Proof of the Collatz Conjecture

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Abstract

Take any positive integer N. If it is odd, multiply it by three and add one. If it is even, divide it by two. Repeatedly do the same operations to the results, forming a sequence. It is found that, whatever the initial number we choose, the sequence will eventually descend and reach number 1, where it enters an eternal closed loop of 1- 4 - 2 - 1. This has been numerically confirmed for initial numbers up to 2^{60}. This is known as the Collatz conjecture which states that the sequence always converges to 1. So far no proof has ever been found that this holds for every positive integer. This problem has been stated by some as perhaps the simplest math problem to state, yet perhaps the most difficult to solve. In this paper, we present a proof that the sequence always converges to 1.

Introduction

The Collatz function is defined as:

$$\mathrm{C(N)} = \begin{cases} 3\mathrm{N}+1\,, & \text{if n is odd} \\ \dfrac{\mathrm{N}}{2}, & \text{if n is even} \end{cases}$$

The Collatz conjecture :

Take any positive number N. If N is odd, multiply it by three and add one. If N is even, divide it by two. Repeatedly do this to form a sequence. The Collatz conjecture says that this sequence will always eventually converge to 1. In this paper, we prove the Collatz conjecture by showing that that all Collatz sequences eventually converge to the number 1.

Collatz Sequence of Odd Terms

A term in a Collatz sequence can be even or odd. We start from the fact that even terms always lead to odd terms because of successive divisions by 2. Therefore, for any initial odd number N, the Collatz sequence can be re-defined as:

$$N \longrightarrow \frac{3N+1}{2^{K1}} \longrightarrow \frac{3\left(\frac{3N+1}{2^{K1}}\right)+1}{2^{K2}} \longrightarrow \frac{3\left(\frac{3\left(\frac{3N+1}{2^{K1}}\right)+1}{2^{K2}}\right)+1}{2^{K3}}$$

$$\longrightarrow \frac{3\left(\frac{3\left(\frac{3\left(\frac{3N+1}{2^{K1}}\right)+1}{2^{K2}}\right)+1}{2^{K3}}\right)+1}{2^{K4}} \longrightarrow \frac{3\left(\frac{3\left(\frac{3\left(\frac{3\left(\frac{3N+1}{2^{K1}}\right)+1}{2^{K2}}\right)+1}{2^{K3}}\right)+1}{2^{K4}}\right)+1}{2^{K5}}$$

$$\longrightarrow \frac{3\left(\frac{3\left(\frac{3\left(\frac{3\left(\frac{3\left(\frac{3N+1}{2^{K1}}\right)+1}{2^{K2}}\right)+1}{2^{K3}}\right)+1}{2^{K4}}\right)+1}{2^{K5}}\right)+1}{2^{K6}} \longrightarrow \cdots$$

All terms of the above sequence are odd numbers because every time an even term occurs in the sequence it is successively divided by 2 until an odd term occurs. That is, when an even term occurs, we don't include it into the sequence but divide it by an integral power of 2 until we get an odd term, which is included into the sequence. Thus, with this we have created an alternative Collatz sequence as shown above with all terms odd.

By expanding each term we get the following:

$$\frac{3\left(\frac{3N+1}{2^{K1}}\right)+1}{2^{K2}} = \frac{3^2N+3+2^{K1}}{2^{K1+K2}}$$

$$\frac{3\left(\frac{3\left(\frac{3N+1}{2^{K1}}\right)+1}{2^{K2}}\right)+1}{2^{K3}} = \frac{3\left(\frac{3^2N+3+2^{K1}}{2^{K1+K2}}\right)+1}{2^{K3}} = \frac{3^3N+3^2+3*2^{K1}+2^{K1+K2}}{2^{K1+K2+K3}}$$

$$\frac{3\left(\frac{3\left(\frac{3\left(\frac{3N+1}{2^{K1}}\right)+1}{2^{K2}}\right)+1}{2^{K3}}\right)+1}{2^{K4}} = \frac{3\left(\frac{3^3N+3^2+3*2^{K1}+2^{K1+K2}}{2^{K1+K2+K3}}\right)+1}{2^{K4}}$$

$$= \frac{3^4N+3^3+3^2*2^{K1}+3*2^{K1+K2}+2^{K1+K2+K3}}{2^{K1+K2+K3+K4}}$$

$$\frac{3\left(\frac{3\left(\frac{3\left(\frac{3N+1}{2^{K1}}\right)+1}{2^{K2}}\right)+1}{2^{K3}}\right)+1}{2^{K4}}\right)+1}{2^{K5}} = \frac{3\left(\frac{3^4N+3^3+3^2*2^{K1}+3*2^{K1+K2}+2^{K1+K2+K3}}{2^{K1+K2+K3+K4}}\right)+1}{2^{K5}}$$

$$= \frac{3^5N + 3^4 + 3^3 * 2^{K1} + 3^2 * 2^{K1+K2} + 3 * 2^{K1+K2+K3} + 2^{K1+K2+K3+K4}}{2^{K1+K2+K3+K4+K5}}$$

and so on.

From the above, we can see that, if we take N to be the first term, then the n^{th} odd Collatz term (C_n) will be:

$$C_n = \frac{3^{n-1}N + 3^{n-2} + 3^{n-3} * 2^{K1} + ... + 3^2 * 2^{K1+K2+...Kn-4} + 3 * 2^{K1+K2+...Kn-3} + 2^{K1+K2+K3+...+Kn-2}}{2^{K1+K2+K3+...+Kn-1}}$$

C_n can be written as:

$$C_n = \frac{3^{n-1}N + 3^{n-2}}{2^{K1+K2+K3+...+Kn-1}} + \frac{3^{n-3} * 2^{K1} + ... + 3^2 * 2^{K1+K2+...Kn-4} + 3 * 2^{K1+K2+...Kn-3} + 2^{K1+K2+K3+...+Kn-2}}{2^{K1+K2+K3+...+Kn-1}}$$

Proof that the Collatz Sequence Always Converges to the Number 1

Now we make the following assumption:

$$\frac{3^{n-1}}{2^{K1+K2+K3+\ ...\ +Kn-1}} \quad , \quad \frac{3^{n-2}}{2^{K1+K2+K3+\ ...\ +Kn-1}} \quad \longrightarrow \quad 0 \text{ as } \ n \ \text{ and } \ K1+K2+K3+\cdots \ \longrightarrow \ \infty$$

