Approximating roots and π with Pythagorean triples

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Abstract

There are several methods for approximating roots using recursive sequences. Algorithms for choosing the first term of such sequences are of interest. Section 1 will introduce an option, based upon the number of digits of the radicand, for selecting the first term. This first term will then be used in a traditional recursive sequence used to approximate roots. Section 2 will apply the method shown in Section 1 to approximate π using Archimedes' method, which then no longer requires a separate algorithm for roots. Section 3 will introduce new recursive sequences for approximating roots using Pythagorean triples. Section 4 will then use the same new method to approximate π.

1 Seed value based on digits

It is well known that if the first approximation is x_0, and $x_\infty = \sqrt{S_d}$, then the recursive sequence 1 will converge to the square root of S_d.

$$x_n = x_{n-1} + \frac{S_d - x_{n-1}^2}{2x_{n-1}} \tag{1}$$

How fast sequence 1 converges depends upon the choice of x_0. Equation 2 closely approximates $x = \sqrt{S_d}$. **It is important to note that floor($\log S_d$), or $\lfloor \log S_d \rfloor$, is simply one less than the number of digits in S_d. This may be important if using this formula in programming, or if doing manual calculations**.

$$x = 1.65^{2.3\lfloor \log S_d \rfloor} \tag{2}$$

The divergence is slow enough that the relative error between $\sqrt{10^{200}}$ and $1.65^{2.3\lfloor \log\left(10^{200}\right)\rfloor}$ $\left(1.65^{2.3\lfloor 200 \rfloor} = 1.65^{460}\right)$ is less than 11%. Next, the exponent is rounded up using the ceiling function so as to provide an integer exponent. The first approximation for sequence 1 is shown in equation 3.

$$x_0 = 1.65^{\lceil 2.3\lfloor \log S_d \rfloor \rceil} \tag{3}$$

No relative error greater than 76% has been found when using equation 3 over the interval from $1 - 10^{200}$. "Near" zero, the maximum errors occur at values just below integer powers of ten. For example, 997 is just below $10^3 = 1000$. The initial seed (to 10 decimal places) for $\sqrt{997}$ would be $1.65^{\lceil 2.3\lfloor \log 997 \rfloor \rceil} = 1.65^5 = 12.2298103125$, when the true value is ≈ 31.5753. Even using this seed value, which has the largest known error, only 6 iterations of sequence 1 are necessary to achieve accuracy to 12 decimal places. Note that at least 5 more digits of accuracy are added with each iteration. Table 1 shows the iteration results for approximating $\sqrt{2}$ and $\sqrt{997}$ to 12 decimal places of accuracy.

Table 1: Demonstration of root approximation

	$\sqrt{2}$	$\sqrt{997}$
True:	1.414213562373	31.575306807694
$x_0 =$	1	12.229810312500
$x_1 =$	1.5	46.875962544891
$x_2 =$	1.416666666666	34.072429568256
$x_3 =$	1.414215686275	31.666812200182
$x_4 =$	1.414213562375	31.575439016089
$x_5 =$	1.414213562373	31.575306807971
$x_6 =$	1.414213562373	31.575306807694

What about the square root of an extremely large number that also has the largest possible error as the initial guess? The first approximation (x_0 in equation 3) of $\sqrt{10^{217}}$ gives the maximum relative error found thus far: $\approx 75\%$. Table 2 shows that only 5 more iterations are needed to arrive at a value accurate to 9 decimal places, with each iteration adding 5 decimal places of accuracy.

Table 2: Largest known error in x_n

	$\sqrt{10^{217}}$
True:	$3.162277660 \times 10^{108}$
$x_0 =$	$5.520419826 \times 10^{108}$
$x_1 =$	$3.665938129 \times 10^{108}$
$x_2 =$	$3.196876426 \times 10^{108}$
$x_3 =$	$3.162464886 \times 10^{108}$
$x_4 =$	$3.162277666 \times 10^{108}$
$x_5 =$	$3.162277660 \times 10^{108}$

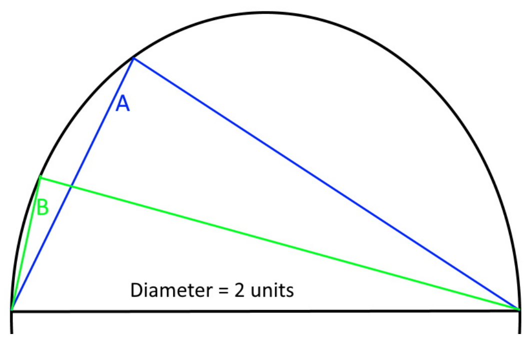

Figure 1: Triangles inscribed in a circle of diameter 2 units

2 π: Archimedes' method

Archimedes' method uses inscribed and circumscribed regular polygons within and around a circle in order to bound the value of π. The disadvantage had been that the sequence generated utilized square roots, and algorithms to calculate square roots depended upon obtaining seed values that were consistently helpful at all scales. Section 1 introduced a simple equation (3) that provides consistently useful seed values at all scales. Using these seed values along with the ancient algorithm of sequence 1, Archimedes' method may again be useful. A simplified version of it is presented here, using only inscribed polygons to approach the value of π.

