Accurate Calculation of the 21 cm Hydrogen Line

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(January 27, 2023)

Abstract

An electron is treated as a point particle, usually without considering the effect of its radius on interactions. This paper abandons this view, In Superfine Splitting of Hydrogen Ground State, this paper takes the charge radius displacement of the electron and the magnetic moment displacement inside the electron and the proton as two simple influencing factors, then deduces a new formula for calculating the 21 cm hydrogen line. The calculated value is compared with the laboratory measurement value, reaching 12 significant digits, which perfectly matches. Similarly. The calculation method in this paper is also applicable to the hyperfine splitting of the muonic hydrogen.

Introduction

When the ground state of hydrogen atom undergoes hyperfine splitting, the spin direction of electron reverse, Generally, we think that the magnetic field of the electron and the magnetic field of the proton will interact at this time. But if we look at it from the electron's point of view, the electron thinks it didn't have a spin flip, and the flip is a proton. When the spin direction of the proton is reversed, its spin direction will be the same as that of the neutron, and the electron will think that it is interacting with the neutron's magnetic field. Therefore, the following article will use the interaction of electron and neutron to derive the formula for the 21 cm line.

In some articles related to the derivation of the 21 cm line, the following formula usually appears [1]:

$$\nu = \frac{4\mathrm{A}}{h} \tag{1}$$

ν is the frequency of the 21 cm line; h is the Planck constant; 4A is the difference in energy levels of two hyperfine splitting of the ground state of a hydrogen atom. Now we begin to derive the exact formula for A.

In combination with the above, we assume that A is the magnetic energy generated by the interaction between the magnetic moment of electron and the magnetic field of neutron, and then:

$$\mathrm{A} = \mu_e \cdot B_n \tag{2}$$

μ_e is the magnetic moment of the electron; B_n is the magnetic field of the neutron. There is a magnetic moment formula $\mu = BV/\mu_0$, B is the magnetic induction intensity, V is the volume of the magnetic field distribution; μ_0 is the vacuum permeability. Use it to get the magnetic induction intensity of the neutron $B_n = \mu_0\mu_n/V$, substitute it into equation (2), and get:

$$\mathrm{A} = \frac{\mu_0\mu_n\mu_e}{V} \tag{3}$$

μ_n is the spin magnetic moment of the neutron. Let $V = \pi a_0^3$, a_0 is the Bohr radius. The spin magnetic moment of the neutron is: $\mu_n = g_n\mu_N S/\hbar$. the spin magnetic moment of the electron is: $\mu_e = g_e\mu_B S/\hbar$. Substitute them into the equation (3), which can be obtained:

$$\mathrm{A} = \frac{\mu_0 e^2}{64\pi^3}\frac{h^2}{a_0^3}\frac{g_n g_e}{m_p m_e} \tag{4}$$

g_n is the spin g-factor of the neutron; g_e is the spin g-factor of the electron; μ_B is the Bohr magneton; μ_N is the nuclear magneton; S is the spin angular momentum of the electron and the neutron. m_p is the mass of the proton; m_e is the mass of the electron; $\hbar$ is the reduced Planck constant; e is the elementary charge; π is the pi.

According to the Bohr model: $a_0 = h/2\pi m_e\alpha c$, α is a fine structural constant, substitute it into equation (4), and We get the formula for A, which is:

$$\mathrm{A} = \frac{e^2}{4\pi\varepsilon_0}\frac{g_n g_e}{4a_0}\frac{m_e}{m_p}\alpha^2 \tag{5}$$

Substitute the equation (5) into the equation (1), we get:

$$\nu = \frac{e^2}{4\pi\varepsilon_0}\frac{g_n g_e}{ha_0}\frac{m_e}{m_p}\alpha^2 \tag{6}$$

ε_0 is the vacuum permittivity. At this point, the calculation formula of the 21 cm line has been basically derived. But if you substitute the value of the physical quantity to calculate, you can find that it is very different from the laboratory measurement. The fundamental reason for this result is that the orbital radii of the Two hyperfine energy levels of the ground state of the hydrogen atom are not the same, and they have slight differences. Let this slight difference be ΔR, substitute it into the equation (6), and we will get:

$$\nu = \frac{e^2}{4\pi\varepsilon_0 h}\frac{g_n g_e}{(a_0 + \Delta R)}\frac{m_e}{m_p}\alpha^2 \tag{7}$$

For ΔR, we set it to come from the charge radius displacement of the electron, and the magnetic moment displacement of the electron and the proton. Since there is no ready-made formula for the charge radius and the radius of the magnetic moment, the following article attempts to derive them with Some new ideas.

