Comparison of two electromagnetic quantization methods

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Abstract:

This paper is a comparison of two different canonical quantization methods of the electromagnetic field, one commonly used, and the other similar to one by Kroll. Both quantization methods will be carried out in the Schrodinger representation.

I. **Introduction**

As is commonly done, the transverse gauge is used and a Fourier expansion is made of the vector potential. The resulting Fourier coefficients are then quantized by two different methods. The first method is similar to a method by Kroll [1], and the second method is commonly done in the Heisenberg representation, see for example Mandel and Wolf [2]. The advantage of the first method is that its quantization produces a wavefunction of the field in the configuration space of the vector potential, while the second method is more straightforward. A canonical transformation is found between the two methods, and the corresponding Wigner distributions [3] of the ground and first excited states are found to be the same.

The first method appears to have started with the paper by Kroll [1], although Wheeler [4] gives the ground state in terms of the magnetic field. Kuchar [5] also uses this ground state when comparing with the gravitational field.

The second method appears to have originated with Dirac [6] and is used by a number of authors, for example see [7-11].

II. **Classical field**

The transverse gauge condition with $\nabla \bullet \mathbf{A} = 0$ and $\phi = 0$ is chosen where **A** is the vector potential and ϕ is the scalar potential. Bold print is used to indicate a vector. In terms of a Fourier expansion the vector potential takes the form

$$\mathbf{A} = \frac{1}{\sqrt{L^3}} \sum_{\mathbf{k}} \mathbf{a}_{\mathbf{k}} e^{i\mathbf{k} \bullet \mathbf{r}} \tag{1}$$

where $\sum_{\mathbf{k}}$ is a sum over all of **k** space, and the field is confined to a box of size L^3, see, for example, Mandel and Wolf [2]. Maxwell's equations then reduce to

$$\frac{d^2 \mathbf{a}_{\mathbf{k}}}{dt^2} + \omega^2 \mathbf{a}_{\mathbf{k}} = 0 \tag{2}$$

and

$$\mathbf{k} \bullet \mathbf{a}_{\mathbf{k}} = 0 \tag{3}$$

where the speed of light is taken to be 1 and $\omega = |\mathbf{k}|$. **A** is real so the condition $\mathbf{a}_{-\mathbf{k}} = \mathbf{a}_{\mathbf{k}}{}^*$ is needed. Define $\boldsymbol{\varepsilon}^1{}_{\mathbf{k}}$ and $\boldsymbol{\varepsilon}^2{}_{\mathbf{k}}$ as two unit vectors perpendicular to each other and to **k** with the convention $\boldsymbol{\varepsilon}^1{}_{-\mathbf{k}} = \boldsymbol{\varepsilon}^1{}_{\mathbf{k}}$ and $\boldsymbol{\varepsilon}^2{}_{-\mathbf{k}} = -\boldsymbol{\varepsilon}^2{}_{\mathbf{k}}$, or $\boldsymbol{\varepsilon}^n{}_{-\mathbf{k}} = (\delta_{n1} - \delta_{n2})\boldsymbol{\varepsilon}^n{}_{\mathbf{k}}$. A * is used to represent a complex conjugate.

Now consider the independent variables. We need to satisfy the conditions

$\mathbf{k} \bullet \mathbf{a}_{\mathbf{k}} = 0$ and $\mathbf{a}_{-\mathbf{k}} = \mathbf{a}_{\mathbf{k}}*$ which can be done in the two following ways:

(1) Set $\mathbf{a}_{\mathbf{k}} = \frac{1}{\sqrt{2}} \sum_{n=1}^{2} a_{n\mathbf{k}} \boldsymbol{\varepsilon}^{n}{}_{\mathbf{k}}$ so as to satisfy the condition $\mathbf{k} \bullet \mathbf{a}_{\mathbf{k}} = 0$, and setting $a_{n\mathbf{k}} = a^{R}{}_{n\mathbf{k}} + ia^{I}{}_{n\mathbf{k}}$, take $a^{R}{}_{n\mathbf{k}}$ and $a^{I}{}_{n\mathbf{k}}$ as independent real variables only for $\mathbf{k} \geq 0$, that is for $\mathbf{k}$ vectors with $k_z \geq 0$. Then require that $a_{n\text{-}\mathbf{k}} = (\delta_{n1} - \delta_{n2}) a_{n\mathbf{k}}*$ so as to satisfy the condition $\mathbf{a}_{-\mathbf{k}} = \mathbf{a}_{\mathbf{k}}*$. We will call this the "a" formulation, and is the one similar to that of Kroll [1]. The $\frac{1}{\sqrt{2}}$ factor is included so that, in terms of the degrees of freedom, the basis functions form an orthogonal set which are normalized to one.