The blue 30°/60°/90° triangle in figure 1 is inscribed within a circle of diameter 2 units. Side A would be one side of a hexagon to be inscribed within this circle. The first approximation for π, given by the first iteration of Archimedes method, would be $\frac{6A}{2}$, which is the perimeter of the hexagon divided by the diameter of the circle. The right hand angle of the green triangle would be half that of the first triangle. This green triangle would then provide side B, which is used in the second iteration by generating an inscribed 12-gon. The second approximation would be $\frac{12B}{2}$. The ratio of the sides of such triangles to a diameter of 2 units can be represented by sequence 4. This is then multiplied by the number of sides of the appropriate polygon for that iteration according to $6 \times 2^{n-1}$.

Table 3: Archimedes' π

π True:	3.141592654
$\pi_1 =$	3.105828541
$\pi_2 =$	3.132628613
$\pi_3 =$	3.139350203
$\pi_4 =$	3.141031951
$\pi_5 =$	3.141452472
$\pi_6 =$	3.141557608
$\pi_7 =$	3.141583892
$\pi_8 =$	3.141590463
$\pi_9 =$	3.141592106
$\pi_{10} =$	3.141592517
$\pi_{11} =$	3.141592619
$\pi_{12} =$	3.141592645
$\pi_{13} =$	3.141592651
$\pi_{14} =$	3.141592653
$\pi_{15} =$	3.141592653
$\pi_{16} =$	3.141592654

$$\begin{aligned} r_n &= \frac{2}{\sqrt{\left(\frac{2+(r_{n-1})\sqrt{\frac{4}{r_{n-1}^2}-1}}{r_{n-1}}\right)^2+1}} \\ r_0 &= 1 \end{aligned} \tag{4}$$

The value of the π approximation given by each iteration is given by sequence 5.

$$\pi_n = \left(6 \times 2^{n-1}\right)(r_n) \tag{5}$$

Table 3 shows Archimedes' π approximation carried to 9 decimal places of accuracy. Note once again that since equation 3 provides the seed values for approximating roots with sequence 1, the approximation of π using this method no longer presents any difficulty.

Note that Archimedes' method requires about 3 iterations to add each decimal place of accuracy. The next few sections will introduce a new method of approximating roots and π using Pythagorean triples.

3 Pythagorean triples to approximate roots

3.1 Generating triples

a and b are the legs, and c is the hypotenuse of a right triangle. a, b, and c must satisfy equation 6.

M and N are any two odd integers with M > N

$$a = MN \qquad b = \frac{M^2 - N^2}{2} \qquad c = \frac{M^2 + N^2}{2} \tag{6}$$

3.2 $\sqrt{2}$

The method consists of seeking Pythagorean triangles where the ratio $\frac{c}{a}$ approximates $\sqrt{2}$. The recursive sequence 7 supplies a sequence of triangles such that that the ratios $\frac{c}{a}$ and $\frac{c}{b}$ within each successive triangle become better approximations of $\sqrt{2}$ than those of the previous triangle.

$$\begin{aligned} M_n &= 2M_{n-1} + M_{n-2} \\ N_n &= M_{n-1} \end{aligned} \tag{7}$$

One of the ratios $\frac{c}{a}$ and $\frac{c}{b}$ from each of these triangles will be greater than $\sqrt{2}$, and one will be less than $\sqrt{2}$. The best approximation is provided by averaging these ratios in each iteration. Algebraic manipulation dispenses with N. Further rearrangement in order to average the ratios $\frac{c}{a}$ and $\frac{c}{b}$, and then to simplify, delivers the pair of sequences in 8.

$$M_0 = 1 \quad M_1 = 3 \qquad M_n = 2M_{n-1} + M_{n-2}$$

$$\sqrt{2}_n =$$

$$\frac{35M_n^4 + 58M_n^3M_{n-1} + 36M_n^2M_{n-1}^2 + 10M_nM_{n-1}^3 + M_{n-1}^4}{24M_n^4 + 44M_n^3M_{n-1} + 24M_n^2M_{n-1}^2 + 4M_nM_{n-1}^3} \tag{8}$$

Table 4 shows the first 7 iterations for $\sqrt{2}$ using the sequences in 8. Note that the first iteration at $\sqrt{2}_1$ is accurate to two decimal places. Each iteration then adds about two decimal places of accuracy.

