\, x}}{e^{2\pi \boldsymbol{i}\, x}-1}\right)^{q}\sum_{j=0}^{q}\frac{(-1)^j (j+n+1)^k}{j!(q-j)!}, \tag{12}$$

but it's actually very cumbersome in this case. It's not hard to approximate $1/(e^{2\pi \boldsymbol{i}\, x}-1)$ or $1/(e^{-2\pi \boldsymbol{i}\, x}-1)$ when x is small, but their powers have patterns that are hard to figure (though not impossible).

The right way to go is to use an integral for the Lerch Φ that holds at the negative integers, which is explained in section (6.3). This integral is this simple formula here (which should hold for every integer k):

$$\begin{aligned}\Phi(e^{2\pi \boldsymbol{i}\, x}, -k,\, n+1) &= \frac{(n+1)^k}{2} + (-2\pi \boldsymbol{i}\, x)^{-k-1}\, e^{-2\pi \boldsymbol{i}\, x(n+1)}\,\Gamma\left(k+1, -2\pi \boldsymbol{i}\, x(n+1)\right)\\ &+ \int_0^{\pi/2}\frac{1-\coth\left(\pi\, n\tan v\right)}{(\cos v)^2}\left((n+1)^2+(\tan v)^2\right)^{k/2}\sin\left(k\arctan\frac{\tan v}{n+1}+2\pi \boldsymbol{i}\, x\tan v\right)dv\end{aligned}$$

It turns out that the limit in (11) comes down to the incomplete gamma function in this formula. That is, for the other parts, we can simply evaluate the expression at $x=0$ (it's not improper anymore).

Evaluated at $x=0$, the integral that doesn't vanish becomes:

$$\begin{aligned}&\int_0^{\pi/2}\frac{1-\coth\left(\pi\tan v\right)}{(\cos v)^2}\left((n+1)^2+(\tan v)^2\right)^{(2k-1)/2}\sin\left((2k-1)\arctan\frac{\tan v}{n+1}\right)dv\\ &\quad= -\frac{\boldsymbol{i}}{2}\int_0^{\pi/2}\frac{1-\coth\left(\pi\tan v\right)}{(\cos v)^2}\left((n+1+\boldsymbol{i}\tan v)^{2k-1}-(n+1-\boldsymbol{i}\tan v)^{2k-1}\right)dv\\ &\qquad= -\frac{\boldsymbol{i}}{2}\int_0^{\infty}(1-\coth\pi x)\left((n+1+\boldsymbol{i}\, x)^{2k-1}-(n+1-\boldsymbol{i}\, x)^{2k-1}\right)dx\end{aligned}$$

As for the limit of the difference between the gamma functions, we have:

$$\lim_{x\to 0}\frac{e^{-2\pi \boldsymbol{i}\, x(n+1)}}{(2\pi \boldsymbol{i}\, x)^{2k}}\left(\Gamma\left(2k, -2\pi \boldsymbol{i}\, x(n+1)\right)-\frac{1}{2k}\Gamma\left(2k+1, -2\pi \boldsymbol{i}\, x(n+1)\right)\right) = -\frac{(n+1)^{2k}}{2k}$$

Therefore, the limit we're looking for is:

$$\begin{aligned}\lim_{x\to 0}\Phi\left(e^{2\pi \boldsymbol{i}\, x}, -2k+1, n+1\right)+\frac{\pi \boldsymbol{i}\, x}{k}\Phi\left(e^{2\pi \boldsymbol{i}\, x}, -2k, n+1\right) &= \frac{(n+1)^{2k-1}}{2}-\frac{(n+1)^{2k}}{2k}\\ &-\frac{\boldsymbol{i}}{2}\int_0^{\infty}(1-\coth\pi x)\left((n+1+\boldsymbol{i}\, x)^{2k-1}-(n+1-\boldsymbol{i}\, x)^{2k-1}\right)dx\end{aligned} \tag{13}$$