(2) Set $\mathbf{a}_{\mathbf{k}} = \mathbf{c}_{\mathbf{k}} e^{-i\omega t} + \mathbf{c}_{-\mathbf{k}} * e^{i\omega t}$ so that $\mathbf{a}_{-\mathbf{k}} = \mathbf{c}_{-\mathbf{k}} e^{-i\omega t} + \mathbf{c}_{\mathbf{k}} * e^{i\omega t} = \mathbf{a}_{\mathbf{k}}*$, satisfying the condition $\mathbf{a}_{-\mathbf{k}} = \mathbf{a}_{\mathbf{k}}*$. Then require that $\mathbf{k} \bullet \mathbf{a}_{\mathbf{k}} = 0$ by setting $\mathbf{c}_{\mathbf{k}} = \sum_{n=1}^{2} c_{n\mathbf{k}} \boldsymbol{\varepsilon}^{n}{}_{\mathbf{k}}$ and define a new real variable $q_{n\mathbf{k}}$ by $q_{n\mathbf{k}} = c_{n\mathbf{k}} e^{-i\omega t} + c_{n\mathbf{k}} * e^{i\omega t}$. In this second method, the independent variables are taken as the $q_{n\mathbf{k}}$ which obey the harmonic oscillator equation $\frac{d^2 q_{n\mathbf{k}}}{dt^2} = -\omega^2 q_{n\mathbf{k}}$. We will call this the "q" formulation, and is similar to that found in numerous texts including Mandel and Wolf [2].

III. **Variational principles**

In the "a" formulation the Lagrangian, conjugate momentum, and Hamiltonian take the form

$$L_a = \frac{1}{8\pi} \sum_{+\mathbf{k}} \sum_{n=1}^{2} \left\{ \frac{da_{n\mathbf{k}}}{dt} \frac{da_{n\mathbf{k}}*}{dt} - \omega^2 a_{n\mathbf{k}} a_{n\mathbf{k}}* \right\} \tag{4a}$$

$$p^{a}{}_{n\mathbf{k}} = p^{Ra}{}_{n\mathbf{k}} + ip^{Ia}{}_{n\mathbf{k}} = \frac{\partial L_a}{\partial \frac{da^{R}{}_{n\mathbf{k}}}{dt}} + i\frac{\partial L_a}{\partial \frac{da^{I}{}_{n\mathbf{k}}}{dt}} = \frac{1}{4\pi}\frac{da_{n\mathbf{k}}}{dt} \tag{4b}$$

$$H_a = \sum_{+\mathbf{k}}\sum_{n=1}^{2}\{p^{Ra}{}_{n\mathbf{k}}\frac{da^{R}{}_{n\mathbf{k}}}{dt} + p^{Ia}{}_{n\mathbf{k}}\frac{da^{I}{}_{n\mathbf{k}}}{dt}\} - L_a$$

$$= \frac{1}{8\pi}\sum_{+\mathbf{k}}\sum_{n=1}^{2}\{(4\pi)^2 p^{a}{}_{n\mathbf{k}} p^{a}{}_{n\mathbf{k}}{}^{*} + \omega^2 a_{n\mathbf{k}} a_{n\mathbf{k}}{}^{*}\} \tag{4c}$$

while in the "q" formulation the Lagrangian, conjugate momentum, and Hamiltonian are

$$L_q = \frac{1}{8\pi}\sum_{\mathbf{k}}\sum_{n=1}^{2}\{(\frac{dq_{n\mathbf{k}}}{dt})^2 - \omega^2 q_{n\mathbf{k}}{}^2\} \tag{5a}$$

$$p^{q}{}_{n\mathbf{k}} = \frac{\partial L_q}{\partial \frac{dq_{n\mathbf{k}}}{dt}} = \frac{1}{4\pi}\frac{dq_{n\mathbf{k}}}{dt} \tag{5b}$$

$$H_q = \sum_{\mathbf{k}}\sum_{n=1}^{2} p^{q}{}_{n\mathbf{k}}\frac{dq_{n\mathbf{k}}}{dt} - L_q = \frac{1}{8\pi}\sum_{\mathbf{k}}\sum_{n=1}^{2}\{(4\pi)^2 p^{q}{}_{n\mathbf{k}}{}^2 + \omega^2 q_{n\mathbf{k}}{}^2\} \tag{5c}$$

In both cases the constant in front of the Lagrangian is chosen so that the Hamiltonian is equal to the total energy $E = \frac{1}{8\pi}\int dv(\mathbf{E}\bullet\mathbf{E} + \mathbf{B}\bullet\mathbf{B})$, for example see Misner, Thorne and Wheeler [12]. $\sum_{+\mathbf{k}}$ means a sum over that part of $\mathbf{k}$ space with $k_z \geq 0$, and $\int dv$ represents an integral over the volume L^3.