Table 4: Pythagorean Triple Approximation for $\sqrt{2}$

True:	1.41421356237	M_n	M_{n-1}
$\sqrt{2}_1$	1.41547619048	3	1
$\sqrt{2}_2$	1.41425070028	7	3
$\sqrt{2}_3$	1.41421465558	17	7
$\sqrt{2}_4$	1.41421359455	41	17
$\sqrt{2}_5$	1.41421356332	99	41
$\sqrt{2}_6$	1.41421356240	239	99
$\sqrt{2}_7$	1.41421356237	577	239

3.3 $\sqrt{3}$ and $\sqrt{5}$

The sequence generating approximations for $\sqrt{3}$ is considerably more complex. Although it may not be a convenient method for calculating this value, it suggests that the ability to approximate all irrational numbers with similar Pythagorean sequences might be a postulate of mathematics. If true, the uses for this general postulate remain to be seen. As before, a recursive sequence is used to generate M and N. These in turn are used in equation 6 to generate the sides of a right triangle. The sequence used to approximate $\sqrt{3}$ will supply triangles that are nearly similar to a right 30°/60°/90° triangle, so that $\sqrt{3} \approx \frac{a}{b}$. M and N will then generate triangles such that the ratio $\frac{a}{b}$ moves closer to $\sqrt{3}$ with each iteration.

$$M_1 = 3 \quad N_1 = 1 \quad M_2 = 5 \quad N_2 = 3$$

$$M_n =$$
$$\left(\frac{1}{2}(-1)^{n+1} + \frac{1}{2}\right)(3N_{n-1}) + \left(\frac{1}{2}(-1)^n + \frac{1}{2}\right)(2N_n - N_{n-2})$$

$$N_n =$$
$$M_{n-1}\left(\frac{1}{2}(-1)^{n+1} + \frac{1}{2}\right) + \left(\frac{1}{2}(-1)^n + \frac{1}{2}\right)(2N_{n-2} + N_{n-1}) \tag{9}$$

Using the sequences in 9 when $n = 3$ provides $M = 9$ and $N = 5$. This in turn, using equation 6, would generate the triangle with sides $a = 45$, $b = 28$, and $c = 53$. $\sqrt{3}_3 = \frac{a}{b} = \frac{45}{28} \approx 1.607$. Each further iteration will move closer to the true value.

$$M_1 = 3\ N_1 = 1\ M_2 = 7\ N_2 = 3\ N_3 = 5\ N_4 = 13$$

$$M_n = 2N_n + N_{n-2} \tag{10}$$

$$N_n = 2M_{n-2} - N_{n-4}$$

The sequences in 10 provide a Pythagorean approximation for $\sqrt{5}$ using the same process.

Table 5: First 6 Pythagorean triples

n	M_n	N_n	a	b	c
1	3	1	3	4	5
2	5	1	5	12	13
3	5	3	15	8	17
4	7	1	7	24	25
5	7	3	21	20	29
6	7	5	35	12	37

4 Pythagorean approximation for π

4.1 The function P used to generate M

The Pythagorean triple approximation for π is similar to Archimedes' method in that it uses triangles inscribed within a circle. In this case the triangles are inscribed within the unit circle. First, a way must be found to list all possible Pythagorean triples in a specified order using an explicit sequence. Any two odd integers M and N, with $M > N$, will form a triple using the equations in 6. The natural order that suggests itself shown in table 5.

M_n is generated as a function of P_n, which is rounding the square root of $2n - 1$ to the nearest integer as in equation 11. P_n will also be used to later generate N_n.

$$\begin{aligned} P_n &= \left\lfloor \sqrt{2n-1} \right\rceil \\ M_n &= 2P_n + 1 = 2\left\lfloor \sqrt{2n-1} \right\rceil + 1 \end{aligned} \tag{11}$$

4.2 N

Finding N_n is more difficult, and will require using the nested functions within the sawtooth function J_n as shown in equation 12. P_n is the varying period that was also used to generate M_n. D_n is the varying horizontal shift, which has the functions g_n, k_n, and w_n nested within it.

