The Tomonaga-Schwinger Equation in flat Space-Time

John French

email: johnfrenchng2@yahoo.com

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Abstract

A derivation of the Tomonaga-Schwinger equation is presented based on defining a wave functional on a general space-like surface such that the expectation values match those in the corresponding Heisenberg picture. This derivation is carried out in flat space-time.

I. Introduction

This paper derives an equation for the wavefunctional of a scalar field on a general space-like surface. This surface is embedded in a flat space-time so there is no space-time curvature. The resulting equation is the same as the Tomonaga-Schwinger equation. Hollands and Wald[1], and Birrell and Davies[2], give reviews of quantizing a scalar field on a general curved space-time but do not address the method used in this paper. Although we take the space-time to be flat there is no use of the Lorentz transformations in the derivation, and the same result could apply to a Galilean space with time added.

Tomonaga[3] and Schwinger[4] developed what has become known as the Tomonaga-Schwinger equation while developing methods for dealing with relativistic Quantum Electrodynamics. In a method similar to Tomonaga and Schwinger, Schweber[5] looks at the Tomonaga-Schwinger equation in the interaction picture. Kramer[6] postulates a generalized Tomonaga-Schwinger equation and uses it to look at quantization on a general space-time surface. Wakita[7], Pradhan[8], Nishijima[9] and Roman[10] look at solving the Tomonaga-Schwinger equation in particular situations.

Our derivation of the Tomonaga-Schwinger equation for a general space-like surface in a flat space-time is accomplished by equating the expectation values of the corresponding Heisenberg quantization with those of a wave functional defined on a curved space-like surface. The interaction Hamiltonian density in the Tomonaga-Schwinger equation is replaced by the full Hamiltonian density, and we use the Schrodinger picture instead of the interaction picture which Tomonaga and Schwinger use.

II. Functional Quantization of a Scalar Field

Consider a real scalar field $\phi(\mathbf{x},t)$ which has a Lagrangian density $\mathscr{L}(\phi,\nabla\phi,\frac{\partial\phi}{\partial t})$, conjugate momentum density $\pi=\frac{\partial\mathscr{L}}{\partial\frac{\partial\phi}{\partial t}}$ and Hamiltonian density $\mathscr{H}(\phi,\pi)=\pi\frac{\partial\phi}{\partial t}-\mathscr{L}$. $\mathbf{x}$ represents the spatial coordinates and t represents time. We quantize this in the functional Schrodinger representation by defining a wave functional $\psi(\phi,t)$ which obeys the equation

$$i\hbar\frac{\partial\psi}{\partial t}=\hat{H}\psi \tag{1}$$

where $\hat{H}=\int dv\hat{\mathscr{H}}_{\mathbf{x}}$ is the Hamiltonian operator with $\hat{\mathscr{H}}_{\mathbf{x}}=\mathscr{H}_{H}(\phi(\mathbf{x}),-i\hbar\frac{\delta}{\delta\phi(\mathbf{x})})$. $\mathscr{H}_{H}$ is the Hamiltonian density $\mathscr{H}$ arranged so that $\hat{\mathscr{H}}_{\mathbf{x}}$ is a Hermitian operator. We use a hat to represent a quantum mechanical operator. For a review of this representation see Jackiw[11]

Break up space into an infinite set of finite volume elements δv with a point $\mathbf{x}$ associate with each element. Then consider a real function $A_{\mathbf{x}}(\phi(\mathbf{x},t),\pi(\mathbf{x},t))$ associated with each point **x** such that $A_{\mathbf{x}}$ is never zero, and only different from 1 for a finite set of points.