Derivation of the charge radius formula

Let's start with the formula for deriving the charge radius of the proton. Protons have spins, and from classical mechanical theory, the spin speed of protons is faster than the speed of light. Since the speed of motion of matter cannot exceed the speed of light, we assume that the spin speed of protons is the speed of light. We further assume that when the proton spins, its spin radius is n times of its charge radius. Because angular momentum is quantized, then n must be equal to integer or semi-integer. The combined proton spin has angular momentum of $\hbar/2$, so the spin angular momentum formula of proton is:

$$m_p c n r_p = \frac{\hbar}{2} \tag{8}$$

r_p is the charge radius of the proton. The value of the proton charge radius is 0.8414 fm [2]. We substitute it into equation (8) to find the value of n. through calculation, we get $n \approx 0.12498$, which is very close to 0.125, that is $1/8$. Since n is equal to an integer or a half integer. So, we take $n = 1/8$, and substitute it into equation (8). Then:

$$m_p c r_p = 4\hbar \tag{9}$$

Calculated by equation (9), the charge radius of the proton obtained is $r_p = 8.412356413422 \times 10^{-16}\ m$. It shows that the mass of the proton spins at its radius of charge $1/8$, and the resulting angular momentum is $\hbar/2$. Of course, we need to find out the charge radius of the electron. Now let's assume that equation (9) applies to all fermions and write equation (9) as generic:

$$R_Q = \frac{4\hbar}{Mc} \tag{10}$$

R_Q is the charge radius of a fermion; M is the mass of fermions. By substituting the mass of the electron into equation (10) for calculation, we get the radius of charge of the electron, which is $r_e = 1.544637071844 \times 10^{-12}\ m$. We then substitute the charge radius of the electron into equation (7), and we get:

$$\nu = \frac{e^2}{4\pi\varepsilon_0 h} \frac{g_n g_e}{(a_0 + r_e)} \frac{m_e}{m_p} \alpha^2 \tag{11}$$

According to the calculation of formula (11), the frequency of the 21 cm line is 1,420,429,203.6672 HZ. (In order to ensure the calculation accuracy, the values of all physical quantities in this paper are from 2018 CODATA.)

The laboratory hydrogen line frequency measured with Hydrogen Pulse is [3]:

$$\nu = 1{,}420{,}405{,}751.786(30)\ \text{HZ} \tag{12}$$

Compared with the two, it can be seen that the results are not very good, indicating that there are still some displacements that are not taken into account.

Derivation of the magnetic moment displacement formula

When the ground state of the hydrogen atom is hyperfinely split, the spin direction of the electron is reversed, and the magnetic field of the electron and the magnetic field of the proton become mutually exclusive, which will lead to the displacement of their magnetic moment positions within the electrons and protons under the action of the magnetic force. In the previous formula (8), we get $n = 1/8$. Now we take the electron as an example. We first derive the electric field force F_e and magnetic field force F_B at the charge radius $1/8$ of the electron, then:

$$F_e = \frac{e^2}{4\pi\varepsilon_0}\frac{1}{(r_e/8)^2}$$

$$F_B = \mu_0 I^2 = \mu_0 \left(\frac{\mu_e}{S}\right)^2 = \mu_0 \frac{\mu_e^2}{\pi^2 (r_e/8)^4} \tag{13}$$

It can be known from the magnetic moment formula that $I = \mu_e/S$, I is the annular current inside the electron; S is the area surrounded by the annular current inside the electron. $S = \pi(r_e/8)^2$. Now, we 're going to derive the ratio of the electric field force to the magnetic field force at the $r_e/8$, combined with the spin magnetic moment of the electron in front: $\mu_e = g_e \mu_B S/\hbar$, there is:

$$\frac{F_e}{F_B} = \frac{\frac{e^2}{4\pi\varepsilon_0}\frac{1}{(r_e/8)^2}}{\mu_0 \frac{\mu_e^2}{\pi^2 (r_e/8)^4}} = \frac{\pi^3 m_e^2 c^2 r_e^2}{4 g_e^2 h^2} \tag{14}$$

Combined with the formula for the fermion charge radius of the equation (10), the equation (14) is simplified to:

$$\frac{F_e}{F_B} = \frac{\pi}{g_e^2} \tag{15}$$

It can be seen from equation (15) that the magnetic field force is g_e^2/π times larger than the electric field force.