IV. **Canonical transformation**

What is the relationship between the two types of independent variables? We have

$$\mathbf{a_k} = \mathbf{c_k} e^{-i\omega t} + \mathbf{c_{-k}} * e^{i\omega t} = \sum_{n=1}^{2} (c_{n\mathbf{k}} \boldsymbol{\varepsilon}^n{}_{\mathbf{k}} e^{-i\omega t} + c_{n\mathbf{-k}} * \boldsymbol{\varepsilon}^n{}_{\mathbf{-k}} e^{i\omega t})$$

$$= \sum_{n=1}^{2} (c_{n\mathbf{k}} \boldsymbol{\varepsilon}^n{}_{\mathbf{k}} e^{-i\omega t} + (\delta_{n1} - \delta_{n2}) c_{n\mathbf{-k}} * \boldsymbol{\varepsilon}^n{}_{\mathbf{k}} e^{i\omega t}) = \frac{1}{\sqrt{2}} \sum_{n=1}^{2} a_{n\mathbf{k}} \boldsymbol{\varepsilon}^n{}_{\mathbf{k}} \qquad (6)$$

Thus

$$a_{n\mathbf{k}} = \sqrt{2}(c_{n\mathbf{k}} e^{-i\omega t} + (\delta_{n1} - \delta_{n2}) c_{n\mathbf{-k}} * e^{i\omega t}) \qquad (7)$$

with the corresponding conjugate momentum

$$p^a{}_{n\mathbf{k}} = \frac{1}{4\pi} \frac{da_{n\mathbf{k}}}{dt} = i \frac{\omega}{4\pi} \sqrt{2}(-c_{n\mathbf{k}} e^{-i\omega t} + (\delta_{n1} - \delta_{n2}) c_{n\mathbf{-k}} * e^{i\omega t}) \qquad (8)$$

We also have

$$q_{n\mathbf{k}} = c_{n\mathbf{k}} e^{-i\omega t} + c_{n\mathbf{k}} * e^{i\omega t} \qquad (9)$$

with

$$p^q{}_{n\mathbf{k}} = \frac{1}{4\pi} \frac{dq_{n\mathbf{k}}}{dt} = i \frac{\omega}{4\pi} (-c_{n\mathbf{k}} e^{-i\omega t} + c_{n\mathbf{k}} * e^{i\omega t}) \qquad (10)$$

Eqs. (7-10) can be combined to yield the canonical transformation

$$a^{R}{}_{n\mathbf{k}} = \frac{1}{\sqrt{2}}(q_{n\mathbf{k}} + (\delta_{n1} - \delta_{n2})q_{n\text{-}\mathbf{k}}) \tag{11a}$$

$$a^{I}{}_{n\mathbf{k}} = \frac{1}{\sqrt{2}}\frac{4\pi}{\omega}(p^{q}{}_{n\mathbf{k}} - (\delta_{n1} - \delta_{n2})p^{q}{}_{n\text{-}\mathbf{k}}) \tag{11b}$$

$$p^{aR}{}_{n\mathbf{k}} = \frac{1}{\sqrt{2}}(p^{q}{}_{n\mathbf{k}} + (\delta_{n1} - \delta_{n2})p^{q}{}_{n\text{-}\mathbf{k}}) \tag{11c}$$

$$p^{aI}{}_{n\mathbf{k}} = \frac{1}{\sqrt{2}}\frac{\omega}{4\pi}(\text{-}q_{n\mathbf{k}} + (\delta_{n1} - \delta_{n2})q_{n\text{-}\mathbf{k}}) \tag{11d}$$

To see that this transformation is canonical it is possible to show that Hamilton's equations in the $q_{n\mathbf{k}}$ and $p^{q}{}_{n\mathbf{k}}$ coordinates transforms into Hamilton's equations in the $a_{n\mathbf{k}}$ and $p^{a}{}_{n\mathbf{k}}$ coordinates using the transformations given by eqs. (11a-d).