To summarize the results, we now have a closed formula:

$$\sum_{j=0}^{k} \frac{(2\pi\boldsymbol{i}\,n)^{-2j}\zeta(2j)}{(2k-2j)!} = \frac{1}{4n(2k-1)!} + \frac{1}{4(2k-1)!}\left(\frac{1}{n+1} - \frac{1}{k}\right)\left(1+\frac{1}{n}\right)^{2k} - \frac{\boldsymbol{i}\,n^{-2k}}{4(2k-1)!}\int_0^\infty (1-\coth\pi x)\left((n+1+\boldsymbol{i}\,x)^{2k-1} - (n+1-\boldsymbol{i}\,x)^{2k-1}\right)\,dx \quad (14)$$

2.5 Zeta relation to partial sums

Going back to the Faulhaber formula:

$$H_{-2k+1}(n) = \frac{n^{2k-1}}{2} + 2(2k-1)!(2\pi\boldsymbol{i})^{-2k}\zeta(2k) - 2(2k-1)!\,n^{2k}\sum_{j=0}^{k}\frac{(2\pi\boldsymbol{i}\,n)^{-2j}\zeta(2j)}{(2k-2j)!}$$

Replacing (14) into the above and simplifying, we have:

$$H_{-2k+1}(n) = -\frac{(n+1)^{2k-1}}{2} + \frac{(n+1)^{2k}}{2k} + 2(2k-1)!(2\pi\boldsymbol{i})^{-2k}\zeta(2k) + \frac{\boldsymbol{i}}{2}\int_0^\infty (1-\coth\pi x)\left((n+1+\boldsymbol{i}\,x)^{2k-1} - (n+1-\boldsymbol{i}\,x)^{2k-1}\right)\,dx$$

As mentioned in [2], for this formula to hold for every k, $\boldsymbol{i}^{-2k}$ needs to be replaced with $\cos k\pi$ (note it's the real part of $(-1)^{-k}$). By doing that and also making $2k-1$ into k, we obtain:

$$H_{-k}(n) = \frac{(n+1)^{k+1}}{k+1} - \frac{(n+1)^k}{2} + 2\,k!(2\pi)^{-k-1}\cos\frac{(k+1)\pi}{2}\zeta(k+1) + \frac{\boldsymbol{i}}{2}\int_0^\infty (1-\coth\pi x)\left((n+1+\boldsymbol{i}\,x)^{k} - (n+1-\boldsymbol{i}\,x)^{k}\right)\,dx$$

Finally, since the equation below,

$$2\,\Gamma(k+1)(2\pi)^{-k-1}\cos\frac{(k+1)\pi}{2}\zeta(k+1) = \zeta(-k),$$

is the Riemann functional equation, the final formula is:

$$H_{-k}(n) = \frac{(n+1)^{k+1}}{k+1} - \frac{(n+1)^k}{2} + \zeta(-k) + \frac{\boldsymbol{i}}{2}\int_0^\infty (1-\coth\pi x)\left((n+1+\boldsymbol{i}\,x)^{k} - (n+1-\boldsymbol{i}\,x)^{k}\right)\,dx \quad (15)$$

One of the advantages of this formula, over (6) and (7), is that it allows one to obtain the partial Taylor series expansion of $H_{-k}(n)$ about $n=0$, even when $\Re(k)<0$. When k is a positive integer, the series expansion about $n = 0$ gives the Faulhaber formula, like (6) and (7).

Relation (15) provides the analytic continuation of $H_{-k}(n)$ to the whole complex plane on both parameters, k and n:

$$\sum_{q=1}^{n} q^k = \zeta(-k) - \zeta(-k, n+1) \tag{16}$$

One exception is $n = -1$ if $\Re(k) < 0$, but there may be other possible singularities.