Now let's find the radius of action of the magnetic force: R_{Be}. The acting radius can be understood as the acting distance of the force. We assume that the action radius of the electric field force is the classical radius of the electron, which is R_e, and there is a relationship: $R_e/R_{Be} = F_e/F_B$, in combination with equation (15), the radius of action of the magnetic field force is:

$$R_{Be} = \frac{R_e g_e^2}{\pi} \tag{16}$$

Equation (10) combined with the classical radius formula of the electron $R_e = e^2/4\pi\varepsilon_0 m_e c^2$, it is not difficult to see that the charge radius of the electron and the classical radius of the electron have a relationship:

$$R_e = \frac{\alpha}{4} r_e \tag{17}$$

Referring to equation (17), we derive the classical radius of proton: $R_p = \alpha r_p/4$. And then referring to equation (16), we derive the radius of action of magnetic force of proton R_{Bp}:

$$R_{Bp} = \frac{R_p g_e^2}{\pi} = \frac{\alpha r_p g_p^2}{4\pi} \tag{18}$$

g_p is the spin g-factor of the proton. Now we have found the radius of action of the magnetic field force of the electron and the proton, but we have not yet found their magnetic moment displacement. In the Bohr model of the hydrogen atom, the orbital of the electron is quantized, and the orbital radius of the electron r_n is related to the Bohr radius: $r_n = n^2 a_0$, and $n = 1.2.3 \cdots$ We assume that the magnetic moment displacement inside electrons and protons is also quantized, and their quantization mode is related to the radius of action of magnetic field force, which is also similar to the Bohr model. Therefore, we obtain the formula of the magnetic moment displacement $R_{\mu e}$ of the electron and the formula of the magnetic moment displacement $R_{\mu p}$ of the proton, as follows:

$$R_{\mu e} = \frac{R_e g_e^2}{n_e^2 \pi}$$

$$R_{\mu p} = \frac{\alpha r_p g_p^2}{n_p^2 4\pi} \tag{19}$$

$n_e, n_p = 1.2.3 \cdots$ From equation (18) we can see that the larger the value of n_e, n_p, the smaller the magnetic moment displacement of the electron and the proton. Now we substitute the magnetic moment displacement formula of the electron into the equation (11), then there is:

$$\nu = \frac{e^2}{4\pi\varepsilon_0 h} \frac{g_n g_e}{\left(a_0 + r_e + \frac{R_e g_e^2}{n_e^2 \pi}\right)} \frac{m_e}{m_p} \alpha^2 \tag{20}$$

Assuming $n_e = 2$, and calculating equation (20), We can get that the frequency of the 21 cm line is 1,420,405,752.0133 HZ. Compared with the laboratory measurement value of equation (12), the calculated value of equation (20) reaches 9 significant digits. Magnetic force can make the magnetic moment of electron displacement, also can make the magnetic moment of proton displacement. Now we include the magnetic moment displacement of proton into the influencing factors, so the complete formula of ΔR is: $\Delta R = r_e + R_{\mu e} + R_{\mu p}$. Substituting it into equation (20), there is:

$$\nu = \frac{e^2}{4\pi\varepsilon_0 h} \frac{g_n g_e}{\left(a_0 + r_e + R_{\mu e} + R_{\mu p}\right)} \frac{m_e}{m_p} \alpha^2$$

Or:

$$\nu = \frac{e^2}{4\pi\varepsilon_0 h} \frac{g_n g_e}{\left(a_0 + r_e + \frac{R_e g_e^2}{n_e^2 \pi} + \frac{\alpha r_p g_p^2}{n_p^2 4\pi}\right)} \frac{m_e}{m_p} \alpha^2 \tag{21}$$