V. **Quantization of the "a" formulation**

The "a" formulation is quantized by replacing the conjugate momentum $p^{aR}{}_{n\mathbf{k}}$ by the operator

$$\hat{p}^{aR}{}_{n\mathbf{k}} = \text{-}i\hbar\frac{\partial}{\partial a^{R}{}_{n\mathbf{k}}}$$

and $p^{aI}{}_{n\mathbf{k}}$ by the operator

$$\hat{p}^{aI}{}_{n\mathbf{k}} = \text{-}i\hbar\frac{\partial}{\partial a^{I}{}_{n\mathbf{k}}}$$

in the Hamiltonian H_a to transform eq. (4c) into the Schrodinger like wave equation

$$i\hbar\frac{\partial\psi}{\partial t} = \hat{H}_a\psi = \frac{1}{8\pi}\sum_{+\mathbf{k}}\sum_{n=1}^{2}\{(4\pi)^2(\hat{p}^{aR}{}_{n\mathbf{k}}{}^2 + \hat{p}^{aI}{}_{n\mathbf{k}}{}^2) + \omega^2 a_{n\mathbf{k}} * a_{n\mathbf{k}}\}\psi \qquad (12)$$

where a ^ represents a quantum mechanical operator.

Then since this is just the Hamiltonian for a set of independent harmonic oscillators, we can define the following creation and annihilation operators for $\mathbf{k} \geq 0$, see for example Saxon [13].

$$\hat{b}^{R}{}_{n\mathbf{k}}{}^{t} = \frac{1}{\sqrt{2\hbar}}(\sqrt{\frac{\omega}{4\pi}}a^{R}{}_{n\mathbf{k}} - i\sqrt{\frac{4\pi}{\omega}}\hat{p}^{aR}{}_{n\mathbf{k}}) \qquad (13a)$$

$$\hat{b}^{R}{}_{n\mathbf{k}} = \frac{1}{\sqrt{2\hbar}}(\sqrt{\frac{\omega}{4\pi}}a^{R}{}_{n\mathbf{k}} + i\sqrt{\frac{4\pi}{\omega}}\hat{p}^{aR}{}_{n\mathbf{k}}) \qquad (13b)$$

$$\hat{b}^{I}{}_{n\mathbf{k}}{}^{t} = \frac{1}{\sqrt{2\hbar}}(\sqrt{\frac{\omega}{4\pi}}a^{I}{}_{n\mathbf{k}} - i\sqrt{\frac{4\pi}{\omega}}\hat{p}^{aI}{}_{n\mathbf{k}}) \qquad (13c)$$

$$\hat{b}^{I}{}_{n\mathbf{k}} = \frac{1}{\sqrt{2\hbar}}(\sqrt{\frac{\omega}{4\pi}}a^{I}{}_{n\mathbf{k}} + i\sqrt{\frac{4\pi}{\omega}}\hat{p}^{aI}{}_{n\mathbf{k}}) \qquad (13d)$$

with the following ground state

$$\psi_{a0} = \prod_{+\mathbf{k}}\prod_{n=1}^{2}\sqrt{\frac{\omega}{4\pi^2\hbar}}\exp(-\frac{\omega}{8\pi\hbar}a_{n\mathbf{k}}a_{n\mathbf{k}}*) \qquad (14)$$

where $\prod_{+\mathbf{k}}$ indicates a product over $\mathbf{k}$ space with $k_z \geq 0$. Note that this can be written as

$$\psi_{a0} = \{\prod_{+\mathbf{k}}\frac{\omega}{4\pi^2\hbar}\}\exp(-\frac{1}{16\pi^3\hbar}\int dv'\int dv\mathbf{B}(\mathbf{r})\bullet\mathbf{B}(\mathbf{r}')\frac{1}{|\mathbf{r}-\mathbf{r}'|^2}) \qquad (15)$$

where $\mathbf{B}(\mathbf{r})$ is the magnetic field. This agrees with Wheeler [4].