The first term is the coefficient of N. This assumption is yet to be proved. The convergence of these terms to zero at infinity (infinite length of the sequence) would reflect the fact that the number of even Collatz operations is significantly greater than the number of odd operations in a sequence, which intuitively means that the sequence always follows a general descent, eventually reaching the number 1. One can intuitively know that the number of even operations is significantly greater than the number of odd operations, and increasingly so at very large or infinitely large numbers. This follows from the fact that Collatz operation on an odd number always results in an even number, whereas an even operation may result in several successive even numbers before an odd number occurs, and this effect is even more pronounced at very large numbers. Note that an odd Collatz operation always gives a number larger than the number on which the operation has been performed and even Collatz operations always give smaller numbers. However, other than such intuitive reasoning, I couldn't conceive of any rigorous proof of the assumption of significantly greater even operations compared to odd operations, that is rigorous in the conventional sense. However, later I will present a proof that is perhaps the only possible rigor for the Collatz conjecture. The proof is based on the fact that after every odd operation, the next term is always an even term and the probability that this even term has large powers of two as a factor (e.g. 2^{1000}) indefinitely increases as the length of the sequence approaches infinity. The rigor of such proof should not be questioned (claiming that it is based on probability) because one can arbitrarily increase this probability making it approach 1 in the limit, that is by assuming infinite length of the sequence.

This assumption is equivalent to:

$$\frac{n}{K1+K2+K3+\ldots} \quad \longrightarrow \quad 0 \quad \text{at infinity}$$

Note that n is the total number of Collatz odd operations and K1+K2+K3+ . . . is the total number of Collatz even operations.

We will make the same assumptions of convergence to zero at infinity for similar terms in our next discussions.

Now as n (and $K1+K2+K3+$ . . . .) approach infinity,

$$C_n \;=\; \cancelto{0}{\frac{3^{n-1}N + 3^{n-2}}{2^{K1+K2+K3+\,...\,+Kn-1}}} \;+\; \frac{3^{n-3} \ast 2^{K1} + \,...\; +3^2 \ast 2^{K1+K2+\,...\,Kn-4} \; +3 \ast 2^{K1+K2+\,...\,Kn-3} + 2^{K1+K2+K3\,+\,...\,+Kn-2}}{2^{K1+K2+K3+\,...\,+Kn-1}}$$

$$C_n \;\longrightarrow\; \frac{3^{n-3} \ast 2^{K1} + 3^{n-4} \ast 2^{K1+K2} + \,...\; +3^2 \ast 2^{K1+K2+\,...\,Kn-4} \; +3 \ast 2^{K1+K2+\,...\,Kn-3} + 2^{K1+K2+K3\,+\,...\,+Kn-2}}{2^{K1+K2+K3+\,...\,+Kn-1}}$$

We now factor out the term 2^{K1} ,

$$C_n \;\longrightarrow\; \frac{2^{K1}\,(3^{n-3} + 3^{n-4} \ast 2^{K2} \; + 3^{n-5} \ast 2^{K2+K3} +.\,.\,. \; +3^2 \ast 2^{K2+\,...\,Kn-4} \; +3 \ast 2^{K2+\,...\,Kn-3} + 2^{K2+K3\,+\,...\,+Kn-2}\,)}{2^{K1+K2+K3+\,...\,+Kn-1}}$$

Cancelling the 2^{K1} term from both the numerator and the denominator,

$$C_n \;\longrightarrow\; \frac{(3^{n-3} + 3^{n-4} \ast 2^{K2} \; + 3^{n-5} \ast 2^{K2+K3} +.\,.\,. \; +3^2 \ast 2^{K2+\,...\,Kn-4} \; +3 \ast 2^{K2+\,...\,Kn-3} + 2^{K2+K3\,+\,...\,+Kn-2}\,)}{2^{K2+K3+\,...\,+Kn-1}}$$

This can be re-written as:

$$C_n \;\longrightarrow\; \frac{(3^{n-3}\,)}{2^{K2+K3+\,...\,+Kn-1}} + \frac{(3^{n-4} \ast 2^{K2} \; + 3^{n-5} \ast 2^{K2+K3} +.\,.\,. \; +3^2 \ast 2^{K2+\,...\,Kn-4} \; +3 \ast 2^{K2+\,...\,Kn-3} + 2^{K2+K3\,+\,...\,+Kn-2}\,)}{2^{K2+K3+\,...\,+Kn-1}}$$

Again, as *n* and K2+K3+ … +Kn-1 approach infinity, the first term diminishes to zero:

$$\frac{(3^{n-3})}{2^{K2+K3+\ .\ .\ .\ +Kn-1}} \quad \longrightarrow \quad 0$$

Therefore:

$$C_n \ \longrightarrow \ \cancelto{0}{\frac{(3^{n-3})}{2^{K2+K3+\ .\ .\ .\ +Kn-1}}} + \frac{(3^{n-4} * 2^{K2} \ + 3^{n-5} * 2^{K2+K3} + .\ .\ . \ +3^2 * 2^{K2+\ .\ .\ .\ Kn-4} \ +3 * 2^{K2+\ .\ .\ .\ Kn-3} + 2^{K2+K3+\ .\ .\ .\ +Kn-2})}{2^{K2+K3+\ .\ .\ .\ +Kn-1}}$$

$$C_n \ \longrightarrow \ \frac{(3^{n-4} * 2^{K2} \ + 3^{n-5} * 2^{K2+K3} + .\ .\ . \ +3^2 * 2^{K2+\ .\ .\ .\ Kn-4} \ +3 * 2^{K2+\ .\ .\ .\ Kn-3} + 2^{K2+K3+\ .\ .\ .\ +Kn-2})}{2^{K2+K3+\ .\ .\ .\ +Kn-1}}$$

