A Compact Notation for Massive Spinors

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Massive angle and square spinors are described as two-vectors with an index denoting their helicity sign category and the property that the order of the components must be swapped for negative sign. Relations between spinors can be written more compactly and several derivations are simplified. Three point amplitudes are investigated and it is shown, that their high energy limit can be obtained more easily for both helicity categories at once.

1. Introduction

The spinor helicity formalism, see for example the reviews in [1-4], is widely used for the calculation of amplitudes in particle physics. Massless states have only two helicities, positive and negative and exactly they are employed for amplitudes avoiding a lot of redundancy. Massive spinor helicity variables were introduced in [5], [6] and [7]. For massless particles the little group is U(1) and for massive particles the little group is SU(2). Massive particles are described by spinors $\lambda_\alpha^I, \tilde{\lambda}_{\dot\alpha}^I$ where $\alpha, \dot\alpha$ denote the $SL(2,\mathbb{C})$ indices and I, J the SU(2) spin indices. Amplitudes within this new formalism were investigated in [8-14].
Amplitudes are usually considered in a certain helicity configuration and other helicity configurations are obtained by parity, cyclicity or other symmetries. In this work we describe massive helicity spinors together as two-vectors $\left| i^I \right)_\sigma = \left(\left| i \right)_\sigma \quad \left| n_i \right)_\sigma \right)$, where σ denotes the helicity category of the spinors, which agrees with the helicity sign of the massless spinor $\left| i \right)_\sigma$ remaining in the high energy limit. For $\sigma = +$ the entries of the two-vectors are in the shown order, which must be swapped for $\sigma = -$. This property also applies to contractions of spinors and their products appearing in amplitudes. With this notation one can derive the high energy limit of amplitudes for both helicity sign categories more easily, as is shown in the following sections.

2. Compact Notation for Massive Spinors

We investigate massive spinors introduced in [7] and take over the notation of [12],[14],[15]. In Appendix A we provide an explicit representation of massive spinors using the two-vector notation of [16] which was later slightly modified in [18],[19]. Massive spinors are given by a pair of massless spinors $\lambda_\alpha^I = \left| i^I \right\rangle$ and $\tilde{\lambda}^{\dot\alpha I} = \left| i^I \right]$ (I=1,2) and we denote them together as $\left| i^I \right)_\sigma$.

$$\left| \mathbf{i} \right)_\sigma = \left| i^I \right)_\sigma = \left| i_\sigma^I \right) = \begin{Bmatrix} \left| i^{\dot\alpha, I} \right] & \sigma = + \\ \left| i_\alpha^I \right\rangle & \sigma = - \end{Bmatrix}, \quad \left(\mathbf{i} \right|_\sigma = \left(i^I \right|_\sigma = \left(i_\sigma^I \right| = \begin{Bmatrix} \left[i_{\dot\alpha}^I \right| & \sigma = + \\ \left\langle i^{\alpha, I} \right| & \sigma = - \end{Bmatrix} \tag{1}$$

The sign σ denotes the helicity category [13] of the massive spinor and corresponds to the helicity sign of the massless spinor remaining in the high energy limit. Contractions are only possible between spinors with the same sign σ

$$\left(\mathbf{i} \ \mathbf{j} \right)_\sigma = \begin{Bmatrix} \left[i^I \ j^J \right] & \sigma = + \\ \left\langle i^I \ j^J \right\rangle & \sigma = - \end{Bmatrix} \tag{2}$$

The momentum of a massive particle with momentum $\mathbf{p_i}$ is given as $\mathbf{p_i} = -\sigma \left| i^I \right)_\sigma \left(i_I \right|_{-\sigma}$ or equivalently

$$\mathbf{p}_i = \sigma\left|\, i^I \right)_{-\sigma}\left(\, i_I \,\right|_\sigma = \begin{Bmatrix} +\left|\, i^I \right\rangle\left[\, i_I \,\right| = \mathbf{p}_{i\alpha\dot{\alpha}} \,,\ \sigma = + \\ -\left|\, i^I \right]\left\langle\, i_I \,\right| = \bar{\mathbf{p}}_i^{\dot{\alpha}\alpha} \,,\ \sigma = - \end{Bmatrix} \tag{3}$$

The relations between massive spinors given in [12], [14] can be now written in a compact form $\left(a, b = \alpha, \beta \text{ or } \dot{\beta}, \dot{\alpha}\right)$.

$$\begin{aligned} &\left(\, i^J \;\; i^K \,\right)_\sigma = \sigma m_i \,\epsilon^{JK} \,,\ \left(\, i_J \;\; i_K \,\right)_\sigma = -\sigma m_i \,\epsilon_{JK} \,,\ \left(\, i^J \;\; i_K \,\right)_\sigma = -\sigma m_i \,\delta^J_K \,,\ \left(\, i_J \;\; i^K \,\right)_\sigma = \sigma m_i \,\delta^K_J \\ &\left(\, i^J \;\; i_J \,\right)_\sigma = -\left(\, i_J \;\; i^J \,\right)_\sigma = -\sigma 2m_i \,,\ \left|\, i^J \right)_\sigma\left(\, i_J \,\right|_\sigma = -\left|\, i_J \right)_\sigma\left(\, i^J \,\right|_\sigma = \sigma m_i \,\delta^b_a \\ &\mathbf{p}_i = \sigma\left|\, i^I \right)_{-\sigma}\left(\, i_I \,\right|_\sigma = -\sigma\left|\, i^J \right)_\sigma\left(\, i_J \,\right|_{-\sigma} ,\ \mathbf{p}_i\left|\, i^I \right)_{-\sigma} = m_i\left|\, i^I \right)_\sigma ,\ \left(\, i^I \,\right|_{-\sigma}\mathbf{p}_i = -m_i\left(\, i^I \,\right|_\sigma ,\ \varepsilon^{IJ} = -\varepsilon_{IJ} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \end{aligned} \tag{4}$$

Looking at the explicit representation in appendix A one asks if a more compact rewriting of all the spin-spinors in (A3) is possible. First we introduce the helicity sign operator h_σ, acting on spinors from the left, defined as $\left(\hat{p} = \vec{p} / |\vec{p}|\right)$:

$$h_\sigma = \hat{p} \cdot \vec{\sigma} = \begin{pmatrix} \left(cc - ss^*\right) & 2cs^* \\ 2cs & -\left(cc - ss^*\right) \end{pmatrix} \tag{5}$$

For conjugate spinors the helicity sign operator is $\bar{h}_\sigma = -h_\sigma$ acting from the right. Then one obtains with (A3) the equations $h_\sigma \left|\, i \right\rangle = -\left|\, i \right\rangle$, $h_\sigma \left|\, i \right] = +\left|\, i \right]$, $h_\sigma \left|\, n_i \right\rangle = +\left|\, n_i \right\rangle$, $h_\sigma \left|\, n_i \right] = -\left|\, n_i \right]$ and similar equations for the conjugate spinors. The explicit expressions for these spinors can be obtained from appendix (A3). With the abbreviations (similar for mirror spinors)

$$\left|\, i \right)_\sigma = \begin{Bmatrix} \left|\, i \right] \ \sigma = + \\ \left|\, i \right\rangle \ \sigma = - \end{Bmatrix} ,\ \left|\, n_i \right)_\sigma = \begin{Bmatrix} \left|\, n_i \right] \ \sigma = + \\ \left|\, n_i \right\rangle \ \sigma = - \end{Bmatrix} \tag{6}$$

these equations can be written in compact form (for later use we also note the action on spinors with sign $-\sigma$)

$$\begin{aligned} &h_\sigma \left|\, i \right)_\sigma = \sigma\left|\, i \right)_\sigma ,\ h_\sigma \left|\, n_i \right)_\sigma = -\sigma\left|\, n_i \right)_\sigma ,\ \left(\, i \right|_\sigma \bar{h}_\sigma = \sigma\left(\, i \right|_\sigma ,\ \left(\, n_i \right|_\sigma \bar{h}_\sigma = -\sigma\left(\, n_i \right|_\sigma \\ &h_\sigma \left|\, i \right)_{-\sigma} = -\sigma\left|\, i \right)_{-\sigma} ,\ h_\sigma \left|\, n_i \right)_{-\sigma} = \sigma\left|\, n_i \right)_{-\sigma} ,\ \left(\, i \right|_{-\sigma} \bar{h}_\sigma = -\sigma\left(\, i \right|_{-\sigma} ,\ \left(\, n_i \right|_{-\sigma} \bar{h}_\sigma = \sigma\left(\, n_i \right|_{-\sigma} \end{aligned} \tag{7}$$