2.6 $\zeta(k)$ formula

Using equation (15) with $n = 0$ we obtain an integral representation for the Riemann zeta function valid in the whole complex plane, except its pole:

$$\zeta(k) = \frac{1}{k-1} + \frac{1}{2} - \frac{\boldsymbol{i}}{2} \int_0^{\infty} (1 - \coth \pi x) \left((1 + \boldsymbol{i}\, x)^{-k} - (1 - \boldsymbol{i}\, x)^{-k} \right) dx \tag{17}$$

The above formula can be used to easily figure out the derivatives of the zeta function at 0. If q is a positive integer:

$$\left. \frac{\mathrm{d}^q\, \zeta(x)}{\mathrm{d}\, x^q} \right|_{x=0} = -q! - \frac{\boldsymbol{i}(-1)^q}{2} \int_0^{\infty} (1 - \coth \pi x) \left(\log^q(1 + \boldsymbol{i}\, x) - \log^q(1 - \boldsymbol{i}\, x) \right) dx$$

3 Hurwitz zeta function

Let's use (15) to demonstrate how to obtain a similar relation for the Hurwitz zeta and the generalized harmonic progressions.

When $v = 1$, we have a special case of the identity (4), which gives:

$$\zeta(-k, b+1) = -\frac{b^{k+1}}{k+1} + k! \sum_{j=0}^{k} \frac{\zeta(-j)\, b^{k-j}}{j!(k-j)!}, \tag{18}$$

3.1 Hurwitz relation to partial sums

First, if we recall the expression in (15), we can change k for j:

$$\sum_{q=1}^{n} q^j = \frac{(n+1)^{j+1}}{j+1} - \frac{(n+1)^j}{2} + \zeta(-j) + \frac{\boldsymbol{i}}{2} \int_0^{\infty} (1 - \coth \pi x) \left((n + 1 + \boldsymbol{i}\, x)^j - (n + 1 - \boldsymbol{i}\, x)^j \right) dx \tag{19}$$

We sum the above over j per identity (18):

$$\sum_{q=1}^{n}\sum_{j=0}^{k}\frac{k!\,q^j\,b^{k-j}}{j!(k-j)!}=\sum_{j=0}^{k}\frac{k!\,b^{k-j}}{j!(k-j)!}\frac{(n+1)^{j+1}}{j+1}-\frac{1}{2}\sum_{j=0}^{k}\frac{k!\,(n+1)^j\,b^{k-j}}{j!(k-j)!}+k!\sum_{j=0}^{k}\frac{\zeta(-j)\,b^{k-j}}{j!(k-j)!}$$
$$+\frac{\boldsymbol{i}}{2}\int_0^\infty(1-\coth\pi x)\sum_{j=0}^{k}\frac{k!\,b^{k-j}}{j!(k-j)!}\left((n+1+\boldsymbol{i}\,x)^j-(n+1-\boldsymbol{i}\,x)^j\right)\,dx$$

Now, simplifying with the Newton's binomial,

$$\sum_{q=1}^{n}(q+b)^k=\frac{-b^{k+1}+(n+1+b)^{k+1}}{k+1}-\frac{(n+1+b)^k}{2}+\frac{b^{k+1}}{k+1}+\zeta(-k,b+1)$$
$$+\frac{\boldsymbol{i}}{2}\int_0^\infty(1-\coth\pi x)\left((n+1+b+\boldsymbol{i}\,x)^k-(n+1+b-\boldsymbol{i}\,x)^k\right)\,dx,$$

we finally find that for all complex $k\neq-1$:

$$\sum_{q=0}^{n}(q+b)^k=\frac{(n+1+b)^{k+1}}{k+1}-\frac{(n+1+b)^k}{2}+\zeta(-k,b)$$
$$+\frac{\boldsymbol{i}}{2}\int_0^\infty(1-\coth\pi x)\left((n+1+b+\boldsymbol{i}\,x)^k-(n+1+b-\boldsymbol{i}\,x)^k\right)\,dx \tag{20}$$

Aside from any singularities, relation (20) provides the analytic continuation of the sum on the left-hand side to the whole complex plane on all parameters, k, b and n:

$$\sum_{q=0}^{n}(q+b)^k=\zeta(-k,b)-\zeta(-k,n+1+b) \tag{21}$$

3.2 $\zeta(-k,b)$ formula

The easiest way to derive a formula for $\zeta(-k,b)$ is simply to set $n=-1$ in relation (20):

$$\zeta(-k,b)=-\frac{b^{k+1}}{k+1}+\frac{b^k}{2}-\frac{\boldsymbol{i}}{2}\int_0^\infty(1-\coth\pi x)\left((b+\boldsymbol{i}\,x)^k-(b-\boldsymbol{i}\,x)^k\right)\,dx \tag{22}$$

This formula is valid in the whole complex plane (except $k=-1$, or $b=0$ if $\Re(k)<0$). It could also be derived using equations (15) and (16).