We then define a function $A_\sigma = \prod_{\mathbf{x}} A_{\mathbf{x}}(\phi(\mathbf{x},\sigma(\mathbf{x})),\pi(\mathbf{x},\sigma(\mathbf{x})))$ where $t=\sigma(\mathbf{x})$ is a space-like surface. $\prod_{\mathbf{x}}$ is a product over all the **x** points. Although the product is over an infinite number of points, A_σ is finite because of the way $A_{\mathbf{x}}$ is defined above. If σ is independent of **x** then the surface is at a constant time t. In this case set $A_\sigma = A_t$ which becomes the Hermitian operator $\hat{A}_t = \prod_{\mathbf{x}} \hat{A}_{\mathbf{x}}$ where $\hat{A}_{\mathbf{x}} = A_{H\mathbf{x}}(\phi(\mathbf{x}),-i\hbar\frac{\delta}{\delta\phi(\mathbf{x})})$. $A_{H\mathbf{x}}$ is the function $A_{\mathbf{x}}$ arranged so that $\hat{A}_{\mathbf{x}}$ is a Hermitian operator. The expectation value of A_t is given by

$$\langle A_t \rangle = \int D\phi \psi^* \hat{A}_t \psi \qquad (2)$$

where $\int D\phi$ is a functional integration and a * represents the complex conjugate.

III. Corresponding Heisenberg Quantization

There is a corresponding Heisenberg picture using states $|\psi\rangle$ and $\langle\psi|$ which are independent of time. ϕ and π become time dependent operators $\hat{\phi}_h(\mathbf{x},t)$ and $\hat{\pi}_h(\mathbf{x},t)$ which act upon $|\psi\rangle$. Since the operators $\hat{\phi}_h$ and $\hat{\pi}_h$ now depend upon time the function A_t becomes the time dependent operator $\hat{A}_h(t) = \prod_{\mathbf{x}} \hat{A}_{\mathbf{x}h}(\mathbf{x},t)$ where $\hat{A}_{\mathbf{x}h}(\mathbf{x},t) = A_{H\mathbf{x}}(\hat{\phi}_h(\mathbf{x},t),\hat{\pi}_h(\mathbf{x},t))$. We also have the Hamiltonian operator defined by $\hat{H}_h(t) = \int dv \hat{\mathscr{H}}_{\mathbf{x}h}(\mathbf{x},t)$ where $\hat{\mathscr{H}}_{\mathbf{x}h}(\mathbf{x},t) = \mathscr{H}_H(\hat{\phi}_h(\mathbf{x},t),\hat{\pi}_h(\mathbf{x},t))$. $\hat{A}_{\mathbf{x}h}(\mathbf{x},t)$ obeys the relation

$$\frac{\partial \hat{A}_{\mathbf{x}h}}{\partial t} = \frac{i}{\hbar}[\hat{H}_h, \hat{A}_{\mathbf{x}h}] = \frac{i}{\hbar}[\int dv' \hat{\mathscr{H}}_{\mathbf{x}'h}, \hat{A}_{\mathbf{x}h}]$$

$$= \frac{i}{\hbar} \sum_{\mathbf{x}'} \delta v [\hat{\mathscr{H}}_{\mathbf{x}'h}, \hat{A}_{\mathbf{x}h}] = \frac{i}{\hbar} \delta v [\hat{\mathscr{H}}_{\mathbf{x}h}, \hat{A}_{\mathbf{x}h}] \qquad (3)$$

since only the $\hat{\mathscr{H}}_{\mathbf{x}'h}$ at $\mathbf{x}' = \mathbf{x}$ may not commute with $\hat{A}_{\mathbf{x}h}$, and we have written $\int dv = \sum_{\mathbf{x}'} \delta v$ with δv independent of **x**'.

The expectation value of A_t is given by

$$\langle A_t \rangle = \langle \psi | \hat{A}_h | \psi \rangle = \langle \psi | \prod_{\mathbf{x}} \hat{A}_{\mathbf{x}h} | \psi \rangle \qquad (4)$$

By using eq. (3) we can define the $\hat{A}_{\mathbf{x}h}$ at different times, and in general define an operator

$$\hat{A}_{\sigma h} = \prod_{\mathbf{x}} \hat{A}_{\mathbf{x}h} (\mathbf{x}, \sigma(\mathbf{x})) \qquad (5)$$

Since $\hat{A}_{\sigma h}$ is no longer a constant function of time there is no longer a corresponding Schrodinger operator, but we can still calculate the expectation value given by

$$\langle A_\sigma \rangle = \langle \psi | \hat{A}_{\sigma h} | \psi \rangle = \langle \psi | \prod_{\mathbf{x}} \hat{A}_{\mathbf{x}h} (\mathbf{x}, \sigma(\mathbf{x})) | \psi \rangle \qquad (6)$$

which defines the expectation value of A_σ on the $t = \sigma(\mathbf{x})$ surface.