$$\begin{aligned} J_n &= \left\lfloor n - D_n - P_n \left\lfloor \frac{1}{2} + \frac{n - D_n}{P_n} \right\rfloor + \frac{P_n}{2} + 1 \right\rfloor \\ P_n &= \left\lfloor \sqrt{2n-1} \right\rceil \\ D_n &= k_n w_n + g_n \\ k_n &= \left\lfloor \frac{P_n + 1}{2} \right\rceil \\ g_n &= \frac{(-1)^{\lceil P_n - .5 \rceil}}{4} \qquad w_n = \frac{(-1)^{\lfloor P_n - .5 \rfloor}}{2} \end{aligned} \tag{12}$$

All possible Pythagorean triples are generated in the systematic order suggested by table 5. This is done by in using equations 11 and 12 to produce the equations in 13.

$$\begin{aligned} M_n &= 2\left\lfloor \sqrt{2n-1} \right\rceil + 1 \\ N_n &= 2J_n - 1 \end{aligned} \tag{13}$$

4.3 Visual demonstration of the method

Note that the variable P is used to generate both M_n and N_n. Also note that P increases in a helpful way: $1, 2, 2, 3, 3, 3, 4, 4, 4, 4, 5, 5, 5, 5, 5$ ... etc. These facts are used to generate distinct groups of triples for use in approximating π using a finite sum of the hypotenuses of a particular group of triangles. The method will first be visually introduced in figures 2 and 3, before presenting the formal sum formula.

The first group generated only has one triangle in it. This is a 3/4/5 triangle. $(3, 4)$ is the only non-quadrantal point used, so it is not necessary to scale this calculation to the unit circle. The length of the line from $(0, 5)$ to $(3, 4)$ is added to the length of the line from $(3, 4)$ to $(5, 0)$. The approximation for π would be $\frac{2}{5}$ of the sum of the lengths.

The second group generated uses two triangles: 5/12/13 and 15/8/17. They must be scaled to the unit circle in order to use them together. They become $\frac{5}{13}/\frac{12}{13}/\frac{13}{13}$ and $\frac{15}{17}/\frac{8}{17}/\frac{17}{17}$. In this case, there are three lengths to add, and the approximation for π would be 2 times the sum of these lengths.

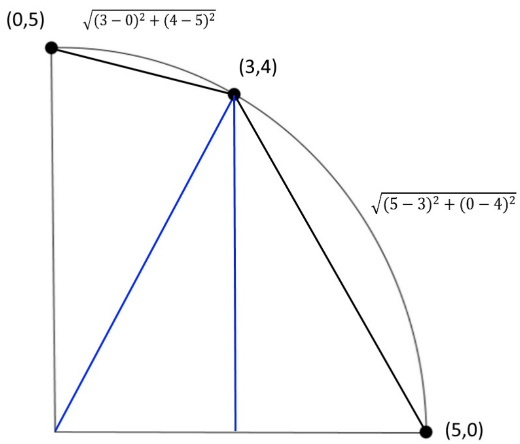

Figure 2: Group 1: Using a single 3/4/5 triangle to approximate π

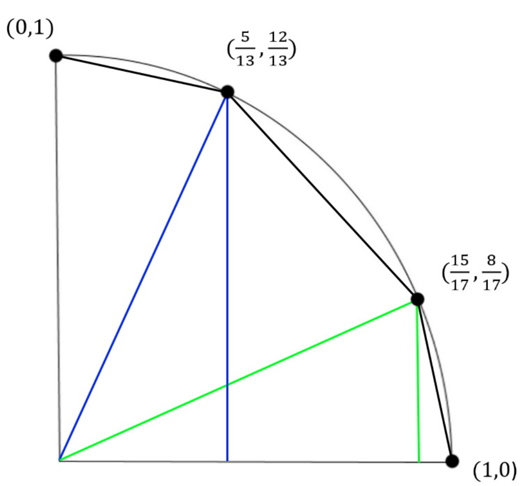

Figure 3: Group 2: Scaling 5/12/13 and 15/8/17 triangles to the unit circle in order to approximate π in the 2nd iteration

4.4 A list of all possible Pythagorean triples

The function $P_n = \left\lfloor \sqrt{2n-1} \right\rceil$ is useful to generate a table listing all Pythagorean triples. It generates a pair, M_n and N_n, that in turn generate the triples a_n, b_n, and c_n. P is also used in the approximation formula in equation 19, where P represents the number of triangles used for the approximation. The more triangles used, the better the approximation becomes. The equations in 14 and 15 will choose the appropriate M along with its associated N values for that particular group of triangles.