4 The polylogarithm

The insight on how to go about deriving this next formula comes from noticing the patterns in these formulas so far. When a transformation was applied to the partial sums of the zeta function, like,

$$\sum_{j=0}^{k}\frac{k!\,b^{k-j}}{j!(k-j)!}\left(\sum_{q=1}^{n}q^j\right),\text{ we obtained }\sum_{q=1}^{n}(q+b)^k,$$

which are the partial sums of the Hurwitz zeta.

So, the next transformation we need in order to obtain the partial sums of the polylogarithm function is:

$$\sum_{q=1}^{n} q^k e^{m\,q} = \sum_{q=1}^{n} q^k \sum_{j=0}^{\infty} \frac{(m\,q)^j}{j!} = \sum_{j=0}^{\infty} \frac{m^j}{j!} \sum_{q=1}^{n} q^{j+k} = \sum_{j=k}^{\infty} \frac{m^{j-k}}{(j-k)!} \sum_{q=1}^{n} q^j$$

We need to apply this transformation to each piece of (19). Let's do it by parts, but first let's get acquainted with the function and its integral. If k is a non-negative integer:

$$\sum_{j=k}^{\infty} \frac{x^j}{(j-k)!} = x^k\, e^x, \text{ and } \sum_{j=k}^{\infty} \frac{x^{j+1}}{(j+1)(j-k)!} = \int_0^x v^k\, e^v\, dv = (-1)^k \left(\Gamma(k+1, -x) - k!\right)$$

Therefore, the first part is:

$$-\frac{1}{2} \sum_{j=k}^{\infty} \frac{m^{j-k}}{(j-k)!} \frac{(n+1)^{j+1}}{j+1} = -(-m)^{-k-1} \left(\Gamma(k+1, -m(n+1)) - k!\right)$$

And the second part is:

$$-\frac{1}{2} \sum_{j=k}^{\infty} \frac{m^{j-k}(n+1)^j}{(j-k)!} = -\frac{(n+1)^k e^{m(n+1)}}{2}$$

The next part deserves an entire section.

4.1 Zeta at the negative integers

For k a non-negative integer, we want a closed-form for the following power series:

$$\sum_{j=k}^{\infty} \frac{m^{j-k}}{(j-k)!} \zeta(-j)$$

One can, therefore, start from the generating function of the zeta function at the negative integers, whose k-th derivative gives the above:

$$\sum_{j=0}^{\infty} \frac{x^j}{j!} \zeta(-j) = -\frac{1}{2} + \sum_{j=1}^{\infty} \frac{x^{2j-1}}{(2j-1)!} \zeta(-2j+1) = -\frac{1}{2} - \sum_{j=1}^{\infty} \frac{B_{2j}\, x^{2j-1}}{(2j)!} = -\frac{1}{2} + \frac{1}{x} - \frac{1}{2} \coth \frac{x}{2}$$

The derivatives of the hyperbolic cotangent can be calculated with the formulae seen in section (2.3):

$$\frac{\mathrm{d}^q}{\mathrm{d}\, x^q} \left(-\frac{1}{2} + \frac{1}{x} - \frac{1}{2} \coth \frac{x}{2} \right) \Bigg|_{x=0} = (-1)^k k!\, m^{-k-1} - \delta_{0\,k} - (-1)^k \operatorname{Li}_{-k}\left(e^{-m}\right)$$

The Kronecker delta is a problem, but fortunately we have the following equivalence:

$$-\delta_{0\,k} - (-1)^k \operatorname{Li}_{-k}\left(e^{-m}\right) = \operatorname{Li}_{-k}\left(e^{m}\right)$$