In summary square (conjugate) i-spinors have positive helicity sign and angle (conjugate) i-spinors negative helicity sign, which is reversed for n_i-spinors. We note that for the spin-spinors in appendix A with upper index I, the first entry always has positive helicity and the second entry has negative helicity, while for spin-spinors with lower index I the situation is reversed. This suggests that the SU(2) indices I,J should run over $\{+,-\}$, see also [14] appendix A. With this convention one could now describe for example $\left|\, i^I \right]$ and $\left|\, i^I \right\rangle$ together as $\left|\, i^I \right)_\sigma = \left|\, i \right)_\sigma \delta^I_\sigma + \left|\, n_i \right)_\sigma \delta^{-I}_\sigma$. It would however be cumbersome to work in amplitudes, containing contractions of these spinors or products of them, with these δ^I_σ terms. We therefore suggest the following notation, mirror spinors are obtained by $\left|\ \ \right) \to \left(\ \ \right|$:

$$\begin{aligned} &\left|\, i^I \right)_\sigma = \left(\left|\, i \right)_\sigma \quad \left|\, n_i \right)_\sigma\right) ,\quad \left|\, i_I \right)_\sigma = \sigma\left(-\left|\, n_i \right)_\sigma \quad \left|\, i \right)_\sigma\right) \\ &\left|\, i^I \right)_{-\sigma} = \left(\left|\, n_i \right)_{-\sigma} \quad \left|\, i \right)_{-\sigma}\right) ,\ \left|\, i_I \right)_{-\sigma} = \sigma\left(-\left|\, i \right)_{-\sigma} \quad \left|\, n_i \right)_{-\sigma}\right) \end{aligned} \tag{8}$$

One can check that all spin-spinors in (A3) are correctly described by the first line. The two-vectors in (8) must be understood in the following way: for $\sigma = +$ the two entries of the vector are in the right order, while for $\sigma = -$ the two entries must be swapped. So we have for example $\left|\, i^I \right)_\sigma = \left(\left|\, i \right)_\sigma \quad \left|\, n_i \right)_\sigma\right) = \left(\left|\, i \right] \quad \left|\, n_i \right]\right)\big|_{\sigma=+}$ or $\left(\left|\, n_i \right\rangle \quad \left|\, i \right\rangle\right)\big|_{\sigma=-}$ and similarly for the mirror spinors. Massive spinors with index $-\sigma$ in (8), which are needed in amplitudes, are also obtained by swapping the two entries. Of course $\left(\, i \;\; i \,\right)_\sigma = \left(\, n_i \;\; n_i \,\right)_\sigma = 0$ and we note the important relation obtained from (A3)

$$\left(i\ n_i \right)_\sigma = m_i \tag{9}$$

With this notation it becomes very easy to prove several of the relations in (4) directly without using the explicit representation given in (A3). For $\sigma = -$ the two components of the vector $\left| i^1 \right)_\sigma$ must be exchanged. The momentum in (3) using $\sigma^2 = 1$ becomes: $\mathbf{p_i} = \sigma \left| i^1 \right)_{-\sigma} \left(i_I \right|_\sigma = \sigma \left(\left| n_i \right)_{-\sigma} \quad \left| i \right)_{-\sigma} \right) \cdot \sigma \left(-\left(n_i \right|_\sigma \quad \left(i \right|_{-\sigma} \right) = \left| i \right)_{-\sigma} \left(i \right|_\sigma - \left| n_i \right)_{-\sigma} \left(n_i \right|_\sigma$, with a dot product between the two-vectors (we write $\sigma = \pm 1$ in terms) and agrees with the explicit form in (A2).

$$\mathbf{p_i} = \sigma \left| i^1 \right)_{-\sigma} \left(i_I \right|_\sigma = \left| i \right)_{-\sigma} \left(i \right|_\sigma - \left| n_i \right)_{-\sigma} \left(n_i \right|_\sigma \tag{10}$$

3. High Energy Limit

We discuss the high energy limit (HE) in the present notation. Up to order $O\left(m^2\right)$ the i-spinors are proportional $\sqrt{E+P} \overset{HE}{\rightarrow} \sqrt{2E}\left(1 - \frac{m^2}{8E^2}\right)$ and leading, while the n_i-spinors proportional to $\sqrt{E-P} \overset{HE}{\rightarrow} \frac{m}{\sqrt{2E}}\left(1 + \frac{m^2}{8E^2}\right) \approx \frac{m}{\sqrt{2E}} = \frac{m}{2E}\sqrt{2E}$ can be neglected at first order in m. The high energy limit of the spinors in (8) up to $O(m^2)$ is therefore

$$\left| i \right)_\sigma \overset{HE}{\rightarrow} \left(1 - \frac{m_i^2}{8E_i^2}\right) \left| i_0 \right)_\sigma , \left| n_i \right)_\sigma \overset{HE}{\rightarrow} \frac{m_i}{2E_i} \left| n_{i0} \right)_\sigma \tag{11}$$

where $\left| i_0 \right)_\sigma$ and $\left| n_{i0} \right)_\sigma$ and its mirrors denote the same spinors as in (A3) but here with a pre factor of $\sqrt{2E_i}$, i.e.

$$\left| i_0 \right] = \sqrt{2E_i} \begin{pmatrix} c_i \\ s_i \end{pmatrix}, \left| i_0 \right\rangle = \sqrt{2E_i} \begin{pmatrix} -s_i^* \\ c_i \end{pmatrix}, \left| n_{i0} \right] = \sqrt{2E_i} \begin{pmatrix} -s_i^* \\ c_i \end{pmatrix} = \left| i_0 \right\rangle, \left| n_{i0} \right\rangle = \sqrt{2E_i} \begin{pmatrix} c_i \\ s_i \end{pmatrix} = \left| i_0 \right]$$
$$\left[i_0 \right| = \sqrt{2E_i} \begin{pmatrix} -s_i \\ c_i \end{pmatrix}, \left\langle i_0 \right| = \sqrt{2E_i} \begin{pmatrix} c_i \\ s_i^* \end{pmatrix}, \left[n_{i0} \right| = -\sqrt{2E_i} \begin{pmatrix} c_i \\ s_i^* \end{pmatrix} = -\left\langle i_0 \right|, \left\langle n_{i0} \right| = -\sqrt{2E_i} \begin{pmatrix} -s_i \\ c_i \end{pmatrix} = -\left[i_0 \right|$$

The high energy limit of spinor contractions up to $O(m^2)$ is thereby given as

$$\left(i\ j \right)_\sigma \overset{HE}{\rightarrow} \left(1 - \frac{m_i^2}{8E_i^2} - \frac{m_j^2}{8E_j^2}\right) \left(i_0\ j_0 \right)_\sigma , \left(n_i\ j \right)_\sigma \overset{HE}{\rightarrow} \frac{m_i}{2E_i} \left(n_{i0}\ j_0 \right)_\sigma \tag{12}$$
$$\left(i\ n_j \right)_\sigma \overset{HE}{\rightarrow} \frac{m_j}{2E_j} \left(i_0\ n_{j0} \right)_\sigma , \left(n_i\ n_j \right)_\sigma \overset{HE}{\rightarrow} \frac{m_i m_j}{4E_i E_j} \left(n_{i0}\ n_{j0} \right)_\sigma$$

Momentum conservation for a general three point amplitude is given by $\mathbf{p_i} + \mathbf{p}_j + \mathbf{p}_k = 0$ or more explicitly $\left| i \right)_{-\sigma} \left(i \right|_\sigma - \left| n_i \right)_{-\sigma} \left(n_i \right|_\sigma + \left| j \right)_{-\sigma} \left(j \right|_\sigma - \left| n_j \right)_{-\sigma} \left(n_j \right|_\sigma + \left| k \right)_{-\sigma} \left(k \right|_\sigma - \left| n_k \right)_{-\sigma} \left(n_k \right|_\sigma = 0$. Multiplying from the left with $\left(j \right|_{-\sigma}, \left(n_j \right|_{-\sigma}, \left(\varsigma \right|_{-\sigma}$ (ς is an arbitrary spinor) and from right with $\left| k \right)_\sigma$, we can obtain with (9) several exact equations, which will turn out to be useful, when we consider amplitudes and especially their high energy limit.