Again, factoring out 2^{K2} ,

$$C_n \ \longrightarrow \ \frac{2^{K2}(3^{n-4} \ + 3^{n-5} * 2^{K3} + 3^{n-6} * 2^{K3+K4} \ + .\ .\ . \ +3^2 * 2^{K3+\ .\ .\ .\ Kn-4} \ +3 * 2^{K3+\ .\ .\ .\ Kn-3} + 2^{K3+K4+\ .\ .\ .\ +Kn-2})}{2^{K2+K3+\ .\ .\ .\ +Kn-1}}$$

and cancelling the 2^{K2} term from both the numerator and the denominator,

$$C_n \ \longrightarrow \ \frac{(3^{n-4} \ + 3^{n-5} * 2^{K3} + 3^{n-6} * 2^{K3+K4} \ + .\ .\ . \ +3^2 * 2^{K3+\ .\ .\ .\ Kn-4} \ +3 * 2^{K3+\ .\ .\ .\ Kn-3} + 2^{K3+K4+\ .\ .\ .\ +Kn-2})}{2^{K3+\ .\ .\ .\ +Kn-1}}$$

Again, this can be re-written as:

$$C_n \ \longrightarrow \ \frac{3^{n-4}}{2^{K3+\ .\ .\ .\ +Kn-1}} + \frac{(3^{n-5} * 2^{K3} + 3^{n-6} * 2^{K3+K4} \ + .\ .\ . \ +3^2 * 2^{K3+\ .\ .\ .\ Kn-4} \ +3 * 2^{K3+\ .\ .\ .\ Kn-3} + 2^{K3+K4+\ .\ .\ .\ +Kn-2})}{2^{K3+\ .\ .\ .\ +Kn-1}}$$

Again applying our assumption about convergence to zero at infinity, that is as *n* and K2+K3+ … +Kn-1 approach infinity, the first term diminishes to zero.

$$C_n \longrightarrow \cancelto{0}{\frac{3^{n-4}}{2^{K3+\,...\,+Kn-1}}} + \frac{(3^{n-5} * 2^{K3} + 3^{n-6} * 2^{K3+K4} \ +... \ +3^2 * 2^{K3+\,...\,Kn-4} \ +3 * 2^{K3+\,...\,Kn-3} + 2^{K3+K4+\,...\,+Kn-2})}{2^{K3+\,...\,+Kn-1}}$$

$$C_n \longrightarrow \frac{(3^{n-5} * 2^{K3} + 3^{n-6} * 2^{K3+K4} \ +... \ +3^2 * 2^{K3+\,...\,Kn-4} \ +3 * 2^{K3+\,...\,Kn-3} + 2^{K3+K4+\,...\,+Kn-2})}{2^{K3+\,...\,+Kn-1}}$$

Next we factor out 2^{K3} and repeat the above procedure, then factor out 2^{K4} , and so on.

It can be shown that eventually we get:

$$C_n \longrightarrow \frac{3}{2^{Kn-2+Kn-1}} + \frac{2^{Kn-2}}{2^{Kn-2+Kn-1}}$$

$$C_n \longrightarrow \frac{3}{2^{Kn-2+Kn-1}} + \frac{1}{2^{Kn-1}}$$

$$C_n \longrightarrow \frac{1}{2^{Kn-1}}\left(\frac{3}{2^{Kn-2}} + 1\right) = \frac{\frac{3}{2^{Kn-2}} + 1}{2^{Kn-1}}$$

Now, since C_n can only be a whole number, then the term:

$$\left(\frac{3}{2^{Kn-2}} + 1\right)$$

must be some factor of 2. The only possible case is if 2^{Kn-2} is equal to 1. That is:

$$2^{Kn-2} = 1$$

Therefore,

$$\left(\frac{3}{2^{Kn-2}} \;+\; 1 \right) = \frac{3}{1} + 1 = 4$$

Therefore:

$$C_n \;\longrightarrow\; \frac{\frac{3}{2^{Kn-2}} \;+\; 1}{2^{Kn-1}} = \frac{4}{2^{Kn-1}}$$

Since C_n can only be a whole number, only three possible values are possible for C_n .

$$2^{Kn-1} = 1 \;\Longrightarrow\; C_n = 4$$

$$2^{Kn-1} = 2 \;\Longrightarrow\; C_n = 2$$

$$2^{Kn-1} = 4 \;\Longrightarrow\; C_n = 1$$

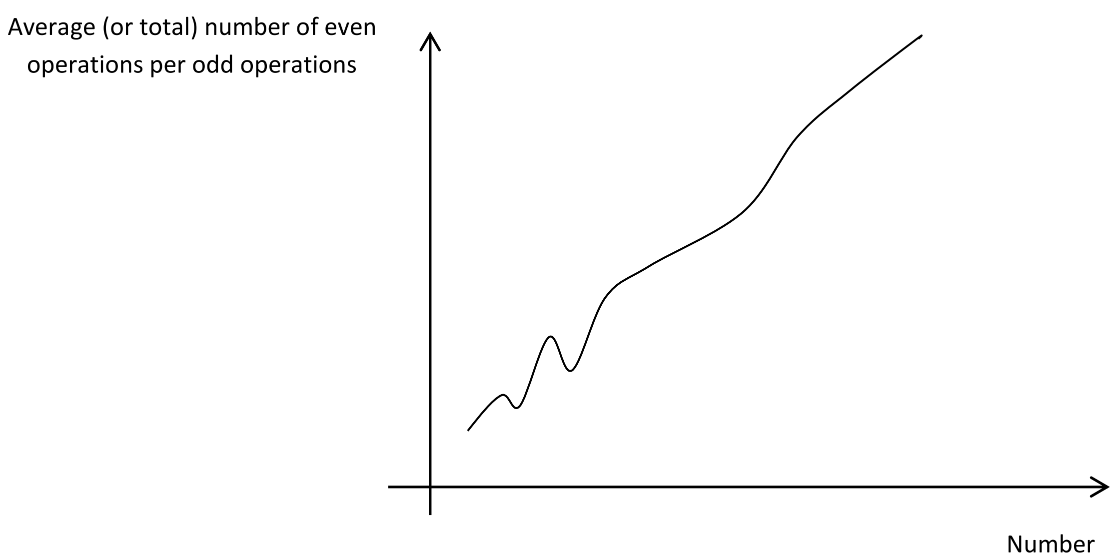
Thus we have proved that all Collatz sequences eventually converge to the 1-4-2-1 loop.