VI. **Coherent state and new operators**

To extend the operators given in eqs. (13a-d) over all of **k** space, consider a coherent state defined by a classical field with the ground state superimposed upon it, see for example Schiff [14]. A classical solution for the vector potential can be written in the form

$$\mathbf{A} = \sum_{\mathbf{k}} \sum_{n=1}^{2} A_{n\mathbf{k}} \boldsymbol{\varepsilon}^{n}{}_{\mathbf{k}} \cos(\mathbf{k} \bullet \mathbf{r} - \omega t + \phi_{n\mathbf{k}}) \qquad (16)$$

with $A_{n\mathbf{k}}$ representing the amplitude and $\phi_{n\mathbf{k}}$ the phase of a wave going in the **k** direction with polarization $\boldsymbol{\varepsilon}^{n}{}_{\mathbf{k}}$. If we now equate eq. (16) with the expression

$$\mathbf{A} = \frac{1}{\sqrt{2L^3}} \sum_{\mathbf{k}} \sum_{n=1}^{2} a^{c}{}_{n\mathbf{k}} \boldsymbol{\varepsilon}^{n}{}_{\mathbf{k}} e^{i\mathbf{k}\bullet\mathbf{r}}$$

which is eq. (1) along with the condition $\mathbf{a}_{\mathbf{k}} = \frac{1}{\sqrt{2}} \sum_{n=1}^{2} a^{c}{}_{n\mathbf{k}} \boldsymbol{\varepsilon}^{n}{}_{\mathbf{k}}$, we find that

$$a^{c}{}_{n\mathbf{k}} = \sqrt{\frac{L^3}{2}} (A^{c}{}_{n\mathbf{k}} + (\delta_{n1} - \delta_{n2}) A^{c}{}_{n\text{-}\mathbf{k}}{}^{*}) \qquad (17)$$

where the c superscript on $a^{c}{}_{n\mathbf{k}}$ denotes that it is a classical function of time and we have set $A^{c}{}_{n\mathbf{k}} = A_{n\mathbf{k}} \exp(i(-\omega t + \phi_{n\mathbf{k}}))$

We then set the probability density ρ_c for the coherent state ψ_c to the probability density of the ground state superimposed upon a classical solution so that

$$\rho_c = \psi_c * \psi_c = \prod_{+\mathbf{k}} (\frac{\omega}{4\pi^2\hbar})^2 \exp(-\sum_{+\mathbf{k}} \sum_{n=1}^{2} \frac{\omega}{4\pi\hbar}((a_{n\mathbf{k}} - a^c{}_{n\mathbf{k}})(a_{n\mathbf{k}} - a^c{}_{n\mathbf{k}})^*)) \qquad (18)$$

The ψ_c that corresponds to this and satisfies eq. (12) is

$$\psi_c = \prod_{+\mathbf{k}} (\frac{\omega}{4\pi^2\hbar}) \exp[\sum_{+\mathbf{k}} \sum_{n=1}^{2} \{-i\omega t - \frac{\omega}{8\pi\hbar}\{a_{n\mathbf{k}} a_{n\mathbf{k}}^* - \sqrt{2L^3}(A^c{}_{n\mathbf{k}} a_{n\mathbf{k}}^* + A^c{}_{n\text{-}\mathbf{k}}(\delta_{n1} - \delta_{n2}) a_{n\mathbf{k}})$$

$$+ L^3 A^c{}_{n\mathbf{k}} A^c{}_{n\text{-}\mathbf{k}}(\delta_{n1} - \delta_{n2}) + \frac{L^3}{2}(A_{n\mathbf{k}}{}^2 + A_{n\text{-}\mathbf{k}}{}^2)\}\}] \qquad (19)$$

up to an overall phase factor.

Now look at the effect of the annihilation operators $\hat{b}^R{}_{n\mathbf{k}}$ and $\hat{b}^I{}_{n\mathbf{k}}$ on ψ_c. We have

$$\hat{b}^R{}_{n\mathbf{k}} \psi_c = \sqrt{\frac{\omega L^3}{16\pi\hbar}} \{A^c{}_{n\mathbf{k}} + (\delta_{n1} - \delta_{n2}) A^c{}_{n\text{-}\mathbf{k}}\} \psi_c \qquad (20a)$$

$$\hat{b}^I{}_{n\mathbf{k}} \psi_c = \sqrt{\frac{\omega L^3}{16\pi\hbar}} \{-iA^c{}_{n\mathbf{k}} + (\delta_{n1} - \delta_{n2}) iA^c{}_{n\text{-}\mathbf{k}}\} \psi_c \qquad (20b)$$