$n_e = 2$ it means that the magnetic moment displacement of the electron is $1/4$ of the radius of action of the magnetic force. Now we need to find n_p. Let's say n_e, n_p has such a relationship:

$$n_e + n_p = \sqrt{\frac{m_p}{m_e}} \tag{22}$$

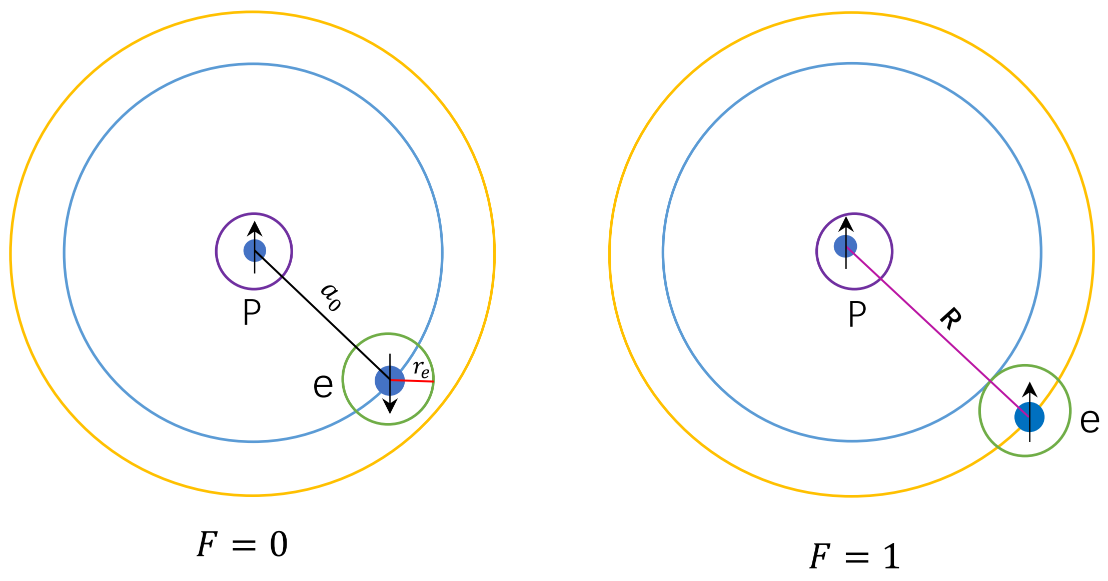
By calculating that there is: $n_e + n_p = 42.85$, since n_e, n_p they are all positive integers, we rounded it up, and there is: $n_e + n_p = 43$, because $n_e = 2$, then there is $n_p = 41$, and by substituting it into equation (21), we get that the frequency of the 21 cm line is:

$$v = 1{,}420{,}405{,}751.7845 \text{ HZ} \tag{23}$$

Comparing the laboratory measurement of equation (12), it can be found that the calculated value of equation (21) in this paper is as high as 12 significant digits, which is consistent with the high accuracy of the laboratory measurement.

The calculation value of the equation (21) requires high accuracy. We take value of g_n is 3.82608545, if you take value of g_n is 3.82608546, your result will be only 9 significant digits, and the value of m_p/m_e is 1836.15267343.

The following is the hyperfine splitting diagram of the ground state of hydrogen atom drawn according to the principle of this paper



Schematic diagram: Hyperfine splitting of ground state of hydrogen atom

Schematic introduction:

P is the proton, which is the purple cricle;

e is the electron, which is the green cricle.

a_0 is the Bohr radius; r_e is the charge radius of the electron.

R is the orbital radius after the electron spin flip. $R = a_0 + \Delta R$.

The blue circle in the diagram is the ground-state orbit of F=0.

The orange circle is the ground-state orbit of F=1.

The blue dot is the position of the magnetic moment inside the electron and the proton.

The direction of the arrow indicates the direction of the spin magnetic moment.

From the above schematic diagram and the previous formula derivation process, we can now understand more details and reasons for the hyperfine splitting of the ground state of hydrogen atom. In addition to the reversal of the spin direction of electron, there is also an electron radius moving outward in the ground state orbit. At the same time, their magnetic moment positions are also moving inside the electron and proton. The distance between the two hyperfine energy levels of the ground state of the hydrogen atom is about one electron radius, and it can also be known that the orbital radius in this paper refers to the distance between the two magnetic moments inside the electron and the proton.