$$\left(j\ i \right)_{-\sigma} \left(i\ k \right)_\sigma = 0 + m_j \left(n_j\ k \right)_\sigma + \left(j\ n_i \right)_{-\sigma} \left(n_i\ k \right)_\sigma - m_k \left(j\ n_k \right)_{-\sigma} \tag{13}$$
$$\left(n_j\ i \right)_{-\sigma} \left(i\ k \right)_\sigma = m_j \left(j\ k \right)_\sigma + \left(n_j\ n_i \right)_{-\sigma} \left(n_i\ k \right)_\sigma - m_k \left(n_j\ n_k \right)_{-\sigma}$$
$$\left(\varsigma\ i \right)_{-\sigma} \left(i\ k \right)_\sigma = -\left(\varsigma\ j \right)_{-\sigma} \left(j\ k \right)_\sigma + \left(\varsigma\ n_i \right)_{-\sigma} \left(n_i\ k \right)_\sigma + \left(\varsigma\ n_j \right)_{-\sigma} \left(n_j\ k \right)_\sigma - m_k \left(\varsigma\ n_k \right)_{-\sigma}$$

In the high energy limit up to order $O\left(m\right)$ only the first summand survives since $\left| n_i \right)_\sigma \sim m_i / \sqrt{2E_i}$. These equations are also valid for permutations in (i, j, k) and for $\sigma \rightarrow -\sigma$.

We write the equations (13) in the important case of $m_i = m_j = m$ and $m_k = 0$:

$$\begin{aligned}
&(j\ i)_{-\sigma}(i\ k)_{\sigma} = m(n_j\ k)_{\sigma} + (j\ n_i)_{-\sigma}(n_i\ k)_{\sigma} \approx 0 + O(m^2) \\
&(n_j\ i)_{-\sigma}(i\ k)_{\sigma} = m(j\ k)_{\sigma} + (n_j\ n_i)_{-\sigma}(n_i\ k)_{\sigma} \approx m(j\ k)_{\sigma} + O(m^3) \\
&(\varsigma\ i)_{-\sigma}(i\ k)_{\sigma} = -(\varsigma\ j)_{-\sigma}(j\ k)_{\sigma} + (\varsigma\ n_i)_{-\sigma}(n_i\ k)_{\sigma} + (\varsigma\ n_j)_{-\sigma}(n_j\ k)_{\sigma} \approx -(\varsigma\ j)_{-\sigma}(j\ k)_{\sigma} + O(m^2)
\end{aligned} \tag{14}$$

In three point amplitudes with two equal mass particles $m_i = m_j = m$ and one massless boson $m_k = 0$ one needs the so called x-factor [7] to write for three point amplitudes in the form $\mathcal{A}_3 = (\mathbf{i}\ \mathbf{j})_{-\sigma} x^{\sigma}$ (σ is the helicity sign of the massless boson k, note that we write $\sigma = \pm 1$ in terms). The x-factor is defined as:

$$x^{\sigma} = \frac{(\varsigma_{-\sigma}\, \mathbf{p_i}\, k_{\sigma})}{m(\varsigma\ k)_{-\sigma}} = \frac{1}{m}\left(\frac{(\varsigma\ i)_{-\sigma}(i\ k)_{\sigma}}{(\varsigma\ k)_{-\sigma}} - \frac{(\varsigma\ n_i)_{-\sigma}(n_i\ k)_{\sigma}}{(\varsigma\ k)_{-\sigma}}\right) \tag{15}$$

From momentum conservation (14) one derives

$$x^{\sigma} \approx \frac{1}{m}\left(\frac{(\varsigma\ i)_{-\sigma}(i\ j)_{\sigma}(i\ k)_{\sigma}}{(\varsigma\ k)_{-\sigma}(i\ j)_{\sigma}} + O(m^2)\right) \approx \frac{1}{m}\left(-\frac{(\varsigma\ k)_{-\sigma}(k\ j)_{\sigma}(i\ k)_{\sigma}}{(\varsigma\ k)_{-\sigma}(i\ j)_{\sigma}} + O(m^2)\right)$$

$$x^{\sigma} \approx \frac{(j\ k)_{\sigma}(k\ i)_{\sigma}}{m(j\ i)_{\sigma}} + O(m) \tag{16}$$

4. Products of Spinor Contractions

In amplitudes with massive spinors one encounters products of spinor contractions and therefore we write some of them down as preparation for their evaluation. First we note the contractions of massive spinors obtained from the expressions in (8).

$$\begin{aligned}
&(\mathbf{i}\ \mathbf{j})_{\sigma} = (i^I\ j^J)_{\sigma} = \begin{pmatrix} (i\ j)_{\sigma} & (i\ n_j)_{\sigma} \\ (n_i\ j)_{\sigma} & (n_i\ n_j)_{\sigma} \end{pmatrix},\ (i_I\ j_J)_{\sigma} = \begin{pmatrix} (n_i\ n_j)_{\sigma} & -(n_i\ j)_{\sigma} \\ -(i\ n_j)_{\sigma} & (i\ j)_{\sigma} \end{pmatrix} \\
&(\mathbf{i}\ \mathbf{j})_{-\sigma} = (i^I\ j^J)_{-\sigma} = \begin{pmatrix} (n_i\ n_j)_{-\sigma} & (n_i\ j)_{-\sigma} \\ (i\ n_j)_{-\sigma} & (i\ j)_{-\sigma} \end{pmatrix},\ (i_I\ j_J)_{-\sigma} = \begin{pmatrix} (i\ j)_{-\sigma} & -(i\ n_j)_{-\sigma} \\ -(n_i\ j)_{-\sigma} & (n_i\ n_j)_{-\sigma} \end{pmatrix}
\end{aligned} \tag{17}$$

These matrices according the comments after (8) should be interpreted as follows: if $\sigma = +$ the entries are in the right order, if $\sigma = -$ then the entries should be swapped according $+ \leftrightarrow -$, that means crosswise in this case. If $\sigma = +$, then the helicity assignments in the matrix $(i^I\ j^J)_{\sigma}$ are according (7) $\begin{pmatrix} ++ & +- \\ -+ & -- \end{pmatrix}$, which is reversed for $\sigma = -$. The row denotes particle i and the column j. Recall from (7) that $|i)_{\sigma}$ and $|n_i)_{-\sigma}$ have helicity sign σ, while $|n_i)_{\sigma}$ and $|i)_{-\sigma}$ have helicity sign $-\sigma$. The contraction of massive and massless spinors is

$$\begin{aligned}
&(i^I\ j)_{\sigma} = \big((i\ j)_{\sigma}\ \ (n_i\ j)_{\sigma}\big), \quad (i\ j^J)_{\sigma} = \big((i\ j)_{\sigma}\ \ (i\ n_j)_{\sigma}\big) \\
&(i^I\ j)_{-\sigma} = \big((n_i\ j)_{-\sigma}\ \ (i\ j)_{-\sigma}\big), \quad (i\ j^J)_{-\sigma} = \big((i\ n_j)_{-\sigma}\ \ (i\ j)_{-\sigma}\big)
\end{aligned} \tag{18}$$

Again the two-vectors are interpreted as follows: if $\sigma = +$ the entries are in the right order, if $\sigma = -$ then the entries should be swapped according $\sigma: + \leftrightarrow -$.