We can separate out the $A^c{}_{n\mathbf{k}}$ and $A^c{}_{n\text{-}\mathbf{k}}$ components if we define the new operator

$$\hat{b}_{an\mathbf{k}} = \frac{1}{\sqrt{2}}(\hat{b}^R{}_{n\mathbf{k}} + i\hat{b}^I{}_{n\mathbf{k}}) \quad \text{for } \mathbf{k} \geq 0 \qquad (21a)$$

$$= \frac{1}{\sqrt{2}}(\hat{b}^R{}_{n\text{-}\mathbf{k}} - i\hat{b}^I{}_{n\text{-}\mathbf{k}})(\delta_{n1} - \delta_{n2}) \quad \text{for } \mathbf{k} \leq 0 \qquad (21b)$$

so that

$$\widehat{b}_{an\mathbf{k}}\Psi_c = \sqrt{\frac{\omega L^3}{8\pi\hbar}} A^c{}_{n\mathbf{k}}\Psi_c \quad (22)$$

for all **k**.

We can then define a new set of creation operators $\widehat{b}_{an\mathbf{k}}{}^t$ such that $[\widehat{b}_{an\mathbf{k}}, \widehat{b}_{an'\mathbf{k}'}{}^t] = \delta_{nn'}\delta_{\mathbf{kk}'}$ if we set

$$\widehat{b}_{an\mathbf{k}}{}^t = \frac{1}{\sqrt{2}}(\widehat{b}^R{}_{n\mathbf{k}}{}^t - i\widehat{b}^I{}_{n\mathbf{k}}{}^t) \text{ for } \mathbf{k} \geq 0 \quad (23a)$$

$$= \frac{1}{\sqrt{2}}(\widehat{b}^R{}_{n\text{-}\mathbf{k}}{}^t + i\widehat{b}^I{}_{n\text{-}\mathbf{k}}{}^t)(\delta_{n1} - \delta_{n2}) \text{ for } \mathbf{k} \leq 0 \quad (23b)$$

The $\frac{1}{\sqrt{2}}$ is included so that the states defined by these operators will be normalized. Kroll [1] makes a similar type of operator, but leaves out the $(\delta_{n1} - \delta_{n2})$ terms. These terms are needed because $\boldsymbol{\varepsilon}^n{}_{-\mathbf{k}} = (\delta_{n1} - \delta_{n2})\boldsymbol{\varepsilon}^n{}_{\mathbf{k}}$.

In terms of the operators given by eqs. (21a-b) and eqs. (22a-b), the Hamiltonian, vector potential, and electromagnetic field momentum take the form

$$\widehat{H}_a = \sum_{\mathbf{k}}\sum_{n=1}^{2}\{\widehat{b}_{an\mathbf{k}}{}^t\widehat{b}_{an\mathbf{k}} + \frac{1}{2}\}\hbar\omega \quad (24a)$$

$$\widehat{\mathbf{A}} = \mathbf{A} = L^{-3/2}\sum_{\mathbf{k}}\sum_{n=1}^{2}\sqrt{\frac{2\pi\hbar}{\omega}}\{\widehat{b}_{an\mathbf{k}}{}^t e^{-i\mathbf{k}\bullet\mathbf{r}} + \widehat{b}_{an\mathbf{k}} e^{i\mathbf{k}\bullet\mathbf{r}}\}\boldsymbol{\varepsilon}^n{}_{\mathbf{k}} \quad (24b)$$

$$\widehat{\mathbf{P}} = \sum_{\mathbf{k}}\sum_{n=1}^{2}\hbar\mathbf{k}\widehat{b}_{an\mathbf{k}}{}^t\widehat{b}_{an\mathbf{k}} \quad (24c)$$

VII. **Quantization of the "q" formulation**

Now look at the quantization of the "q" formulation. The system is quantized by replacing the conjugate momentum $p^q{}_{n\mathbf{k}}$ by the operator $\hat{p}^q{}_{n\mathbf{k}} = -i\hbar \dfrac{\partial}{\partial q_{n\mathbf{k}}}$ in the

Hamiltonian H_q to transform eq. (5c) into the Schrodinger like wave equation

$$i\hbar \frac{\partial\psi}{\partial t} = \hat{H}_q \psi = \frac{1}{8\pi}\sum_{\mathbf{k}}\sum_{n=1}^{2}\{(4\pi)^2 \hat{p}^q{}_{n\mathbf{k}}{}^2 + \omega^2 q_{n\mathbf{k}}{}^2\}\psi \tag{25}$$