IV. Wave Functional on space-like surface and the Tomonaga-Schwinger Equation

Now define a wave functional $\psi'(\phi,\sigma(\mathbf{x}))$ on the surface $t=\sigma(\mathbf{x})$ such that it has the same expectation value of A_σ as in the Heisenberg formulation. We then need

$$\int D\phi\psi'^*(\phi,\sigma(\mathbf{x}))\hat{A}_\sigma\psi'(\phi,\sigma(\mathbf{x})) = \langle\psi|\hat{A}_{\sigma h}|\psi\rangle \qquad (7)$$

where $\hat{A}_\sigma = \prod_{\mathbf{x}} \hat{A}_{\mathbf{x}}$ so it is the same form of operator as $\hat{A}_t$ but now operates on $\psi'(\phi,\sigma(\mathbf{x}))$.

Now consider another surface $t=\sigma'(\mathbf{x})$ close to the first one, and set $\delta\sigma(\mathbf{x}) = \sigma'(\mathbf{x}) - \sigma(\mathbf{x})$.

Then to first order we have

$$\int D\phi\psi'^*(\phi,\sigma'(\mathbf{x}))\hat{A}_{\sigma'}\psi'(\phi,\sigma'(\mathbf{x})) - \int D\phi\psi'^*(\phi,\sigma(\mathbf{x}))\hat{A}_\sigma\psi'(\phi,\sigma(\mathbf{x}))$$

$$= \langle\psi|\hat{A}_{\sigma' h}|\psi\rangle - \langle\psi|\hat{A}_{\sigma h}|\psi\rangle$$

$$= \langle\psi|\prod_{\mathbf{x}} \hat{A}_{\mathbf{x}h}(\mathbf{x},\sigma'(\mathbf{x}))|\psi\rangle - \langle\psi|\prod_{\mathbf{x}} \hat{A}_{\mathbf{x}h}(\mathbf{x},\sigma(\mathbf{x}))|\psi\rangle$$

$$= \langle\psi|\prod_{\mathbf{x}} \{\hat{A}_{\mathbf{x}h}(\mathbf{x},\sigma(\mathbf{x})) + \delta\sigma(\mathbf{x})\frac{\partial}{\partial t}\hat{A}_{\mathbf{x}h}(\mathbf{x},\sigma(\mathbf{x}))\}|\psi\rangle - \langle\psi|\prod_{\mathbf{x}} \hat{A}_{\mathbf{x}h}(\mathbf{x},\sigma(\mathbf{x}))|\psi\rangle$$

$$= \sum_{\mathbf{x}'}\langle\psi|\delta\sigma(\mathbf{x}')\frac{\partial}{\partial t}\hat{A}_{\mathbf{x}'h}(\mathbf{x}',\sigma(\mathbf{x}'))\prod_{\mathbf{x}\neq\mathbf{x}'} \hat{A}_{\mathbf{x}h}(\mathbf{x},\sigma(\mathbf{x}))|\psi\rangle \qquad (8)$$

Using eq. (3), eq. (8) becomes

$$\int D\phi\psi'^*(\phi,\sigma'(\mathbf{x}))\hat{A}_{\sigma'}\psi'(\phi,\sigma'(\mathbf{x})) - \int D\phi\psi'^*(\phi,\sigma(\mathbf{x}))\hat{A}_{\sigma}\psi'(\phi,\sigma(\mathbf{x}))$$

$$= \frac{i}{\hbar}\delta v\sum_{\mathbf{x}'}\langle\psi|\delta\sigma(\mathbf{x}')[\hat{\mathscr{H}}_{\mathbf{x}'h},\hat{A}_{\mathbf{x}'h}(\mathbf{x}',\sigma(\mathbf{x}'))]\prod_{\mathbf{x}\neq\mathbf{x}'}\hat{A}_{\mathbf{x}h}(\mathbf{x},\sigma(\mathbf{x}))|\psi\rangle$$