4.2 Polylogarithm relation to partial sums

The last part is simple:

$$\frac{\boldsymbol{i}}{2}\int_0^\infty (1-\coth\pi x)\sum_{j=k}^{\infty}\frac{m^{j-k}}{(j-k)!}\left((n+1+\boldsymbol{i}\,x)^j-(n+1-\boldsymbol{i}\,x)^j\right)\,dx$$
$$=\frac{\boldsymbol{i}\,e^{m(n+1)}}{2}\int_0^\infty (1-\coth\pi x)\left(e^{m\,\boldsymbol{i}\,x}(n+1+\boldsymbol{i}\,x)^k-e^{-m\,\boldsymbol{i}\,x}(n+1-\boldsymbol{i}\,x)^k\right)\,dx$$

When everything is put together, we obtain:

$$\sum_{q=1}^{n} q^k e^{m\,q} = -\frac{(n+1)^k e^{m(n+1)}}{2} - (-m)^{-k-1}\Gamma(k+1,-m(n+1)) + \mathrm{Li}_{-k}\left(e^m\right)$$
$$+\frac{\boldsymbol{i}\,e^{m(n+1)}}{2}\int_0^\infty (1-\coth\pi x)\left(e^{m\,\boldsymbol{i}\,x}(n+1+\boldsymbol{i}\,x)^k-e^{-m\,\boldsymbol{i}\,x}(n+1-\boldsymbol{i}\,x)^k\right)\,dx \quad (23)$$

This relation provides the analytic continuation of the sum on the left-hand side to the whole complex plane on all parameters, k, m and n:

$$\sum_{q=1}^{n} q^k e^{m\,q} = e^{m(n+1)}\Phi\left(e^m,-k,n+1\right) + \mathrm{Li}_{-k}\left(e^m\right) \quad (24)$$

4.3 $\mathrm{Li}_{-k}\left(e^m\right)$ formula

The simplest formula for $\mathrm{Li}_{-k}\left(e^m\right)$ is obtained by setting $n=0$ in relation (23):

$$\mathrm{Li}_{-k}\left(e^m\right) = \frac{e^m}{2} + (-m)^{-k-1}\Gamma(k+1,-m)$$
$$-\frac{\boldsymbol{i}\,e^m}{2}\int_0^\infty (1-\coth\pi x)\left(e^{m\,\boldsymbol{i}\,x}(1+\boldsymbol{i}\,x)^k-e^{-m\,\boldsymbol{i}\,x}(1-\boldsymbol{i}\,x)^k\right)\,dx, \quad (25)$$

which should be valid in the whole complex plane, except when $e^m=1$.

5 Lerch Φ function

For the Lerch Φ function the process is the very same, but this time we use identity (3), instead of (4).

5.1 Lerch Φ relation to partial sums

Summing both sides of equation (23) (with k replaced by j), over j per the transformation (3):

$$\sum_{q=1}^{n} e^{m\,q} \sum_{j=0}^{k} \frac{k!\,q^j\,b^{k-j}}{j!(k-j)!} = -\frac{e^{m(n+1)}}{2} \sum_{j=0}^{k} \frac{k!\,(n+1)^j\,b^{k-j}}{j!(k-j)!}$$
$$- \sum_{j=0}^{k} \frac{k!\,b^{k-j}}{j!(k-j)!} (-m)^{-j-1} \Gamma(j+1, -m(n+1)) + \sum_{j=0}^{k} \frac{k!\,\mathrm{Li}_{-j}\left(e^m\right)\,b^{k-j}}{j!(k-j)!}$$
$$+ \frac{\boldsymbol{i}\,e^{m(n+1)}}{2} \int_0^\infty (1-\coth \pi x) \sum_{j=0}^{k} \frac{k!\,b^{k-j}}{j!(k-j)!} \left(e^{m\,\boldsymbol{i}\,x}(n+1+\boldsymbol{i}\,x)^j - e^{-m\,\boldsymbol{i}\,x}(n+1-\boldsymbol{i}\,x)^j\right)\,dx,$$