Now again following Saxon [5], define the creation and annihilation operators

$$\hat{b}_{qn\mathbf{k}}{}^t = \frac{1}{\sqrt{2\hbar}}\left(\sqrt{\frac{\omega}{4\pi}} q_{n\mathbf{k}} - i\sqrt{\frac{4\pi}{\omega}}\hat{p}^q{}_{n\mathbf{k}}\right) \tag{26a}$$

$$\hat{b}_{qn\mathbf{k}} = \frac{1}{\sqrt{2\hbar}}\left(\sqrt{\frac{\omega}{4\pi}} q_{n\mathbf{k}} + i\sqrt{\frac{4\pi}{\omega}}\hat{p}^q{}_{n\mathbf{k}}\right) \tag{26b}$$

In terms of the operators given by eqs. (26a-b), the Hamiltonian, vector potential and electromagnetic field momentum take the following form

$$\hat{H}_q = \sum_{\mathbf{k}}\sum_{n=1}^{2}\{\hat{b}_{qn\mathbf{k}}{}^t \hat{b}_{qn\mathbf{k}} + \frac{1}{2}\}\hbar\omega \tag{27a}$$

$$\hat{\mathbf{A}} = L^{-3/2}\sum_{\mathbf{k}}\sum_{n=1}^{2}\sqrt{\frac{2\pi\hbar}{\omega}}\{\hat{b}_{qn\mathbf{k}}{}^t e^{-i\mathbf{k}\bullet\mathbf{r}} + \hat{b}_{qn\mathbf{k}} e^{i\mathbf{k}\bullet\mathbf{r}}\}\boldsymbol{\varepsilon}^n{}_{\mathbf{k}} \tag{27b}$$

$$\hat{\mathbf{P}} = \sum_{\mathbf{k}}\sum_{n=1}^{2}\hbar\mathbf{k}\hat{b}_{qn\mathbf{k}}{}^t \hat{b}_{qn\mathbf{k}} \tag{27c}$$

Thus the Hamiltonian, vector potential, and field momentum take the same form in terms of the creation and annihilation operators in both the “a” and “q” formulations.

VIII. **Comparison of ground states and first excited states**

For the "a" formulation, the ground state solution is given by eq. (14). and the first excited states are

$$\psi_{an\mathbf{k}} = \hat{b}_{an\mathbf{k}}{}^{t}\psi_{a0} = \frac{1}{2}\sqrt{\frac{\omega}{\pi\hbar}}(a^{R}{}_{n\mathbf{k}} - ia^{I}{}_{n\mathbf{k}})\psi_{a0} \tag{28a}$$

and

$$\psi_{an\text{-}\mathbf{k}} = \hat{b}_{an\text{-}\mathbf{k}}{}^{t}\psi_{a0} = \frac{1}{2}\sqrt{\frac{\omega}{\pi\hbar}}(a^{R}{}_{n\mathbf{k}} + ia^{I}{}_{n\mathbf{k}})(\delta_{n1} - \delta_{n2})\psi_{a0} \tag{28b}$$

with $k_z \geq 0$.

In the "q" formulation, the ground state takes the form

$$\psi_{q0} = \prod_{\mathbf{k}}\prod_{n=1}^{2}\left(\frac{\omega}{4\pi^{2}\hbar}\right)^{1/4}\exp(-\frac{\omega}{8\pi\hbar}q_{n\mathbf{k}}{}^{2}) \tag{29}$$

where $\prod_{\mathbf{k}}$ indicates a product over all of **k** space. The first excited state is given by

$$\psi_{qn\mathbf{k}} = \hat{b}_{qn\mathbf{k}}{}^{t}\psi_{q0} = \sqrt{\frac{\omega}{2\pi\hbar}}q_{n\mathbf{k}}\psi_{q0} \tag{30}$$

To compare these states in the two formulations, look at their corresponding Wigner phase space distributions [3]. In the "a" formulation this can be written as

$$\rho_a = \prod_{+\mathbf{k}} \prod_{n=1}^{2} [\frac{1}{(\pi\hbar)^2} \int dx^R{}_{n\mathbf{k}} \int dx^I{}_{n\mathbf{k}} \exp\{-\frac{2i}{\hbar}(x^R{}_{n\mathbf{k}} p^{aR}{}_{n\mathbf{k}} + x^I{}_{n\mathbf{k}} p^{aI}{}_{n\mathbf{k}})\}]$$

$$X\psi^*(a^R{}_{n\mathbf{k}} - x^R{}_{n\mathbf{k}}, a^I{}_{n\mathbf{k}} - x^I{}_{n\mathbf{k}}, t)\psi(a^R{}_{n\mathbf{k}} + x^R{}_{n\mathbf{k}}, a^I{}_{n\mathbf{k}} + x^I{}_{n\mathbf{k}}, t) \quad (31)$$

and in the “q” formulation as

$$\rho_q = \prod_{\mathbf{k}} \prod_{n=1}^{2} [\frac{1}{\pi\hbar} \int dx_{n\mathbf{k}} \exp(-\frac{2i}{\hbar} x_{n\mathbf{k}} p^q{}_{n\mathbf{k}})]\psi^*(q_{n\mathbf{k}} - x_{n\mathbf{k}}, t)\psi(q_{n\mathbf{k}} + x_{n\mathbf{k}}, t) \quad (32)$$