Proof of convergence to zero at infinity

So far, we have reduced the problem of the Collatz conjecture to a problem of proving that those terms indicated as converging to zero at infinity indeed converge to zero. Next I present the proof of convergence to zero of the terms we have assumed as converging to zero at infinity, and perhaps the rigor of the proof is the only possible one for the Collatz conjecture.

We start from the fact that the number of even Collatz operations in a given Collatz sequence is significantly greater than the number of odd Collatz operations. And this effect indefinitely increases at infinitely large numbers. That is, if one starts with a very, very large number, the average (and total) number of even operations will be very large compared to the average number of odd operations, resulting in steep descent of the sequence. In fact, the steepness of the general descent indefinitely increases as the starting value approaches infinity. The steepness of the general descent will gradually decrease as the sequence descents towards finitely large numbers and the descent of the sequence becomes less and less pronounced at relatively small numbers, but eventually reaching 1 in all cases.

Why does the ratio of the number (average or total) of even operations to the number of odd operations increases vastly if the sequence diverged to infinity? The reason for this is that at infinitely large numbers, unlike at smaller numbers, even integers can have very large powers of two as a factor, say 2^{10} , 2^{100} , 2^{1000} etc. Thus, since every Collatz operation on an odd number results in an even number, once the sequence lands on an even number it may go through tens or hundreds or thousands of successive even operations before ending in an odd number, that is before another odd operation occurs again, with successive divide by 2 operations, vastly dropping the sequence. There can never be successive odd operations, but there can always be successive even operations. Therefore, although every odd operation results in a term greater than the term before it, that operation always results in an even operation, which may vastly drop the sequence, vastly more than it was lifted by the single odd operation. In other words, at relatively small numbers, an even number can have 2, 2^2 , 2^3 , 2^4 , etc. as a factor and at very large numbers this could be all the way from 2, 2^2 , . . . to 2^{100} and still at immensely large numbers 2, 2^2 , . . . to 2^{10000} and so on , following the same trend indefinitely. Therefore, as the starting odd number increases, the average and total ratio of even to odd operations also increases accordingly. Thus, after just one odd operation, there could be hundreds of successive even operations before the next odd operation occurs, always resulting in a steep average descent of the sequence.



Note that the above graph is qualitative, meant only to explain the concept.

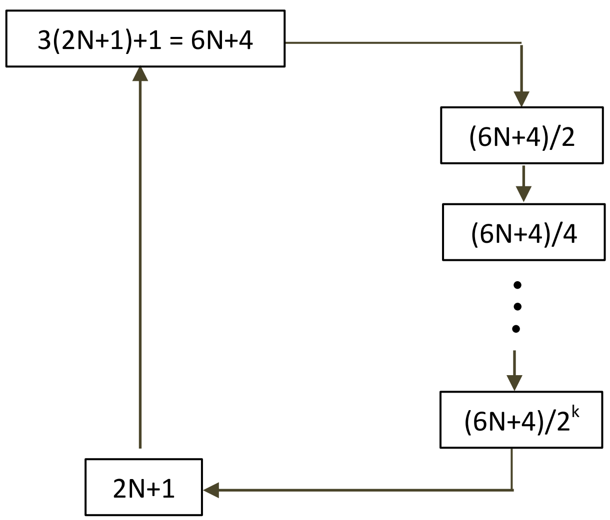
Now, there are two possibilities for the Collatz conjecture to be disproved.

1. The sequence indefinitely diverges to infinity.
2. The sequence may enter some closed loop (there might be more than one loop.

Now assume that the Collatz sequence indefinitely goes to infinity. However, based on our argument above that the (average) number of even Collatz operations for every single odd operation also increases indefinitely, hence leading to the convergence of the terms, confirming our assumption. This is kind of proof by contradiction and we can see the beauty of mathematics here. We started by assuming that a Collatz sequence could go to infinity, but ended in a sequence that is in general descent, ending in the 1-4-2-1-… closed loop.

Next consider the second case of some possible closed loops somewhere at very large numbers. (There is only one closed loop known for initial numbers up to 2^{60}). The question is: what could the ratio of even operations to odd operations be in such a loop? Consider the following closed loop.

We start with an odd integer, $2N + 1$, where N is an integer greater than or equal to zero. Since it is odd, we multiply it by 3 and add 1. To form a loop, successive Collatz operations (divide by 2) on the result must give the original integer,



$$\frac{3(2N+1)+1}{2^k} = 2N+1$$

$$\Longrightarrow \quad 3\,(2N+1)+1 = (2N+1)\,2^k \quad , \quad k \text{ is a positive integer}$$

$$\Longrightarrow \; 6N+4 = (2N+1)\,2^k$$

$$\Longrightarrow \frac{6N+4}{2N+1} = 2^k$$

$$\Longrightarrow \frac{2(\,3N+2\,)}{2N+1} = 2^k$$

$$\Longrightarrow \frac{(\,3N+2\,)}{2N+1} = 2^{k-1}$$

$$\Longrightarrow \frac{(2N+1\,)+(N+1)}{2N+1} = 2^{k-1}$$

$$\Longrightarrow 1+\frac{N+1}{2N+1} = 2^{k-1}$$

Since the right hand side is always a positive integer that is a power of 2, the left hand side must also be an integer (not a fraction) and a power of 2 , for both sides to be equal. This is possible only if *N* = 0.

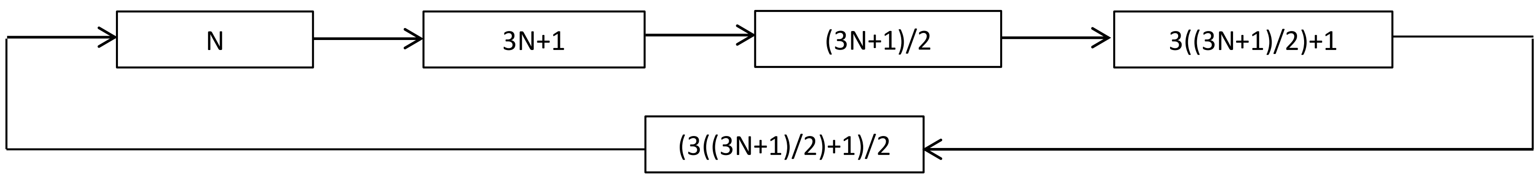
$$\Longrightarrow N = 0$$

Since *N* = 0, our initial odd number *2N+1* will be:

$$\Longrightarrow \; 2N+1 = 2*0+1 = 1$$

Thus we have proved that no other closed loop exists other than the 1-4-2-1 loop which has the same form. However, what we have proved is the absence of any other loop that has the form of the 1-4-2-1 loop, that is a Collatz operation on an odd number gives an even number, which, when divided by 2^k gives the original odd number. Therefore, this proof does not rule out possibility of more complex loops, for example operation on an odd number gives an even number, and operation on the even number gives another odd number, etc. eventually returning to the original odd number.