$$= \frac{i}{\hbar}\delta v\sum_{\mathbf{x}'}\langle\psi|\delta\sigma(\mathbf{x}')[\hat{\mathscr{H}}_{\mathbf{x}'h},\hat{A}_{\sigma h}]|\psi\rangle \tag{9}$$

since $\hat{\mathscr{H}}_{\mathbf{x}'h}$ commutes with $\prod_{\mathbf{x}\neq\mathbf{x}'}\hat{A}_{\mathbf{x}h}(\mathbf{x},\sigma(\mathbf{x}))$. We have now set $\hat{\mathscr{H}}_{x'h} = \mathscr{H}_H(\hat{\phi}_h(\mathbf{x}',\sigma(\mathbf{x}')),\hat{\pi}_h(\mathbf{x}',\sigma(\mathbf{x}')))$.

Define $\delta\psi'(\phi,\sigma(\mathbf{x})) = \psi'(\phi,\sigma'(\mathbf{x})) - \psi'(\phi,\sigma(\mathbf{x}))$ so that

$$\int D\phi\psi'^*(\phi,\sigma'(\mathbf{x}))\hat{A}_{\sigma'}\psi'(\phi,\sigma'(\mathbf{x})) - \int D\phi\psi'^*(\phi,\sigma(\mathbf{x}))\hat{A}_{\sigma}\psi'(\phi,\sigma(\mathbf{x}))$$

$$= \int D\phi\delta\psi'^*(\phi,\sigma(\mathbf{x}))\hat{A}_{\sigma}\psi'(\phi,\sigma(\mathbf{x})) + \int D\phi\psi'^*(\phi,\sigma(\mathbf{x}))\hat{A}_{\sigma}\delta\psi'(\phi,\sigma(\mathbf{x})) \tag{10}$$

ignoring second order terms, and using the fact that $\int D\phi$ is a linear operator. We have also used the fact that $\hat{A}_{\sigma'} = \hat{A}_{\sigma}$.

Since $\delta\sigma(\mathbf{x}')[\hat{\mathscr{H}}_{\mathbf{x}'h},\hat{A}_{\sigma h}]$ is an operator we need the Heisenberg expectation values of this operator in eq. (9) to equal the expectation value in the Functional Schrodinger picture so that

$$\langle\psi|\delta\sigma(\mathbf{x}')[\hat{\mathscr{H}}_{\mathbf{x}'h},\hat{A}_{\sigma h}]|\psi\rangle = \int D\phi\psi'^*(\phi,\sigma(\mathbf{x}))\delta\sigma(\mathbf{x}')[\hat{\mathscr{H}}_{\mathbf{x}'},\hat{A}_{\sigma}]\psi'(\phi,\sigma(\mathbf{x}))$$

$$= \int D\phi\psi'^*\delta\sigma(\mathbf{x}')[\hat{\mathscr{H}}_{\mathbf{x}'}, \hat{A}_\sigma]\psi' \tag{11}$$

where now $\psi' = \psi'(\phi, \sigma(\mathbf{x}))$.

Using eqs. (10) and (11) in eq. (9) , expanding $[\hat{\mathscr{H}}_{\mathbf{x}'}, \hat{A}_\sigma]$ out, and setting $\delta\psi' = \delta\psi'(\phi, \sigma(\mathbf{x}))$, eq. (9) becomes

$$\int D\phi\delta\psi^* \hat{A}_\sigma\psi + \int D\phi\psi^* \hat{A}_\sigma\delta\psi$$

$$= \frac{i}{\hbar}\delta v\sum_{\mathbf{x}}\{\int D\phi\psi^*\delta\sigma(\mathbf{x})\hat{\mathscr{H}}_{\mathbf{x}}\hat{A}_\sigma\psi - \int D\phi\psi^*\delta\sigma(\mathbf{x})\hat{A}_\sigma \hat{\mathscr{H}}_{\mathbf{x}}\psi\} \tag{12}$$

where we have used the fact that $\int D\phi$ is a linear operator and dropped the primes on ψ and **x** since they are longer needed. Now define $\psi'' = \delta\sigma(\mathbf{x})\hat{A}_\sigma\psi$ so that

$$\int D\phi\psi^*\delta\sigma(\mathbf{x})\hat{\mathscr{H}}_{\mathbf{x}}\hat{A}_\sigma\psi = \int D\phi\psi^*\hat{\mathscr{H}}_{\mathbf{x}}\psi'' \tag{13}$$

and then since $\hat{\mathscr{H}}_{\mathbf{x}}$ is Hermitian we need

$$\int D\phi\psi^*\hat{\mathscr{H}}_{\mathbf{x}}\psi'' = \int D\phi(\hat{\mathscr{H}}_{\mathbf{x}}\psi)^*\psi'' \tag{14}$$