Next we consider products of spinor contractions, which appear in massive amplitudes and start with

$$(\mathbf{i}\ \mathbf{j})_{\sigma}^2 = \begin{pmatrix} (i\ j)_{\sigma} & (i\ n_j)_{\sigma} \\ (n_i\ j)_{\sigma} & (n_i\ n_j)_{\sigma} \end{pmatrix} \circ \begin{pmatrix} (i\ j)_{\sigma} & (i\ n_j)_{\sigma} \\ (n_i\ j)_{\sigma} & (n_i\ n_j)_{\sigma} \end{pmatrix} = \begin{pmatrix} ++ & +0 & +- \\ 0+ & 00 & 0- \\ -+ & -0 & -- \end{pmatrix}$$

Here the right matrix shows the helicity assignments of particles (i j) in the case of $\sigma=+$, in the case of $\sigma=-$ the entries must be swapped according $+\leftrightarrow-$. The "multiplication" $\circ$ of the two matrices is defined as follows: select one entry of each matrix, so that one gets the correct helicity of the corresponding boson displayed in the right matrix, and then multiply them. Rows are for i and columns for j. If there are several possibilities simply take the mean of them, this automatically gives the required symmetrisation of the little group indices. Examples are: $++=(\mathrm{i\ j})_\sigma(\mathrm{i\ j})_\sigma$, $-+=(\mathrm{n_i\ j})_\sigma\cdot(\mathrm{n_i\ j})_\sigma$, $0-=\frac{1}{2}\left((\mathrm{i\ n_j})_\sigma(\mathrm{n_i\ n_j})_\sigma+(\mathrm{n_i\ n_j})_\sigma\cdot(\mathrm{i\ n_j})_\sigma\right)=(\mathrm{i\ n_j})_\sigma(\mathrm{n_i\ n_j})_\sigma$,
$00=\frac{1}{2}\left((\mathrm{i\ n_j})_\sigma\cdot(\mathrm{n_i\ j})_\sigma+(\mathrm{i\ j})_\sigma\cdot(\mathrm{n_i\ n_j})_\sigma\right)$. As final result one obtains

$$(\mathbf{i\ j})_\sigma^2=\begin{pmatrix}(\mathrm{i\ j})_\sigma^2 & (\mathrm{i\ j})_\sigma(\mathrm{i\ n_j})_\sigma & (\mathrm{i\ n_j})_\sigma^2\\ (\mathrm{i\ j})_\sigma(\mathrm{n_i\ j})_\sigma & \frac{1}{2}\left((\mathrm{i\ j})_\sigma(\mathrm{n_i\ n_j})_\sigma+(\mathrm{i\ n_j})_\sigma(\mathrm{n_i\ j})_\sigma\right) & (\mathrm{i\ n_j})_\sigma(\mathrm{n_i\ n_j})_\sigma\\ (\mathrm{n_i\ j})_\sigma^2 & (\mathrm{n_i\ j})_\sigma(\mathrm{n_i\ n_j})_\sigma & (\mathrm{n_i\ n_j})_\sigma^2\end{pmatrix}\qquad(19)$$

For $\sigma\to-\sigma$ the entries must be swapped according $+\leftrightarrow-$. In a similar manner one could derive

$$(\mathbf{i\ j})_{-\sigma}(\mathbf{i\ j})_\sigma=\begin{pmatrix}(\mathrm{n_i\ n_j})_{-\sigma} & (\mathrm{n_i\ j})_{-\sigma}\\ (\mathrm{i\ n_j})_{-\sigma} & (\mathrm{i\ j})_{-\sigma}\end{pmatrix}\circ\begin{pmatrix}(\mathrm{i\ j})_\sigma & (\mathrm{i\ n_j})_\sigma\\ (\mathrm{n_i\ j})_\sigma & (\mathrm{n_i\ n_j})_\sigma\end{pmatrix}==\begin{pmatrix}++ & +0 & +-\\ 0+ & 00 & 0-\\ -+ & -0 & --\end{pmatrix}$$

used for Higgs coupling with two massive spin 1 bosons and we get as result for the matrix $(\mathbf{i\ j})_{-\sigma}(\mathbf{i\ j})_\sigma$:

$$\begin{pmatrix}(\mathrm{n_i\ n_j})_{-\sigma}(\mathrm{i\ j})_\sigma & \frac{1}{2}\left((\mathrm{n_i\ n_j})_{-\sigma}(\mathrm{i\ n_j})_\sigma+(\mathrm{n_i\ j})_{-\sigma}(\mathrm{i\ j})_\sigma\right) & (\mathrm{n_i\ j})_{-\sigma}(\mathrm{i\ n_j})_\sigma\\ \frac{1}{2}\left((\mathrm{n_i\ n_j})_{-\sigma}(\mathrm{n_i\ j})_\sigma+(\mathrm{i\ n_j})_{-\sigma}(\mathrm{i\ j})_\sigma\right) & 00 & \frac{1}{2}\left((\mathrm{n_i\ j})_{-\sigma}(\mathrm{n_i\ n_j})_\sigma+(\mathrm{i\ j})_{-\sigma}(\mathrm{i\ n_j})_\sigma\right)\\ (\mathrm{i\ n_j})_{-\sigma}(\mathrm{n_i\ j})_\sigma & \frac{1}{2}\left((\mathrm{i\ n_j})_{-\sigma}(\mathrm{n_i\ n_j})_\sigma+(\mathrm{i\ j})_{-\sigma}(\mathrm{n_i\ j})_\sigma\right) & (\mathrm{i\ j})_{-\sigma}(\mathrm{n_i\ n_j})_\sigma\end{pmatrix}\quad(20)$$

$$00=\frac{1}{4}\left((\mathrm{i\ j})_{-\sigma}(\mathrm{i\ j})_\sigma+(\mathrm{i\ n_j})_{-\sigma}(\mathrm{i\ n_j})_\sigma+(\mathrm{n_i\ j})_{-\sigma}(\mathrm{n_i\ j})_\sigma+(\mathrm{n_i\ n_i})_{-\sigma}(\mathrm{n_i\ n_i})_\sigma\right).$$

Now we investigate a term needed for the coupling of two massive fermions and a massive boson $(\mathbf{i\,j\,k})=(\bar{\mathbf{f}}\,\mathbf{f}\,\mathbf{V})$.

$$(\mathbf{j\ k})_{-\sigma}(\mathbf{k\ i})_\sigma=\begin{pmatrix}\left(\overset{+}{\mathrm{n_j}}\ \overset{+}{\mathrm{n_k}}\right)_{-\sigma} & \left(\overset{+}{\mathrm{n_j}}\ \overset{-}{\mathrm{k}}\right)_{-\sigma}\\ \left(\overset{-}{\mathrm{j}}\ \overset{+}{\mathrm{n_k}}\right)_{-\sigma} & \left(\overset{-}{\mathrm{j}}\ \overset{-}{\mathrm{k}}\right)_{-\sigma}\end{pmatrix}\circ\begin{pmatrix}\left(\overset{+}{\mathrm{k}}\ \overset{+}{\mathrm{i}}\right)_\sigma & \left(\overset{+}{\mathrm{k}}\ \overset{-}{\mathrm{n_i}}\right)_\sigma\\ \left(\overset{-}{\mathrm{n_k}}\ \overset{+}{\mathrm{i}}\right)_\sigma & \left(\overset{-}{\mathrm{n_k}}\ \overset{-}{\mathrm{n_i}}\right)_\sigma\end{pmatrix}$$

As a memo we have here written as superscripts the helicity signs of each spinor in the case of $\sigma=+$, which may be helpful in selecting the correct entries giving the matrix with particle helicity assignments below. We can write down two matrices, one for $(\mathbf{i}^+)$ and one for $(\mathbf{i}^-)$ both of the form $\begin{pmatrix}++ & +0 & +-\\ -+ & -0 & --\end{pmatrix}$, where the row corresponds to fermion **j** and the column to boson **k**. For $(\mathbf{i}^+)$ one has to take the first column of matrix 2 and "multiply" it in the usual way with matrix 1, for $(\mathbf{i}^-)$ take instead the second column of matrix 2. This gives the following matrices:

$$(\mathbf{i}^+)=\begin{pmatrix}(\mathrm{n_j\ n_k})_{-\sigma}(\mathrm{k\ i})_\sigma & \frac{1}{2}\left((\mathrm{n_j\ n_k})_{-\sigma}(\mathrm{n_k\ i})_\sigma+(\mathrm{n_j\ k})_{-\sigma}(\mathrm{k\ i})_\sigma\right) & (\mathrm{n_j\ k})_{-\sigma}(\mathrm{n_k\ i})_\sigma\\ (\mathrm{j\ n_k})_{-\sigma}(\mathrm{k\ i})_\sigma & \frac{1}{2}\left((\mathrm{j\ n_k})_{-\sigma}(\mathrm{n_k\ i})_\sigma+(\mathrm{j\ k})_{-\sigma}(\mathrm{k\ i})_\sigma\right) & (\mathrm{j\ k})_{-\sigma}(\mathrm{n_k\ i})_\sigma\end{pmatrix}\qquad(21)$$

$$(\mathbf{i}^-)=\begin{pmatrix}(\mathrm{n_j\ n_k})_{-\sigma}(\mathrm{k\ n_i})_\sigma & \frac{1}{2}\left((\mathrm{n_j\ n_k})_{-\sigma}(\mathrm{n_k\ n_i})_\sigma+(\mathrm{n_j\ k})_{-\sigma}(\mathrm{k\ n_i})_\sigma\right) & (\mathrm{n_j\ k})_{-\sigma}(\mathrm{n_k\ n_i})_\sigma\\ (\mathrm{j\ n_k})_{-\sigma}(\mathrm{k\ n_i})_\sigma & \frac{1}{2}\left((\mathrm{j\ n_k})_{-\sigma}(\mathrm{n_k\ n_i})_\sigma+(\mathrm{j\ k})_{-\sigma}(\mathrm{k\ n_i})_\sigma\right) & (\mathrm{j\ k})_{-\sigma}(\mathrm{n_k\ n_i})_\sigma\end{pmatrix}$$