one concludes that:

$$\sum_{q=0}^{n} (q+b)^k\,e^{m\,q} = -\frac{(n+1+b)^k e^{m(n+1)}}{2} - (-m)^{-k-1} e^{-m\,b}\,\Gamma(k+1, -m(n+1+b)) + \Phi(e^m, -k, b)$$
$$+ \frac{\boldsymbol{i}\,e^{m(n+1)}}{2} \int_0^\infty (1-\coth \pi x) \left(e^{m\,\boldsymbol{i}\,x}(n+1+b+\boldsymbol{i}\,x)^k - e^{-m\,\boldsymbol{i}\,x}(n+1+b-\boldsymbol{i}\,x)^k\right)\,dx \quad (26)$$

Aside from singularities, relation (26) provides the analytic continuation of the sum on the left-hand side to the whole complex plane on all parameters, k, m, b and n:

$$\sum_{q=0}^{n} (q+b)^k\,e^{m\,q} = \Phi(e^m, -k, b) - e^{m(n+1)}\,\Phi(e^m, -k, n+1+b) \quad (27)$$

5.2 $\Phi(e^m, -k, b)$ formula

Making $n = -1$ in relation (26) is the easiest way to derive a Lerch Φ formula valid in the whole complex plane (except occasional singularities):

$$\Phi(e^m, -k, b) = \frac{b^k}{2} + (-m)^{-k-1} e^{-m\,b}\,\Gamma(k+1, -m\,b)$$
$$- \frac{\boldsymbol{i}}{2} \int_0^\infty (1-\coth \pi x) \left(e^{m\,\boldsymbol{i}\,x}(b+\boldsymbol{i}\,x)^k - e^{-m\,\boldsymbol{i}\,x}(b-\boldsymbol{i}\,x)^k\right)\,dx \quad (28)$$

6 Different formulae

These ones may not hold everywhere (except for the parameter k). The Lerch Φ formula used in section (2.4) came from this method.

6.1 Hurwitz zeta function

If we recall the expression in (6), we can use it and repeat the steps outlined previously. After all is put together, we find that for all complex $k \neq -1$:

$$\sum_{q=0}^{n}(q+b)^k = \frac{(n+b)^{k+1}}{k+1} + \frac{(n+b)^k}{2} + \zeta(-k,b)$$
$$- n\int_0^{\pi/2} \frac{1-\coth\left(\pi\, n\tan v\right)}{(\cos v)^2}\left((n+b)^2+(n\tan v)^2\right)^{k/2}\sin\left(k\arctan\frac{n\tan v}{n+b}\right)dv,$$

It's not possible to turn this integral into the one from equation (20), and the reason is cause two of the terms outside of the integral differ between the two formulas.

6.2 The Polylogarithm

Using equation (6), this is the polylogarithm formula we end up with:

$$\sum_{q=1}^{n} q^k e^{m\,q} = \frac{n^k\, e^{m\,n}}{2} - (-m)^{-k-1}\Gamma(k+1,-m\,n) + \mathrm{Li}_{-k}\left(e^m\right)$$
$$- n^{k+1}e^{m\,n}\int_0^{\pi/2}\left(1-\coth\left(\pi\, n\tan v\right)\right)\frac{\sin\left(k\,v+m\,n\tan v\right)}{(\cos v)^{k+2}}\,dv$$

6.3 Lerch Φ function

Using the polylogarithm relation from the previous section we obtain:

$$\sum_{q=0}^{n}(q+b)^k e^{m\,q} = \frac{(n+b)^k\, e^{m\,n}}{2} - (-m)^{-k-1}e^{-m\,b}\,\Gamma(k+1,-m(n+b)) + \Phi\left(e^m,-k,b\right)$$
$$-n\,e^{m\,n}\int_0^{\pi/2} \frac{1-\coth\left(\pi\, n\tan v\right)}{(\cos v)^2}\left((n+b)^2+(n\ \tan v)^2\right)^{k/2}\sin\left(k\arctan\frac{n\tan v}{n+b}+m\,n\tan v\right)dv$$

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