Using eqs. (13) and (14), eq. (12) takes the form

$$\int D\phi(\delta\psi + \frac{i}{\hbar}\delta v \sum_{\mathbf{x}} \hat{\mathscr{H}}_{\mathbf{x}}\, \psi\delta\sigma(\mathbf{x})\}^* \hat{A}_\sigma\psi$$

$$= -\int D\phi\psi^* \hat{A}_\sigma\{\delta\psi + \frac{i}{\hbar}\delta v \sum_{\mathbf{x}} \hat{\mathscr{H}}_{\mathbf{x}}\, \psi\delta\sigma(\mathbf{x})\} \qquad (15)$$

where we have exchanged $\hat{A}_\sigma$ and $\hat{\mathscr{H}}_{\mathbf{x}}\psi$ with $\delta\sigma(\mathbf{x})$ and used the fact that $\delta\sigma(\mathbf{x})$ is real and that $\int D\phi$ is a linear operator. Now set

$$\psi''' = \delta\psi + \frac{i}{\hbar}\delta v \sum_{\mathbf{x}} \hat{\mathscr{H}}_{\mathbf{x}}\, \psi\delta\sigma(\mathbf{x}) \qquad (16)$$

so that eq. (15) becomes

$$\int D\phi\psi'''^* \hat{A}_\sigma\psi = -\int D\phi\psi^* \hat{A}_\sigma\psi''' \qquad (17)$$

Since $\hat{A}_\sigma$ is an Hermitian operator we can write this as

$$\int D\phi(\hat{A}_\sigma\psi''')^* \psi = -\int D\phi\psi^* \hat{A}_\sigma\psi''' \qquad (18)$$

We will obtain this if we set $\psi''' = ic\psi$ where c is a real constant, since the expectation value of A_σ is real. Thus eq. (16) becomes

$$\delta\psi + \frac{i}{\hbar}\delta v\sum_{\mathbf{x}}\hat{\mathscr{H}}_{\mathbf{x}}\,\psi\delta\sigma(\mathbf{x}) = ic\psi \tag{19}$$

c can be absorbed into ψ by a phase shift.

In the limit of the surfaces only being different in a small area, so that $\delta\sigma(\mathbf{x})$ is only non-zero in this small area, we can approximate eq. (19) by

$$\delta\psi + \frac{i}{\hbar}\hat{\mathscr{H}}_{\mathbf{x}}\psi\delta\Omega = 0 \tag{20}$$

where $\delta\Omega = \delta v\sum_{\mathbf{x}}\delta\sigma(\mathbf{x})$ and we are taking $\hat{\mathscr{H}}_{\mathbf{x}}$ to be constant over that small area.

Dividing by $\delta\Omega$ we obtain

$$i\hbar\frac{\delta\psi}{\delta\Omega} = \hat{\mathscr{H}}_{\mathbf{x}}\psi \tag{21}$$

which is the Tomonaga-Schwinger equation. Note that the $\hat{\mathscr{H}}_{\mathbf{x}}$ is not a Hamiltonian density operator associated with equations for ϕ defined on the surface $\sigma(\mathbf{x})$ but the one associated with the flat space equations.

V. Conclusion

The Tomonaga-Schwinger equation that we come up with is not in the interaction picture and $\hat{\mathscr{H}}_{\mathbf{x}}$ is not the interaction Hamiltonian density as it is in the Tomonaga and Schwinger derivations. It might be possible extend this idea to more general frames such as an accelerating frame, or to use it to find an electromagnetic field quantization in a general frame.

There are some problems with the interpretation of the Tomonaga-Schwinger equation , (for example see Wakita[7]), but this paper does not look at those issues and just gives a plausible derivation of it.

VI. References

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