The amplitudes built from the matrices in (21) depend on the considered boson: if the boson V is a Z-boson, then one has different couplings g_L and g_R to left-handed and right-handed fermions, if V is a W-boson, then it couples only to left-handed fermions. We consider here and in the next section only the general structure of the matrices in (21), so you have to insert the appropriate couplings yourself, for details see [10].

The high energy limit of the matrices in (17)-(21) up to $O(m^2)$ is now obtained by neglecting terms proportional $n^{3,4}$. Finally one should insert the expressions in equation (12). It is however much simpler to eliminate at first dependencies on $|n_i)$ and $|\varsigma)$ spinors employing equations (13), (14) and (16) and to use (12) at the end, as we shall see when we discuss some examples for amplitudes in the next section. The nice feature is here, that we obtain expressions valid for all helicity signs and the symmetrisation of $SU(2)$ indices goes nearly automatically.

5. Three Point Amplitudes

Now we discuss several three point amplitudes and their high energy limit using the formalism of the previous sections, neglecting symmetry factors.

We begin with two fermions of equal mass m and a massless spin one boson (photon, gluon). The amplitude and its high energy limit with (16) and (17) are now obtained as

$$\mathcal{A}_3\left(f,\bar{f},A^\sigma\right)=g(\mathbf{1}\ \mathbf{2})_{-\sigma}x^\sigma=g\begin{pmatrix}(n_1\ n_2)_{-\sigma} & (n_1\ 2)_{-\sigma}\\ (1\ n_2)_{-\sigma} & (1\ 2)_{-\sigma}\end{pmatrix}\frac{(\varsigma_{-\sigma}\,\mathbf{p_1}\,3_\sigma)}{m(\varsigma\ 3)_{-\sigma}}\approx$$

$$g\begin{pmatrix}0 & (n_1\ 2)_{-\sigma}\\ (1\ n_2)_{-\sigma} & (1\ 2)_{-\sigma}\end{pmatrix}\left(\frac{(2\ 3)_\sigma(3\ 1)_\sigma}{m(2\ 1)_\sigma}+O(m)\right)\approx\frac{g}{m(2\ 1)_\sigma}\begin{pmatrix}0 & m(1\ 3)_\sigma(3\ 1)_\sigma\\ m(2\ 3)_\sigma(2\ 3)_\sigma & 0\end{pmatrix}$$

$$\mathcal{A}_3\left(f,\bar{f},A^\sigma\right)=g(\mathbf{1}\ \mathbf{2})_{-\sigma}x^\sigma\approx g\begin{pmatrix}0 & -(3\ 1)_\sigma^2/(2\ 1)_\sigma\\ (2\ 3)_\sigma^2/(2\ 1)_\sigma & 0\end{pmatrix}\qquad(22)$$

where we used $(n_1\ n_2)_{-\sigma}\approx 0$ from (11) and from equation (14) $(1\ 2)_{-\sigma}(2\ 3)_\sigma\approx 0+O(m^2)$, $(n_1\ 2)_{-\sigma}(2\ 3)_\sigma\approx m(1\ 3)_\sigma+O(m^3)$, $(1\ n_2)_{-\sigma}(3\ 1)_\sigma\approx m(2\ 3)_\sigma+O(m^3)$. After elimination of the n_i-spinors the i-spinors should be replaced by their massless counterparts the i_0-spinors up to $O(m^2)$, which we omitted for better readability. Possible and physically irrelevant sign differences in the amplitude compared to [10] are due to (9) with another sign convention used there.

The amplitude with two equal mass m fermions and a massless graviton is then obtained in agreement with [10] from the final amplitude above by multiplying with $x^\sigma\cdot m/M_P$, where M_P is the Planck mass.

$$\mathcal{A}_3\left(f,\bar{f},G^{2\sigma}\right)=\frac{m}{M_P}(\mathbf{1}\ \mathbf{2})_{-\sigma}x^{2\sigma}\approx\begin{pmatrix}0 & -(3\ 1)_\sigma^2/(2\ 1)_\sigma\\ (2\ 3)_\sigma^2/(2\ 1)_\sigma & 0\end{pmatrix}\frac{(2\ 3)_\sigma(3\ 1)_\sigma}{m(2\ 1)_\sigma}\frac{m}{M_P}$$

$$\mathcal{A}_3\left(f,\bar{f},G^{2\sigma}\right)=\frac{m}{M_P}(\mathbf{1}\ \mathbf{2})_{-\sigma}x^{2\sigma}\approx\frac{1}{M_P}\begin{pmatrix}0 & -(3\ 1)_\sigma^3(2\ 3)_\sigma/(2\ 1)_\sigma^2\\ (2\ 3)_\sigma^3(3\ 1)_\sigma/(2\ 1)_\sigma^2 & 0\end{pmatrix}\qquad(23)$$

Now we look at two spin one bosons of equal mass m and a massless spin one boson $(\bar{W}W\gamma)=(\mathbf{12}3)$ and employ (16) and (19), where $+,-$ entries were swapped and $\sigma\to-\sigma$ used. Terms proportional n_i^N for $N\geq 3$ can be neglected. The terms $0-=(1\ 2)_\sigma(n_1\ 2)_\sigma$ and $-0=(1\ 2)_\sigma(1\ n_2)_\sigma$ in the matrix obtained from (19) must be multiplied with $(2\ 3)_\sigma/(2\ 3)_\sigma$ and because of (14) they have order $O(m^3)$ and can be neglected. The amplitude becomes:

$$\mathcal{A}_3\left(W,\bar{W},\gamma^\sigma\right)=\frac{g}{m}x^\sigma(\mathbf{1}\ \mathbf{2})_{-\sigma}^2=\begin{pmatrix}++ & +0 & +-\\ 0+ & 00 & 0-\\ -+ & -0 & --\end{pmatrix}\approx g\begin{pmatrix}0 & 0 & (n_1\ 2)_{-\sigma}^2\\ 0 & \frac{1}{2}(1\ n_2)_{-\sigma}(n_1\ 2)_{-\sigma} & 0\\ (1\ n_2)_{-\sigma}^2 & 0 & (1\ 2)_{-\sigma}^2\end{pmatrix}\frac{(2\ 3)_\sigma(3\ 1)_\sigma}{m^2(2\ 1)_\sigma}$$