Note that the 1-4-2-1 loop is of the form Odd (1) → Even (4)→ Even(2)→Odd(1). One would have to check infinitely different possible forms and check if such a closed loop exists. As an illustration, consider a random form of a possible closed loop, and check if such a closed loop exists: Odd → Even →Odd → Even→Odd . The loop would look like this:



Note that, although the initial number is odd, we have designated it by N, instead of 2N+1, to simplify the calculations. For the closed loop to work:

$$N = \frac{3\left(\frac{3N+1}{2}\right)+1}{2}$$

$$2N-1 = 3\left(\frac{3N+1}{2}\right)$$

$$4N-2 = 9N+3$$

$$N = -1$$

Since N cannot be negative, such a loop does not exist. This is only an illustration and it is impossible to check the infinitely possible forms of closed loops.

Now consider the following form: Odd → Even →Odd → Even→ Even→Odd :
(Note that these examples are being discussed only as illustrations and not as any proofs)

$$N = \frac{3\,(\frac{3N+1}{2}) + 1}{4}$$

$$4N - 1 = 3\,(\frac{3N+1}{2})$$

$$8N - 2 = 9N + 3$$

$$N = -5$$

Again, since N cannot be negative, a closed loop with this form does not exist. At this point, one may wonder if generalization of the above calculations is possible for a complete proof.

The question is: is there a way to prove that no closed loops other than the 1-4-2-1 loop exists? Our argument is again based on the "number theory" I have already proposed: the ratio between the number of even Collatz operations to the number of odd Collatz operations in such hypothetical closed loop would be large, ensuring convergence of the terms to zero at infinity. Even if this ratio was 2, as in the 1-4-2-1 loop, this would lead to the convergence of the above terms to zero at infinity, which eventually leads to the 1-4-2-1 loop. We started by assuming a possible loop at some very large number, but ended in the 1-4-2-1 loop!

Having said this, there may still be some gap in my argument that needs to be closed. Can we prove that there is no closed loop at some large number (or at any infinitely large number) in which the number of even Collatz operations is not significantly greater than the number of odd Collatz operations such that the assumptions of convergence we have made won't hold? It should be noted that even a ratio of 2 would be enough to prove our assumption, whereas at infinitely large numbers average (and total) ratios could be tens, hundreds, thousands, millions, etc.

Proof that no closed loops exist other than the 1-4-2-1 loop

The arguments made so far make any other closed loop other than the 1-4-2-1 loop unlikely, but these arguments may not be considered to be a rigorous mathematical proof. Next I present a more complete proof.

We propose a property of a closed loop as follows: *any closed loop is decoupled from the sequences and the numbers in the sequences leading to it*. Since multiple Collatz sequences lead to the closed loop, the closed loop must be independent from any Collatz sequence (and its terms) leading to that loop.

This necessitates the vanishing of the coefficient of N at infinity, in the following equation:

$$C_n = \frac{3^{n-1}N + 3^{n-2}}{2^{K1+K2+K3+\ ...\ +Kn-1}} + \frac{3^{n-3} * 2^{K1} + \ ...\ +3^2 * 2^{K1+K2+\ ...\ Kn-4} +3 * 2^{K1+K2+\ ...\ Kn-3} + 2^{K1+K2+K3+\ ...\ +Kn-2}}{2^{K1+K2+K3+\ ...\ +Kn-1}}$$

Note that the coefficient of N here is:

$$\frac{3^{n-1}}{2^{K1+K2+K3+\ ...\ +Kn-1}}$$

which in turn brings us back to our previous assumption about the convergence of terms to zero at infinity, which in turn leads to the 1-4-2-1 loop! So we began with an assumption that another closed loop could exist somewhere in the darkness of infinity, which demanded the vanishing of the coefficient of N, which led back to the only known closed loop of 1-4-2-1. This should be a complete mathematical proof (by induction) that no other closed loop exists.

In other words, assume that a closed loop exists at some immensely large number. The idea presented here is that any closed loop requires that the coefficient of N in the above equation should diminish to zero in the limit. Assume that the sequence is not decoupled from the starting number even at infinity, that is for infinite length of the sequence. By contradiction, there cannot be such a closed loop because C_n cannot be shown to converge to constant numbers (possible terms in the supposed closed loop), in the same way that we have proved that every Collatz sequence converges to the number 4, 2 or 1. In summary, any closed loop requires the coefficient of N to diminish to zero at infinity, but this leads to the number 4 which is one of the terms in the 1-4-2-1 loop. This proves that no other closed loop can exist.

Argument based on probability

We make an additional argument based on probability. However, this is not meant to imply that the proof we presented so far is not sufficient. Assume some very large or infinitely large odd number N. Since N is odd, the next term in the Collatz sequence will be 3N+1, which is an even number. The next term will be (3N+1)/2, which can be even or odd. The sequence will ascend for every odd operation and descend for every even operation. A closed loop of some form could possibly result from a number of odd and even operations. The sequence would first go in the general ascent and then a general descent, to form a closed loop. My argument is that the *difference* between the initial number and the closest possible number to the initial number is the same order of magnitude as the numbers themselves, whereas a closed form requires *exactly* zero difference. The closed loop will fail even if there is a difference of 1. My argument is that the probability of a closed loop essentially diminishes to zero at infinitely large numbers because the difference between the initial number and any closest number to it in the sequence will also be of the same order of magnitude as the numbers themselves. If no closed loop other than 1-4-2-1 exists at relatively small numbers, how can one expect a closed loop at very large numbers?!