$$\begin{aligned} &P = \text{number of triangles chosen} \\ &M_n = 2P+1 \\ &N(x)_n = 2J(x)-1 \end{aligned} \tag{14}$$

$$\begin{aligned} s_1 &= \frac{1}{2}P^2 - \frac{1}{2}P + 1 \\ f_1 &= \frac{1}{2}P^2 + \frac{1}{2}P \\ J(x) &= \left\lfloor x - D - P\left\lfloor \frac{1}{2} + \frac{x-D}{P} \right\rfloor + \frac{P}{2} + 1 \right\rfloor \\ D &= kw + g \\ k &= \left\lfloor \frac{P+1}{2} \right\rceil \\ g &= \frac{(-1)^{\lceil P-.5 \rceil}}{4} \qquad w_n = \frac{(-1)^{\lfloor P-.5 \rfloor}}{2} \end{aligned} \tag{15}$$

4.5 Scaling to circle of radius 1

The equations in 16 will use the output of M and its N values to generate triangles with side lengths a,b, and c. These triangles will then be scaled to the unit circle, giving triangles with sides $\frac{a}{c}$, $\frac{b}{c}$, and $(\frac{c}{c} = 1)$.

$$\begin{aligned} a(x) &= MN(x) \\ b(x) &= \frac{M^2 - (N(x))^2}{2} \\ c(x) &= \frac{M^2 + (N(x))^2}{2} \\ A_u(x) &= \frac{a(x)}{c(x)} \\ B(x) &= \frac{b(x)}{c(x)} \end{aligned} \tag{16}$$

4.6 The first and last length

The equations in 17 deliver the starting first length and the last length used for that particular approximation. **Note that $x = s_1$ in function S; $x = n$ in function U; $x = f_1$ in function F.**

$$\begin{aligned} S &= \sqrt{(A_u(s_1))^2 + (1 - B(s_1))^2} \\ F &= \sqrt{(1 - A_u(f_1))^2 + (B(f_1))^2} \end{aligned} \tag{17}$$

4.7 The sum of all the middle lengths

The sum formula in equation 18 gives the sum of all of the middle lengths used for that particular approximation.

$$U = \sum_{.5P^2-.5P+1}^{.5P^2+.5P-1} \sqrt{(A_u(n+1) - A_u(n))^2 + (B(n+1) - B(n))^2} \tag{18}$$

4.8 The composite approximation function

The approximation for π using P triangles is given by equation 19.

$$\begin{aligned} A_p &\approx \pi \quad \text{using P triangles} \\ A_p &= 2(S + U + F) \end{aligned} \tag{19}$$

Table 6 shows the composite sum formula ($A_p = 2(S + U + F)$) approaching the true value of π as P (the number of triangles used) increases.

The hypertext Pythagorean π approximation is a link to a web page where this approximation may be used or demonstrated. On this site the variable P is the number of triangles to be used in the approximation. P is the only variable that should be changed. The output A_p will be the approximation using P triangles.

Table 6: $A_p = 2(S+U+F)$; Where P = the number of triangles used

π True:	3.141592654
A_1 =	3.053765446
A_2 =	3.099952629
A_3 =	3.117962783
A_4 =	3.126484857
A_5 =	3.131132028
...	...
A_{10} =	3.138519318
...	...
A_{100} =	3.141556959
...	...
A_{1000} =	3.141592291
...	...
A_{10000} =	3.141592650

5 Conclusion

Section 1 introduced an equation delivering seed values for all recursive sequences used for calculating square roots. This is useful because it removes ambiguity in algorithms used to accomplish this task. Section 2 demonstrated this by recommending its use for Archimedes' approximation of π. The remaining sections of the paper introduced a new method entirely.

Other methods might converge to π faster, so why is this method useful? This method shows that patterns in the generation of Pythagorean triples can be used to approximate both irrational and transcendental numbers. Because the method is new, it is difficult to guess what insights might be gained by exploring it further.

All mathematics is a type of counting. Counting only involves integers. In some sense the expression $\sqrt{2}$ might be interpreted as "undefined," or "no solution," because there is no integer which can be squared to equal 2. Taking the square root of such a number is an act of imagination. Pretend that 2 can be divided into 196 equal parts (since 196 is close to 200). Since the square root of 196 is 14, one might then declare that the square root of 2 is 1.4.

Irrational numbers might better be thought of as an algorithm or method rather than a "number." Saying "the side length of a square whose area is 2" sends the mind on a quest to approximate this quantity. It is a never-ending quest. Transcendental numbers like π send their seekers on similar quests.

The method of Pythagorean approximation introduced in this paper may lead to further insight about how all such numbers are related to one another. This might in turn lead to better approximation methods in the future.