With equation (14) $(1\ 2)_{-\sigma}(2\ 3)_{\sigma}=m(n_1\ 3)_{\sigma}+(1\ n_2)_{-\sigma}(n_2\ 3)_{\sigma}=O(m^2)$ and $n_i\sim m$ one sees that all remaining terms in the left matrix have the correct order to cancel the $1/m^2$ from the factor x^{σ}/m. Now we calculate the single entries in the matrix above using equation (14):

$$+-=\frac{(n_1\ 2)_{-\sigma}(2\ 3)_{\sigma}(n_1\ 2)_{-\sigma}(2\ 3)_{\sigma}(3\ 1)_{\sigma}}{m^2(2\ 1)_{\sigma}(2\ 3)_{\sigma}}\approx\frac{m(1\ 3)_{\sigma}m(1\ 3)_{\sigma}(3\ 1)_{\sigma}}{m^2(2\ 1)_{\sigma}(2\ 3)_{\sigma}}=-\frac{(3\ 1)_{\sigma}^3}{(1\ 2)_{\sigma}(2\ 3)_{\sigma}}$$

$$00=\frac{(n_1\ 2)_{-\sigma}(2\ 3)_{\sigma}(1\ n_2)_{-\sigma}(3\ 1)_{\sigma}}{2m^2(2\ 1)_{\sigma}}=\frac{m(1\ 3)_{\sigma}m(2\ 3)_{\sigma}}{2m^2(2\ 1)_{\sigma}}=\frac{1}{2}\frac{(2\ 3)_{\sigma}(3\ 1)_{\sigma}}{(1\ 2)_{\sigma}}$$

$$-+=\frac{(n_2\ 1)_{-\sigma}(3\ 1)_{\sigma}(n_2\ 1)_{-\sigma}(1\ 3)_{\sigma}(2\ 3)_{\sigma}}{m^2(2\ 1)_{\sigma}(1\ 3)_{\sigma}}\approx\frac{-m(2\ 3)_{\sigma}m(2\ 3)_{\sigma}(2\ 3)_{\sigma}}{m^2(2\ 1)_{\sigma}(1\ 3)_{\sigma}}=-\frac{(2\ 3)_{\sigma}^3}{(1\ 2)_{\sigma}(3\ 1)_{\sigma}}$$

$$--=\frac{(1\ 2)_{-\sigma}(2\ 3)_{\sigma}(1\ 2)_{-\sigma}(3\ 1)_{\sigma}}{m^2(2\ 1)_{\sigma}}\approx\frac{\left(m(n_1\ 3)_{\sigma}+(1\ n_2)_{-\sigma}(n_2\ 3)_{\sigma}\right)(1\ 2)_{-\sigma}(3\ 1)_{\sigma}}{m^2(2\ 1)_{\sigma}}$$

$$\approx\frac{(1\ 2)_{-\sigma}^2(3\ 1)_{\sigma}}{(3\ 2)_{-\sigma}(2\ 1)_{\sigma}}+\frac{(1\ 2)_{-\sigma}^2(3\ 2)_{\sigma}}{(3\ 1)_{-\sigma}(2\ 1)_{\sigma}}=\frac{(1\ 2)_{-\sigma}^3}{(2\ 3)_{-\sigma}(3\ 1)_{-\sigma}}\frac{(1\ 3)_{-\sigma}(3\ 1)_{\sigma}+(2\ 3)_{-\sigma}(3\ 2)_{\sigma}}{(1\ 2)_{-\sigma}(2\ 1)_{\sigma}}$$

$$\approx\frac{(1\ 2)_{-\sigma}^3}{(2\ 3)_{-\sigma}(3\ 1)_{-\sigma}}\frac{2(p_1+p_2)p_3}{2p_1p_2}=\frac{(1\ 2)_{-\sigma}^3}{(2\ 3)_{-\sigma}(3\ 1)_{-\sigma}}\frac{-2(p_1+p_2)^2}{2p_1p_2}=-\frac{2(1\ 2)_{-\sigma}^3}{(2\ 3)_{-\sigma}(3\ 1)_{-\sigma}}$$

where (14) and (13) for $--$ and at last $2p_a\cdot p_b\approx(a\ b)_{-\sigma}(b\ a)_{\sigma}$ with $p_3=-(p_1+p_2)$ were used. The result is

$$\mathcal{A}_3\left(W,\bar{W},\gamma^{\sigma}\right)\approx g\begin{pmatrix}0 & 0 & -(3\ 1)_{\sigma}^3/(1\ 2)_{\sigma}(2\ 3)_{\sigma}\\ 0 & \frac{1}{2}(2\ 3)_{\sigma}(3\ 1)_{\sigma}/(1\ 2)_{\sigma} & 0\\ -(2\ 3)_{\sigma}^3/(1\ 2)_{\sigma}(3\ 1)_{\sigma} & 0 & -2(1\ 2)_{-\sigma}^3/(2\ 3)_{-\sigma}(3\ 1)_{-\sigma}\end{pmatrix}\qquad(24)$$

The amplitude for two massive bosons V interacting with a massless graviton G is derived from the amplitude above by multiplying the final result with $x^{\sigma}\cdot\frac{m}{M_P}\approx\frac{(2\ 3)_{\sigma}(3\ 1)_{\sigma}}{M_P(2\ 1)_{\sigma}}$ giving $\mathcal{A}_3\left(\bar{V},V,G\right)\approx\frac{x^{\sigma}}{m}(\mathbf{1}\ \mathbf{2})_{-\sigma}^2\cdot x^{\sigma}\frac{m}{M_P}$ and the lower right entry vanishes due to (14a).

$$\mathcal{A}_3\left(V,\bar{V},G^{2\sigma}\right)=\frac{1}{M_P}(\mathbf{1}\ \mathbf{2})_{-\sigma}^2\cdot x^{2\sigma}\approx\frac{1}{M_P}\begin{pmatrix}0 & 0 & (3\ 1)_{\sigma}^4/(1\ 2)_{\sigma}^2\\ 0 & -\frac{1}{2}(2\ 3)_{\sigma}^2(3\ 1)_{\sigma}^2/(1\ 2)_{\sigma}^2 & 0\\ (2\ 3)_{\sigma}^4/(1\ 2)_{\sigma}^2 & 0 & 0\end{pmatrix}\qquad(25)$$

As an example for employing (18) we take a fermion with mass m_1, a massless fermion 2 and a boson with mass m_3

$$\mathcal{A}_3(\mathbf{1},2,\mathbf{3})=\frac{g}{m_3}(\mathbf{3}\ \mathbf{1})_{-\sigma}(2\ \mathbf{3})_{\sigma}=\begin{pmatrix}(n_3\ n_1)_{-\sigma} & (n_3\ 1)_{-\sigma}\\ (3\ n_1)_{-\sigma} & (3\ 1)_{-\sigma}\end{pmatrix}\circ\begin{pmatrix}(2\ 3)_{\sigma} & (2\ n_3)_{\sigma}\end{pmatrix}=\begin{pmatrix}++ & +0 & +-\\ -+ & -0 & --\end{pmatrix}$$

The row denotes particle 1 and the column particle 3 for $\sigma=+$. The matrix then becomes

$$\mathcal{A}_3=\begin{pmatrix}(n_3\ n_1)_{-\sigma}(2\ 3)_{\sigma} & \frac{1}{2}\left((n_3\ n_1)_{-\sigma}(2\ n_3)_{\sigma}+(3\ n_1)_{-\sigma}(2\ 3)_{\sigma}\right) & (3\ n_1)_{-\sigma}(2\ n_3)_{\sigma}\\ (n_3\ 1)_{-\sigma}(2\ 3)_{\sigma} & \frac{1}{2}\left((3\ 1)_{-\sigma}(2\ 3)_{\sigma}+(n_3\ 1)_{-\sigma}(2\ n_3)_{\sigma}\right) & (3\ 1)_{-\sigma}(2\ n_3)_{\sigma}\end{pmatrix}$$

Neglecting higher orders in n_i, using (13),(14) and multiplying $-+$ and $--$ with $(1\ 2)_{\pm\sigma}/(1\ 2)_{\pm\sigma}$ one obtains

$$\mathcal{A}_3(\mathbf{1},2,\mathbf{3})=\frac{g}{m_3}(\mathbf{3}\ \mathbf{1})_{-\sigma}(2\ \mathbf{3})_{\sigma}\approx g\begin{pmatrix}0 & \frac{m_1}{2m_3}(1\ 2)_{\sigma} & 0\\ -(2\ 3)_{\sigma}^2/(1\ 2)_{\sigma} & 0 & (3\ 1)_{-\sigma}^2/(1\ 2)_{-\sigma}\end{pmatrix}\qquad(26)$$

This is the matrix for $\sigma=+$. For $\sigma=-$ the entries should be swapped according: $++\leftrightarrow--$, $+0\leftrightarrow-0$, $+-\leftrightarrow-+$.