To illustrate this, let us look at the Collatz sequences in the Appendix. Consider the sequence starting by the number 564358543425 (first column). The closest number in the sequence to this number is 602662175044. The difference between these numbers is 38303631619, which is the same order of magnitude as the numbers themselves. Now consider the number 110725 in the same sequence. The closest number in the sequence to it is 83044, the difference now being 27685, which is again the same order of magnitude as the numbers themselves. Next consider the number 59 in the sequence. The closest number in the sequence to it is 67, with the difference being 8. In other words, in the first case, a possible closed loop was ‘missed’ by a margin of 38303631619, in the second case by a margin of 27685, and in the third case by a margin of only 8.

From the above, we can see that the probability of a closed loop at very large numbers is much smaller than at small numbers. In fact, the only known closed loop of the Collatz sequence occurs at the smallest number, which is the number1! If no closed loop exists up to 100 or 1000, how can one expect a closed loop at 1000,000,000,000,000 !

Could such argument based on probability be accepted as a mathematical proof? Could the fact that no other closed loop exists at small numbers (where it is highly probable) rule out the existence of any closed loop at large numbers, where the probability is much smaller?

Conclusion

In this paper, we have been able to prove the Collatz sequence, based on an assumption that certain terms diminish to zero at infinity. This reduces the problem of proving the Collatz conjecture to proving these assumptions. Based on the proposed theory that the number of even Collatz operations is significantly greater than the number of odd Collatz operations in a sequence, we have been able to show that the Collatz sequence will always converge to 1, that no sequence exists that diverges to infinity and no other closed loop other than the 1-4-2-1 loop. The Collatz conjecture has been solved in this paper by introducing an unconventional kind of rigor.
The proof presented in this paper is based on the fact that the probability that an even term in the sequence can have arbitrarily large powers of 2 as a factor can be made to be arbitrarily close to 1 by assuming infinite length of the sequence. I believe that the new kind of rigor applied in this paper is perhaps the only possible one for the Collatz conjecture.

Glory be to Almighty God Jesus Christ and His Mother Our Lady Saint Virgin Mary

APPENDIX

The following are Collatz sequences for four randomly selected initial numbers:
564358543425 , 76736783426, 456734256789, and 87654768546 .

We can see that the number of even operations is always greater than the number of odd operations. For the first number there are 116 odd operations vs. 220 even operations before the sequence reaches number 1, for the second number 84 odd operations vs.168 even operations, for the third number 56 odd operations vs 126 even operations, and for the last number 82 odd operations vs.165 even operations. This explains why the Collatz sequence always converges.

Sequence	**0 if even , 1 if odd**	**Sequence**	**0 if even , 1 if odd**	**Sequence**	**0 if even , 1 if odd**	**Sequence**	**0 if even , 1 if odd**
564358543425	1	76736783426	0	456734256789	1	87654768546	0
1693075630276	0	38368391713	1	1370202770368	0	43827384273	1
846537815138	0	1.15105E+11	0	685101385184	0	131482152820	0
423268907569	1	57552587570	0	342550692592	0	65741076410	0
1269806722708	0	28776293785	1	171275346296	0	32870538205	1
634903361354	0	86328881356	0	85637673148	0	98611614616	0
317451680677	1	43164440678	0	42818836574	0	49305807308	0
952355042032	0	21582220339	1	21409418287	1	24652903654	0
476177521016	0	64746661018	0	64228254862	0	12326451827	1
238088760508	0	32373330509	1	32114127431	1	36979355482	0
119044380254	0	97119991528	0	96342382294	0	18489677741	1
59522190127	1	48559995764	0	48171191147	1	55469033224	0
178566570382	0	24279997882	0	144513573442	0	27734516612	0
89283285191	1	12139998941	1	72256786721	1	13867258306	0
267849855574	0	36419996824	0	216770360164	0	6933629153	1
133924927787	1	18209998412	0	108385180082	0	20800887460	0

401774783362	0	9104999206	0	54192590041	1	10400443730	0
200887391681	1	4552499603	1	162577770124	0	5200221865	1
602662175044	0	13657498810	0	81288885062	0	15600665596	0
301331087522	0	6828749405	1	40644442531	1	7800332798	0
150665543761	1	20486248216	0	121933327594	0	3900166399	1
451996631284	0	10243124108	0	60966663797	1	11700499198	0
225998315642	0	5121562054	0	182899991392	0	5850249599	1
112999157821	1	2560781027	1	91449995696	0	17550748798	0
338997473464	0	7682343082	0	45724997848	0	8775374399	1
169498736732	0	3841171541	1	22862498924	0	26326123198	0
84749368366	0	11523514624	0	11431249462	0	13163061599	1
42374684183	1	5761757312	0	5715624731	1	39489184798	0
127124052550	0	2880878656	0	17146874194	0	19744592399	1
63562026275	1	1440439328	0	8573437097	1	59233777198	0
190686078826	0	720219664	0	25720311292	0	29616888599	1
95343039413	1	360109832	0	12860155646	0	88850665798	0
286029118240	0	180054916	0	6430077823	1	44425332899	1
143014559120	0	90027458	0	19290233470	0	133275998698	0
71507279560	0	45013729	1	9645116735	1	66637999349	1
35753639780	0	135041188	0	28935350206	0	199913998048	0
17876819890	0	67520594	0	14467675103	1	99956999024	0
8938409945	1	33760297	1	43403025310	0	49978499512	0
26815229836	0	101280892	0	21701512655	1	24989249756	0
13407614918	0	50640446	0	65104537966	0	12494624878	0
6703807459	1	25320223	1	32552268983	1	6247312439	1
20111422378	0	75960670	0	97656806950	0	18741937318	0
10055711189	1	37980335	1	48828403475	1	9370968659	1
30167133568	0	113941006	0	146485210426	0	28112905978	0
15083566784	0	56970503	1	73242605213	1	14056452989	1
7541783392	0	170911510	0	219727815640	0	42169358968	0
3770891696	0	85455755	1	109863907820	0	21084679484	0