For the “a” formulation ground state, eq. (14), the corresponding Wigner distribution is

$$\rho_{a0} = \left(\prod_{\mathbf{k}} \frac{1}{(\pi\hbar)^2}\right) \exp\{-\frac{1}{\hbar} \sum_{+\mathbf{k}} \sum_{n=1}^{2} (\frac{\omega}{4\pi} a_{n\mathbf{k}} a_{n\mathbf{k}}{}^* + \frac{4\pi}{\omega} p^a{}_{n\mathbf{k}} p^a{}_{n\mathbf{k}}{}^*)\} \quad (33)$$

and for the “q” formulation ground state, eq. (29), it takes the form

$$\rho_{q0} = \left(\prod_{\mathbf{k}} \frac{1}{(\pi\hbar)^2}\right) \exp\{-\frac{1}{\hbar} \sum_{\mathbf{k}} \sum_{n=1}^{2} (\frac{\omega}{4\pi} q_{n\mathbf{k}}{}^2 + \frac{4\pi}{\omega} p^q{}_{n\mathbf{k}}{}^2)\} \quad (34)$$

Under the canonical transformation, eq. (11a-d), ρ_{a0} transforms into ρ_{q0}, even though the wavefunctions themselves occupy different subspaces of the phase space.

For the first excited states, the Wigner distribution corresponding to $\psi_{an\mathbf{k}}$ is

$$\rho_{an\mathbf{k}} = \{\frac{\omega}{4\pi\hbar} a_{n\mathbf{k}} a_{n\mathbf{k}}{}^* + \frac{4\pi}{\hbar\omega} p^a{}_{n\mathbf{k}} p^a{}_{n\mathbf{k}}{}^* - 1 - \frac{2}{\hbar}(a^R{}_{n\mathbf{k}} p^{aI}{}_{n\mathbf{k}} - a^I{}_{n\mathbf{k}} p^{aR}{}_{n\mathbf{k}})\}\rho_{a0} \quad (35a)$$

and for $\psi_{an\text{-}\mathbf{k}}$ is

$$\rho_{an-\mathbf{k}} = \{\frac{\omega}{4\pi\hbar} a_{n\mathbf{k}} a_{n\mathbf{k}}{}^* + \frac{4\pi}{\hbar\omega} p^a{}_{n\mathbf{k}} p^a{}_{n\mathbf{k}}{}^* - 1 + \frac{2}{\hbar}(a^R{}_{n\mathbf{k}} p^{aI}{}_{n\mathbf{k}} - a^I{}_{n\mathbf{k}} p^{aR}{}_{n\mathbf{k}})\}\rho_{a0} \qquad (35b)$$

for $k_z \geq 0$, while the Wigner distribution for $\psi_{qn\mathbf{k}}$ is

$$\rho_{qn\mathbf{k}} = \{\frac{\omega}{2\pi\hbar} q_{n\mathbf{k}}{}^2 + \frac{8\pi}{\hbar\omega} p^q{}_{n\mathbf{k}}{}^2 - 1\}\rho_{q0} \qquad (36)$$

for all **k**. Upon using the canonical transformations eq. (11a-d), we find that $\rho_{an\mathbf{k}} = \rho_{qn\mathbf{k}}$ and $\rho_{an\text{-}\mathbf{k}} = \rho_{qn\text{-}\mathbf{k}}$ so the Wigner phase space distributions of the first excited states are the same.

IX. **Conclusions**

The “a” method has the advantage that the wave function lives in the configuration space of the vector potential while the “q” method has the advantage of being more direct and simpler. By using a coherent state with the “a” method, a way for justifying the new creation and annihilation operators is found. Since the two methods are related by a canonical transformation they are just different ways of looking at the same situation.

The phase space distributions of the ground and first excited states are found to be the same for the two methods, but it is not apparent that a corresponding unitary transformation can be found between the two wave functions.

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