Next we investigate amplitudes with two massive fermions with masses m_1, m_2 and a massive boson 3 with mass m_3. The possible terms are given by $\mathcal{A}_3(\mathbf{1},\mathbf{2},\mathbf{3}) = \frac{g}{m_3}\langle \mathbf{2}\ \mathbf{3}\rangle_{\sigma'}\langle \mathbf{3}\ \mathbf{1}\rangle_{\sigma}$ with $\sigma' = \pm\sigma$ and we consider here only $\sigma' = -\sigma$. The matrices to be used for this amplitude were already written in (21) and we have here $(\mathbf{i},\mathbf{j},\mathbf{k}) = (\mathbf{1},\mathbf{2},\mathbf{3})$. One can write two matrices one for $(\mathbf{1}^+)$ and one for $(\mathbf{1}^-)$ which are for $\sigma = +$ both of the form $\begin{pmatrix} ++ & +0 & +- \\ -+ & -0 & -- \end{pmatrix}$. Again for $\sigma = -$ the entries must be swapped according $+ \leftrightarrow -$. The row corresponds to fermion 2 and the column to boson 3. Since the amplitude should be of order O(m) we can neglect for the high energy limit all terms $\propto n^N$ for $N \geq 2$ and the term $\langle 2\ 3\rangle_{-\sigma}\langle 3\ 1\rangle_{\sigma} = O(m^2)$ due to momentum conservation in equation (13), leaving us with:

$$\mathcal{A}_3(\mathbf{1}^{+,-},\mathbf{2},\mathbf{3}) \approx \frac{g}{m_3}\cdot\begin{pmatrix} 0 & \frac{1}{2}\langle n_2\ 3\rangle_{-\sigma}\langle 3\ 1\rangle_{\sigma} & 0 \\ \langle 2\ n_3\rangle_{-\sigma}\langle 3\ 1\rangle_{\sigma} & 0 & \langle 2\ 3\rangle_{-\sigma}\langle n_3\ 1\rangle_{\sigma} \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & \frac{1}{2}\langle 2\ 3\rangle_{-\sigma}\langle 3\ n_1\rangle_{\sigma} & 0 \end{pmatrix}$$

Using the relations in (13) one obtains

$$\mathcal{A}_3(\mathbf{1}^{+,-},\mathbf{2},\mathbf{3}) \approx g\cdot\begin{pmatrix} 0 & -\frac{m_2}{2m_3}\langle 1\ 2\rangle_{\sigma} & 0 \\ \langle 3\ 1\rangle_{\sigma}^2/\langle 1\ 2\rangle_{\sigma} & 0 & -\langle 2\ 3\rangle_{-\sigma}^2/\langle 1\ 2\rangle_{-\sigma} \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & \frac{m_1}{2m_3}\langle 1\ 2\rangle_{-\sigma} & 0 \end{pmatrix} \quad (27)$$

As mentioned below (21) we do not consider here the different couplings of W,Z-bosons to left- and right-handed fermions, but only the general structure of the matrices in (21) and (27). Nevertheless one sees that the terms in (27) are in agreement with the terms in table IV of [10].

Finally we discuss the coupling of two massive spin 1 bosons (i,j)= (1,2) of mass m with a massive spin 0 boson (3) of mass m_3 (Higgs). Neglecting higher powers of n and the centre term because of $(p_1+p_2)^2 = 2p_1p_2 = p_3^2 \approx 0$ in the HE limit, one obtains from equation (20):

$\begin{pmatrix} 0 & \langle n_1\ 2\rangle_{-\sigma}\langle 1\ 2\rangle_{\sigma}/2 & 0 \\ \langle 1\ n_2\rangle_{-\sigma}\langle 1\ 2\rangle_{\sigma}/2 & 0 & \langle 1\ 2\rangle_{-\sigma}\langle 1\ n_2\rangle_{\sigma}/2 \\ 0 & \langle 1\ 2\rangle_{-\sigma}\langle n_1\ 2\rangle_{\sigma}/2 & 0 \end{pmatrix}$. Now we multiply $\langle n_i\ j\rangle_{-\sigma}$ with $\langle j\ 3\rangle_{\sigma}/\langle j\ 3\rangle_{\sigma}$ and use $\langle n_i\ j\rangle_{-\sigma}\langle j\ 3\rangle_{\sigma} = m\langle i\ 3\rangle_{\sigma}$ (14b) and their pendants for $\sigma \to -\sigma$ to obtain for $\mathcal{A}_3(V,V,h) = \frac{1}{m_3}\langle \mathbf{1}\ \mathbf{2}\rangle_{-\sigma}\langle \mathbf{1}\ \mathbf{2}\rangle_{\sigma}$

$$\mathcal{A}_3(V,V,h) = \frac{m}{2m_3}\begin{pmatrix} 0 & -\langle 1\ 2\rangle_{\sigma}\langle 3\ 1\rangle_{\sigma}/\langle 2\ 3\rangle_{\sigma} & 0 \\ \langle 1\ 2\rangle_{\sigma}\langle 2\ 3\rangle_{\sigma}/\langle 3\ 1\rangle_{\sigma} & 0 & \langle 1\ 2\rangle_{-\sigma}\langle 2\ 3\rangle_{-\sigma}/\langle 3\ 1\rangle_{-\sigma} \\ 0 & -\langle 1\ 2\rangle_{-\sigma}\langle 3\ 1\rangle_{-\sigma}/\langle 2\ 3\rangle_{-\sigma} & 0 \end{pmatrix} \quad (28)$$

This agrees with the results in table V in [20].

For three massive bosons there are several possibilities and we show one example for the first term of
$\langle \mathbf{1}\ \mathbf{2}\rangle_{-\sigma}\langle \mathbf{2}\ \mathbf{3}\rangle_{-\sigma}\langle \mathbf{3}\ \mathbf{1}\rangle_{\sigma}, \langle \mathbf{1}\ \mathbf{2}\rangle_{-\sigma}\langle \mathbf{2}\ \mathbf{3}\rangle_{\sigma}\langle \mathbf{3}\ \mathbf{1}\rangle_{-\sigma}, \langle \mathbf{1}\ \mathbf{2}\rangle_{\sigma}\langle \mathbf{2}\ \mathbf{3}\rangle_{-\sigma}\langle \mathbf{3}\ \mathbf{1}\rangle_{-\sigma} \propto m_k m_l \mathcal{A}_3(h_1,h_2,h_3)$
Consider $\langle \mathbf{1}\ \mathbf{2}\rangle_{-\sigma}\langle \mathbf{2}\ \mathbf{3}\rangle_{-\sigma}\langle \mathbf{3}\ \mathbf{1}\rangle_{\sigma}$ for $(h_1,h_2,h_3) = (\sigma,-\sigma,\sigma)$. Since we know from equation (7) that $h(|i\rangle_{\sigma}) = \sigma$, $h(|n_i\rangle_{\sigma}) = -\sigma$, $h(|i\rangle_{-\sigma}) = -\sigma$, $h(|n_i\rangle_{-\sigma}) = \sigma$ and we have here $h_1 = \sigma$, $h_2 = -\sigma$ we can write the first bracket as $\langle \mathbf{1}\ \mathbf{2}\rangle_{-\sigma} = \langle n_1\ 2\rangle_{-\sigma}$ and similar for the other two brackets to arrive at (see table VI in [10])
$\langle \mathbf{1}\ \mathbf{2}\rangle_{-\sigma}\langle \mathbf{2}\ \mathbf{3}\rangle_{-\sigma}\langle \mathbf{3}\ \mathbf{1}\rangle_{\sigma} = \langle n_1\ 2\rangle_{-\sigma}\langle 2\ n_3\rangle_{-\sigma}\langle 3\ 1\rangle_{\sigma} =$
$\langle n_1\ 2\rangle_{-\sigma}\langle 2\ 3\rangle_{\sigma}\langle 2\ n_3\rangle_{-\sigma}\langle 1\ 2\rangle_{\sigma}\langle 3\ 1\rangle_{\sigma}/\langle 2\ 3\rangle_{\sigma}\langle 1\ 2\rangle_{\sigma} \approx -m_1 m_3\langle 3\ 1\rangle_{\sigma}^3/\langle 1\ 2\rangle_{\sigma}\langle 2\ 3\rangle_{\sigma}$