1885445848	0	256367266	0	54931953910	0	10542339742	0
942722924	0	128183633	1	27465976955	1	5271169871	1
471361462	0	384550900	0	82397930866	0	15813509614	0
235680731	1	192275450	0	41198965433	1	7906754807	1
707042194	0	96137725	1	123596896300	0	23720264422	0
353521097	1	288413176	0	61798448150	0	11860132211	1
1060563292	0	144206588	0	30899224075	1	35580396634	0
530281646	0	72103294	0	92697672226	0	17790198317	1
265140823	1	36051647	1	46348836113	1	53370594952	0
795422470	0	108154942	0	139046508340	0	26685297476	0
397711235	1	54077471	1	69523254170	0	13342648738	0
1193133706	0	162232414	0	34761627085	1	6671324369	1
596566853	1	81116207	1	104284881256	0	20013973108	0
1789700560	0	243348622	0	52142440628	0	10006986554	0
894850280	0	121674311	1	26071220314	0	5003493277	1
447425140	0	365022934	0	13035610157	1	15010479832	0
223712570	0	182511467	1	39106830472	0	7505239916	0
111856285	1	547534402	0	19553415236	0	3752619958	0
335568856	0	273767201	1	9776707618	0	1876309979	1
167784428	0	821301604	0	4888353809	1	5628929938	0
83892214	0	410650802	0	14665061428	0	2814464969	1
41946107	1	205325401	1	7332530714	0	8443394908	0
125838322	0	615976204	0	3666265357	1	4221697454	0
62919161	1	307988102	0	10998796072	0	2110848727	1
188757484	0	153994051	1	5499398036	0	6332546182	0
94378742	0	461982154	0	2749699018	0	3166273091	1
47189371	1	230991077	1	1374849509	1	9498819274	0
141568114	0	692973232	0	4124548528	0	4749409637	1
70784057	1	346486616	0	2062274264	0	14248228912	0
212352172	0	173243308	0	1031137132	0	7124114456	0
106176086	0	86621654	0	515568566	0	3562057228	0

53088043	1	43310827	1	257784283	1	1781028614	0
159264130	0	129932482	0	773352850	0	890514307	1
79632065	1	64966241	1	386676425	1	2671542922	0
238896196	0	194898724	0	1160029276	0	1335771461	1
119448098	0	97449362	0	580014638	0	4007314384	0
59724049	1	48724681	1	290007319	1	2003657192	0
179172148	0	146174044	0	870021958	0	1001828596	0
89586074	0	73087022	0	435010979	1	500914298	0
44793037	1	36543511	1	1305032938	0	250457149	1
134379112	0	109630534	0	652516469	1	751371448	0
67189556	0	54815267	1	1957549408	0	375685724	0
33594778	0	164445802	0	978774704	0	187842862	0
16797389	1	82222901	1	489387352	0	93921431	1
50392168	0	246668704	0	244693676	0	281764294	0
25196084	0	123334352	0	122346838	0	140882147	1
12598042	0	61667176	0	61173419	1	422646442	0
6299021	1	30833588	0	183520258	0	211323221	1
18897064	0	15416794	0	91760129	1	633969664	0
9448532	0	7708397	1	275280388	0	316984832	0
4724266	0	23125192	0	137640194	0	158492416	0
2362133	1	11562596	0	68820097	1	79246208	0
7086400	0	5781298	0	206460292	0	39623104	0
3543200	0	2890649	1	103230146	0	19811552	0
1771600	0	8671948	0	51615073	1	9905776	0
885800	0	4335974	0	154845220	0	4952888	0
442900	0	2167987	1	77422610	0	2476444	0
221450	0	6503962	0	38711305	1	1238222	0
110725	1	3251981	1	116133916	0	619111	1
332176	0	9755944	0	58066958	0	1857334	0
166088	0	4877972	0	29033479	1	928667	1
83044	0	2438986	0	87100438	0	2786002	0

41522	0	1219493	1	43550219	1	1393001	1
20761	1	3658480	0	130650658	0	4179004	0
62284	0	1829240	0	65325329	1	2089502	0
31142	0	914620	0	195975988	0	1044751	1
15571	1	457310	0	97987994	0	3134254	0
46714	0	228655	1	48993997	1	1567127	1
23357	1	685966	0	146981992	0	4701382	0
70072	0	342983	1	73490996	0	2350691	1
35036	0	1028950	0	36745498	0	7052074	0
17518	0	514475	1	18372749	1	3526037	1
8759	1	1543426	0	55118248	0	10578112	0
26278	0	771713	1	27559124	0	5289056	0
13139	1	2315140	0	13779562	0	2644528	0
39418	0	1157570	0	6889781	1	1322264	0
19709	1	578785	1	20669344	0	661132	0
59128	0	1736356	0	10334672	0	330566	0
29564	0	868178	0	5167336	0	165283	1
14782	0	434089	1	2583668	0	495850	0
7391	1	1302268	0	1291834	0	247925	1
22174	0	651134	0	645917	1	743776	0
11087	1	325567	1	1937752	0	371888	0
33262	0	976702	0	968876	0	185944	0
16631	1	488351	1	484438	0	92972	0
49894	0	1465054	0	242219	1	46486	0
24947	1	732527	1	726658	0	23243	1
74842	0	2197582	0	363329	1	69730	0
37421	1	1098791	1	1089988	0	34865	1
112264	0	3296374	0	544994	0	104596	0
56132	0	1648187	1	272497	1	52298	0
28066	0	4944562	0	817492	0	26149	1
14033	1	2472281	1	408746	0	78448	0

42100	0	7416844	0	204373	1	39224	0
21050	0	3708422	0	613120	0	19612	0
10525	1	1854211	1	306560	0	9806	0
31576	0	5562634	0	153280	0	4903	1
15788	0	2781317	1	76640	0	14710	0
7894	0	8343952	0	38320	0	7355	1
3947	1	4171976	0	19160	0	22066	0
11842	0	2085988	0	9580	0	11033	1
5921	1	1042994	0	4790	0	33100	0
17764	0	521497	1	2395	1	16550	0
8882	0	1564492	0	7186	0	8275	1
4441	1	782246	0	3593	1	24826	0
13324	0	391123	1	10780	0	12413	1
6662	0	1173370	0	5390	0	37240	0
3331	1	586685	1	2695	1	18620	0
9994	0	1760056	0	8086	0	9310	0
4997	1	880028	0	4043	1	4655	1
14992	0	440014	0	12130	0	13966	0
7496	0	220007	1	6065	1	6983	1
3748	0	660022	0	18196	0	20950	0
1874	0	330011	1	9098	0	10475	1
937	1	990034	0	4549	1	31426	0
2812	0	495017	1	13648	0	15713	1
1406	0	1485052	0	6824	0	47140	0
703	1	742526	0	3412	0	23570	0
2110	0	371263	1	1706	0	11785	1
1055	1	1113790	0	853	1	35356	0
3166	0	556895	1	2560	0	17678	0
1583	1	1670686	0	1280	0	8839	1
4750	0	835343	1	640	0	26518	0
2375	1	2506030	0	320	0	13259	1

