SOLUTION TO BROCARD'S PROBLEM

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Abstract

It was proved that for a factorial to be a solution to Brocard's problem, it must be representable as a product of two natural numbers differing by 2. It was then proved that only factorials, which are known solutions to Brocard's problem, can be represented as a product of two natural numbers differing by 2. It follows from this that Brocard's Diophantine equation $n! = t^2 - 1$ has no solutions other than the classical $(n, t) = (4, 5), (5, 11), (7, 71)$ by an elementary method.

Keywords: Brocard's problem, factorial, Diophantine equations.

1 Introduction

The Brocard Diophantine equation consists of finding positive integers n and t satisfying:

$$n! = t^2 - 1 \tag{1}$$

The problem was formulated by H. Brocard in 1876 and 1885 [1, 2] and later considered by S. Ramanujan [3, 4]. Only three solutions are known: $(n, t) = (4, 5), (5, 11), (7, 71)$. Many results are based on computational checks and expanding the boundaries for the absence of solutions [5, 6, 7].

In this work, the problem is reduced to proving a lemma, according to which for any $n!$ with $n \geq 9$, the minimum natural quotient from the division of numbers of the form $a(a + 2)$ by $n!$ is greater than $n + 1$.

2 Representations of Factorials

This work deals with positive integers. The key is the identity:

$$t^2 = (t - 1)(t + 1) + 1 \tag{2}$$

This identity motivates the study of factorial representations as products of two integers differing by 2. If $n! = t^2 - 1$, then by (2) we have:

$$n! = a(a + 2) \tag{3}$$

Known solutions correspond to: $4! = 4 \cdot 6$; $5! = 10 \cdot 12$; $7! = 70 \cdot 72$. Thus, if a factorial greater than 7! exists that satisfies equation (1), it must be representable as a product of two natural numbers differing by 2.

3 Lemma

Lemma 1. *Let $n \geq 9$. There do not exist integers a and m such that:*

$$a(a+2) = m \cdot n!, \quad 1 \leq m \leq n+1$$

Proof. Suppose that for some $n \geq 9$, there exists $m \in \{1, 2, \ldots, n+1\}$ and an integer a such that:

$$a(a+2) = m \cdot n! \tag{4}$$

Let $t = a+1$, then:

$$a(a+2) = (t-1)(t+1) = t^2 - 1$$

And the equality is rewritten as:

$$t^2 = 1 + m \cdot n! \tag{5}$$

This means it is necessary for the number $1 + m \cdot n!$ to be a perfect square. We represent (4) in the form:

$$(t-1)(t+1) = m \cdot n! \tag{6}$$

Now we use a key observation regarding large prime divisors. By Bertrand's Postulate, for any $n \geq 2$, there exists a prime p such that $n/2 < p \leq n$. For $n \geq 9$, p occurs in $n!$ to the first power since $2p > n$. From (1), $t^2 \equiv 1 \pmod{p}$, so $t \equiv \pm 1 \pmod{p}$. Thus p divides either $t-1$ or $t+1$, but not both.

Case 1. p does not divide m. Then one factor (say $t-1$) must be divisible by p, and the other not. Let $t - 1 = p \cdot s$. From (6):

$$s(t+1) = m \cdot \frac{n!}{p} \tag{7}$$

The left side is a product of two numbers not containing p. The right side contains $n!/p$, including all large primes $q \in (n/2, n]$ where $q \neq p$. We obtain an upper bound for large divisors on the right-hand side; for a sufficiently large n, this contradicts the fact that n!/p contains too large prime factors to fit into such small factors on the left. The proof of this statement is given in Appendix A. From $t^2 = 1 + m \cdot n!$, we have $t + 1 = \sqrt{1 + m \cdot n!} + 1$. Since $m \leq n+1$:

$$t + 1 \leq \sqrt{(n+1)! + 1} + 1$$

However, for $n \geq 9$:

$$\frac{n!}{p} > \sqrt{(n+1)!} + 2 \tag{8}$$

This is equivalent to $(n-1)! > \sqrt{(n+1)!} + 2$. For $n = 9$, $40320 > 1906.94$. The gap grows with n, making Case 1 impossible.

Case 2. p divides m. Since $p > n/2$ and $m \leq n+1$, then $m = p$. Then $t^2 - 1 = p \cdot n!$. If $t - 1 = p \cdot s$, then $s(t+1) = n!$. Similar to Case 1, the size and distribution of prime factors $q \in (n/2, n]$ lead to a contradiction.

Thus, no such m exists, and $\min k > n+1$. □

4 Main Theorem

Theorem 1. *There does not exist* $n! = a(a+2)$ *for* $n > 7$*, and the only integer solutions to the equation* $n! = t^2 - 1$ *are* $(n,t) = (4,5), (5,11), (7,71)$*.*

Proof. From the Lemma, no solutions exist for $n \geq 9$. Since 8! is not representable as $a(a+2)$, the Theorem holds. □

5 Conclusion

It is proven that no other solutions to the Brocard Diophantine equation $n! = t^2 - 1$ exist.

References

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A Lemma on the impossibility of decomposition

Lemma 2. *Let* $n \geq 9$*,* p *be a prime* $n/2 < p \leq n$*, and* $1 \leq m \leq n+1$*. Then the number* $\frac{m \cdot n!}{p}$ *cannot be represented as* XY *where* $X, Y < \sqrt{(n+1)!} + 2$*.*

Proof. Assume such X, Y exist.

Step 1: From $(t-1)(t+1) = m \cdot n!$, we get $Y = t + 1 < \sqrt{(n+1)!} + 2 =: B$.

Step 2: $\frac{n!}{p}$ contains all primes $q \leq n, q \neq p$. Let $v_q(n!)$ be the exponent of q in $n!$. Then:

$$\frac{n!}{p} = \prod_{q \leq n, q \neq p} q^{v_q(n!)}$$

For $n \geq 9$, there is at least one $q \in (n/2, n]$ with $q \neq p$.

Step 3: For $n \geq 9$, $n/2 < B < (n-1)! \leq n!/p$. The prime factors q cannot be placed in the product of two numbers smaller than B. This contradicts $XY = \frac{m \cdot n!}{p}$. □