So far in this paper, we have deduced the formula of hyperfine splitting of the ground state of hydrogen atom. From the above derivation process, we can see that the hyperfine splitting of hydrogen atomic ground state only needs to consider the displacement of the electron charge radius and the influence of the magnetic moment displacement inside the electron and proton. All other factors need not be considered. We can obtain a high-precision formula for calculating the frequency of hydrogen lines. In addition to describing the hyperfine splitting of the ground state of the hydrogen atom, the formula in this paper can also be used to analyze the hyperfine splitting of the muonic hydrogen.

Below we try to calculate the hyperfine splitting of the muonic hydrogen ground state.

Since the mass of the muon is larger, the orbital radius of the muon is smaller and closer to the proton after binding to the proton. Therefore, the case of the muonic hydrogen is more complicated than that of the electronic hydrogen, but the formula is still applicable. Now we refer to the equation (21) of the electron hydrogen and make some minor changes:

$$\nu_{\mu\mathrm{H}} = \frac{1}{2}\frac{e^2}{4\pi\varepsilon_0 h}\frac{g_p g_\mu}{\left(a_\mu + r_\mu + r_p + R_{\mu\mu} + R_{\mu p}\right)}\frac{m_\mu}{m_p}\alpha^2 \tag{24}$$

Here are only physical quantities that have not appeared before:

m_μ is the quality of the muon;

a_μ is the Bohr radius of the muonic hydrogen;

r_μ is the charge radius of the muon;

$R_{\mu\mu}$ is the magnetic moment displacement of the muon;

g_μ is the spin g-factor of the muon;

$\nu_{\mu\mathrm{H}}$ is the transition frequency between the hyperfine energy levels of the muonic hydrogen ground state.

From equation (24), it can be found that the object of action of the magnetic moment of the lepton has become a proton, which is not the same as the case of the hydrogen atom. In addition, the charge radius displacement of the proton is added, and the energy of the hyperfine structure is reduced by half.

According to the photon energy formula: $E = h\nu$, we transform the formula (24) into the formula of the $E_{1S-\mathrm{hfs}}(\mu\mathrm{H})$, which is the energy corresponding to $\nu_{\mu\mathrm{H}}$, then there is:

$$E_{1S-\mathrm{hfs}}(\mu\mathrm{H}) = \frac{e^2}{8\pi\varepsilon_0}\frac{g_p g_\mu}{\left(a_\mu + r_\mu + r_p + R_{\mu\mu} + R_{\mu p}\right)}\frac{m_\mu}{m_p}\alpha^2 \tag{25}$$

Below we give some concrete formulas of new physical quantities.

$a_\mu = \frac{\hbar}{m_\mu \alpha c}$ By referring to Bohr radius formula.

$r_\mu = \frac{4\hbar}{m_\mu c}$ By referring to equation (10).

$R_{\mu\mu} = \frac{\alpha r_\mu g_\mu^2}{n_\mu^2 \pi}$ By referring to equation (19).

Combined with the above, substitute them into equation (25), we get:

$$E_{1S-\mathrm{hfs}}(\mu\mathrm{H}) = \frac{e^2}{8\pi\varepsilon_0}\frac{g_p g_\mu}{\left(a_\mu + r_\mu + r_p + \frac{\alpha r_\mu g_\mu^2}{n_\mu^2\pi} + \frac{\alpha r_p g_p^2}{n_p^2\pi}\right)}\frac{m_\mu}{m_p}\alpha^2 \tag{26}$$

According to the above, there is $n_\mu + n_p = \sqrt{\frac{m_p}{m_\mu}} \approx 3$. Since the mass of the proton is greater than the mass of the muon, we take $n_\mu = 1,\ n_p = 2$ and substitute them into equation (26) for calculation, and the result we get is:

$$E_{1S-\mathrm{hfs}}(\mu\mathrm{H}) = 182.729\ \mathrm{meV} \tag{27}$$

The latest laboratory measurement is [4]:

$$E_{1S-\mathrm{hfs}}(\mu\mathrm{H}) = 182.634\ \mathrm{meV} \tag{28}$$

Comparing the above two, the result of this paper is generally consistent. But this is enough to show that the formula in this paper can be generalized. Since the situation of the muonic hydrogen is more complicated than that of the electronic hydrogen, this paper only considers the charge radius displacement and the magnetic moment displacement, and all other influencing factors are not considered. Therefore, there are many methods to eliminate the small differences in this paper. So, here this paper will not give any examples.

References

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