6. Summary

In summary we have described massive angle and square spinors together using an index connected to their helicity category agreeing with the helicity sign of the spinor remaining in the high energy limit. This allows writing many relations between the spinors in a compact form. Massive spinors are defined as two-vectors $|i^I\rangle_\sigma = (|i\rangle_\sigma \;\; |n_i\rangle_\sigma)$ with the at first sight strange property, that the entries are in right order for $\sigma = +$, but must be swapped for $\sigma = -$. This property holds also for contractions and products of them and allows writing down amplitudes for different helicity categories at once. The high energy limit of three particle amplitudes $\mathcal{A}_3$ is then obtained with much less effort as we have shown in the previous sections and for both helicity categories together. Since the n_i spinors scale with $m_i / \sqrt{E}$, one sees immediately which terms can be neglected in an amplitude. Also remember that we work during the entire process with spinors $|i\rangle_\sigma \propto \sqrt{E_i + P_i}$ and $|n_i\rangle_\sigma \propto \sqrt{E_i - P_i}$ and first after elimination of the spinors $|n_i\rangle_\sigma$ and $|\varsigma\rangle_\sigma$ (coming from the x-factor) one should replace $|i\rangle_\sigma \to |i_0\rangle_\sigma$, which we omit for better readability. The nice feature is that we don't need the explicit high energy expansions in (11) and (12), simplifying the derivation of the high energy limit of amplitudes considerably. One also sees from the relations between spinors and the previous examples the crucial role played by the helicity category σ respectively the helicity sign in amplitudes.

Addendum:
This second version of the paper is improved compared to the first version in the following respects. With the slight change in representation of massive spinors introduced in [17],[18] i.e. $|n] \to -|n]$, $[n| \to -[n|$ the spinor contractions in (17) with only upper $SU(2)$ indices and their products in (18) to (21) don't contain any signs or σ, which additionally makes their evaluation easier. Comments to chiral interactions of W and Z-bosons, the vertex with Higgs and two massive spin 1 bosons and one example for the interaction of three massive spin 1 bosons were added.

Appendix A: Spinor Representations

The explicit representation for massive spinors in [12][14] is based on the metric $(+---)$ and momentum

$$p^{\mu} = \left(E \quad P\sin(\theta)\cos(\phi) \quad P\sin(\theta)\sin(\phi) \quad P\cos(\theta)\right) = \left(E \quad Pc\left(s^{*}+s\right) \quad Pic\left(s^{*}-s\right) \quad P\left(cc-ss^{*}\right)\right) \qquad \text{(A1)}$$

With the Pauli matrices we can write the momentum in bispinor form $p = p_{\mu}\sigma^{\mu}$ or $\overline{p} = p_{\mu}\overline{\sigma}^{\mu}$ using $c = \cos(\theta/2)$, $s = \sin(\theta/2)\exp(i\phi)$, $s^{*} = \sin(\theta/2)\exp(-i\phi)$ with $cc + ss^{*} = 1$ resulting in

$$p = \begin{pmatrix} E - \sigma P(cc - ss^{*}) & -\sigma 2Pcs^{*} \\ -\sigma 2Pcs & E + \sigma P(cc - ss^{*}) \end{pmatrix} \qquad \text{(A2)}$$

We write massive spinors in the 2-vector notation used in [16] see also [17], which is better readable than enumerating all eight 2x2 matrices. We choose here the representation of [18],[19] which is identical to [17] with the exception that $|n] \to -|n]$ and $[n| \to -[n|$. This avoids minus signs and σ in spinors with upper indices appearing in amplitudes. Lowercase index spinors are obtained by $|i_I\rangle = \epsilon_{IJ}|i^J\rangle$ and mirror spinors by $|\ \rangle \to \langle\ |$ and $|\ ] \to [\ |$. One can obtain the expressions for $|i], |i\rangle, |n_i\rangle, |n_i]$ and its mirrors from the following equations.

$$\left|i^I\right] = \left(|i] \quad |n_i]\right) = \left(\sqrt{E_i+P_i}\begin{pmatrix} c_i \\ s_i\end{pmatrix} \quad \sqrt{E_i-P_i}\begin{pmatrix} -s_i^{*} \\ c_i\end{pmatrix}\right) \qquad \left|i^I\right\rangle = \left(|n_i\rangle \quad |i\rangle\right) = \left(\sqrt{E_i-P_i}\begin{pmatrix} c_i \\ s_i\end{pmatrix} \quad \sqrt{E_i+P_i}\begin{pmatrix} -s_i^{*} \\ c_i\end{pmatrix}\right) \qquad \text{(A3)}$$

$$\left[i^I\right| = \left([i| \quad [n_i|\right) = \left(\sqrt{E_i+P_i}\begin{pmatrix} -s_i \\ c_i\end{pmatrix} \quad -\sqrt{E_i-P_i}\begin{pmatrix} c_i \\ s_i^{*}\end{pmatrix}\right) \qquad \left\langle i^I\right| = \left(\langle n_i| \quad \langle i|\right) = \left(-\sqrt{E_i-P_i}\begin{pmatrix} -s_i \\ c_i\end{pmatrix} \quad \sqrt{E_i+P_i}\begin{pmatrix} c_i \\ s_i^{*}\end{pmatrix}\right)$$

$$\left|i_I\right] = \left(-|n_i] \quad |i]\right) = \left(-\sqrt{E_i-P_i}\begin{pmatrix} -s_i^{*} \\ c_i\end{pmatrix} \quad \sqrt{E_i+P_i}\begin{pmatrix} c_i \\ s_i\end{pmatrix}\right) \qquad \left|i_I\right\rangle = \left(-|i\rangle \quad |n_i\rangle\right) = \left(-\sqrt{E_i+P_i}\begin{pmatrix} -s_i^{*} \\ c_i\end{pmatrix} \quad \sqrt{E_i-P_i}\begin{pmatrix} c_i \\ s_i\end{pmatrix}\right)$$

$$\left[i_I\right| = \left(-[n_i| \quad [i|\right) = \left(\sqrt{E_i-P_i}\begin{pmatrix} c_i \\ s_i^{*}\end{pmatrix} \quad \sqrt{E_i+P_i}\begin{pmatrix} -s_i \\ c_i\end{pmatrix}\right) \qquad \left\langle i_I\right| = \left(-\langle i| \quad \langle n_i|\right) = \left(-\sqrt{E_i+P_i}\begin{pmatrix} c_i \\ s_i^{*}\end{pmatrix} \quad -\sqrt{E_i-P_i}\begin{pmatrix} -s_i \\ c_i\end{pmatrix}\right)$$

With $|i)_{\sigma}, |n_i)_{\sigma}$ defined in (6), the momentum is $\mathbf{p_i} = \sigma\left|i^I\right)_{-\sigma}\left(i_I\right|_{\sigma} = |i)_{-\sigma}(i|_{\sigma} - |n_i)_{-\sigma}(n_i|_{\sigma}$ leading to (A2) for $\sigma = \pm$ and one can check, that the equations in (4) and $(i\ n_i)_{\sigma} = m_i$ in (9) are satisfied.

An interesting equivalent representation was given in [15], [5], [6]. With momenta in bispinor form $p = p_{\alpha\dot{\alpha}} = p_{\mu}\sigma^{\mu}$ one decomposes a massive momentum with $p^2 = m^2$ in terms of two null momenta k, q: $p = k + \frac{m^2}{2k\cdot q}q$. Massive spinors then can be written as (raising and lowering of SU(2) indices I,J goes with the Levi-Civita symbols ϵ^{IK} and ϵ_{IK})

$$|p^I\rangle = \left(\frac{m}{\langle k\ q\rangle}|q\rangle \quad |k\rangle\right), \; |p^I] = \left(|k] \quad \frac{m}{[k\ q]}|q]\right), \; |p_I\rangle = \left(-|k\rangle \quad \frac{m}{\langle k\ q\rangle}|q\rangle\right), \; |p_I] = \left(-\frac{m}{[k\ q]}|q] \quad |k]\right)$$

Conjugate spinors are obtained with $|a\rangle \to \langle a|, |a] \to [a|$. One can then prove (4) and the connection with the representation above is $|n\rangle = \frac{m}{\langle k\ q\rangle}|q\rangle$, $|n] = \frac{m}{[k\ q]}|q]$. With this we get as in (9): $\langle k\ n\rangle = m$, $[k\ n] = m$.

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