7126	0	1253015	1	160	0	39778	0
3563	1	3759046	0	80	0	19889	1
10690	0	1879523	1	40	0	59668	0
5345	1	5638570	0	20	0	29834	0
16036	0	2819285	1	10	0	14917	1
8018	0	8457856	0	5	1	44752	0
4009	1	4228928	0	16	0	22376	0
12028	0	2114464	0	8	0	11188	0
6014	0	1057232	0	4	0	5594	0
3007	1	528616	0	2	0	2797	1
9022	0	264308	0	1	1	8392	0
4511	1	132154	0	4	0	4196	0
13534	0	66077	1	2	0	2098	0
6767	1	198232	0	1	1	1049	1
20302	0	99116	0	4	0	3148	0
10151	1	49558	0	2	0	1574	0
30454	0	24779	1	1	1	787	1
15227	1	74338	0	4	0	2362	0
45682	0	37169	1	2	0	1181	1
22841	1	111508	0	1	1	3544	0
68524	0	55754	0	4	0	1772	0
34262	0	27877	1	2	0	886	0
17131	1	83632	0	1	1	443	1
51394	0	41816	0	4	0	1330	0
25697	1	20908	0	2	0	665	1
77092	0	10454	0	1	1	1996	0
38546	0	5227	1	4	0	998	0
19273	1	15682	0	2	0	499	1
57820	0	7841	1	1	1	1498	0
28910	0	23524	0	4	0	749	1
14455	1	11762	0	2	0	2248	0

43366	0	5881	1	1	1	1124	0
21683	1	17644	0	4	0	562	0
65050	0	8822	0	2	0	281	1
32525	1	4411	1	1	1	844	0
97576	0	13234	0	4	0	422	0
48788	0	6617	1	2	0	211	1
24394	0	19852	0	1	1	634	0
12197	1	9926	0	4	0	317	1
36592	0	4963	1	2	0	952	0
18296	0	14890	0	1	1	476	0
9148	0	7445	1	4	0	238	0
4574	0	22336	0	2	0	119	1
2287	1	11168	0	1	1	358	0
6862	0	5584	0	4	0	179	1
3431	1	2792	0	2	0	538	0
10294	0	1396	0	1	1	269	1
5147	1	698	0	4	0	808	0
15442	0	349	1	2	0	404	0
7721	1	1048	0	1	1	202	0
23164	0	524	0	4	0	101	1
11582	0	262	0	2	0	304	0
5791	1	131	1	1	1	152	0
17374	0	394	0	4	0	76	0
8687	1	197	1	2	0	38	0
26062	0	592	0	1	1	19	1
13031	1	296	0	4	0	58	0
39094	0	148	0	2	0	29	1
19547	1	74	0	1	1	88	0
58642	0	37	1	4	0	44	0
29321	1	112	0	2	0	22	0
87964	0	56	0	1	1	11	1

43982	0	28	0	4	0	34	0
21991	1	14	0	2	0	17	1
65974	0	7	1	1	1	52	0
32987	1	22	0	4	0	26	0
98962	0	11	1	2	0	13	1
49481	1	34	0	1	1	40	0
148444	0	17	1	4	0	20	0
74222	0	52	0	2	0	10	0
37111	1	26	0	1	1	5	1
111334	0	13	1	4	0	16	0
55667	1	40	0	2	0	8	0
167002	0	20	0	1	1	4	0
83501	1	10	0	4	0	2	0
250504	0	5	1	2	0	1	1
125252	0	16	0	1	1	4	0
62626	0	8	0	4	0	2	0
31313	1	4	0	2	0	1	1
93940	0	2	0	1	1	4	0
46970	0	1	1	4	0	2	0
23485	1	4	0	2	0	1	1
70456	0	2	0	1	1	4	0
35228	0	1	1	4	0	2	0
17614	0	4	0	2	0	1	1
8807	1	2	0	1	1	4	0
26422	0	1	1	4	0	2	0
13211	1	4	0	2	0	1	1
39634	0	2	0	1	1	4	0
19817	1	1	1	4	0	2	0
59452	0	4	0	2	0	1	1
29726	0	2	0	1	1	4	0
14863	1	1	1	4	0	2	0

44590	0	4	0	2	0	1	1
22295	1	2	0	1	1	4	0
66886	0	1	1	4	0	2	0
33443	1	4	0	2	0	1	1
100330	0	2	0	1	1	4	0
50165	1	1	1	4	0	2	0
150496	0	4	0	2	0	1	1
75248	0	2	0	1	1	4	0
37624	0	1	1	4	0	2	0
18812	0	4	0	2	0	1	1
9406	0	2	0	1	1	4	0
4703	1	1	1	4	0	2	0
14110	0	4	0	2	0	1	1
7055	1	2	0	1	1	4	0
21166	0	1	1	4	0	2	0
10583	1	4	0	2	0	1	1
31750	0	2	0	1	1	4	0
15875	1	1	1	4	0	2	0
47626	0	4	0	2	0	1	1
23813	1	2	0	1	1	4	0
71440	0	1	1	4	0	2	0
35720	0	4	0	2	0	1	1
17860	0	2	0	1	1	4	0
8930	0	1	1	4	0	2	0
4465	1	4	0	2	0	1	1
13396	0	2	0	1	1	4	0
6698	0	1	1	4	0	2	0
3349	1	4	0	2	0	1	1
10048	0	2	0	1	1	4	0
5024	0	1	1	4	0	2	0
2512	0	4	0	2	0	1	1

1256	0	2	0	1	1	4	0
628	0	1	1	4	0	2	0
314	0	4	0	2	0	1	1
157	1	2	0	1	1	4	0
472	0	1	1	4	0	2	0
236	0	4	0	2	0	1	1
118	0	2	0	1	1	4	0
59	1	1	1	4	0	2	0
178	0	4	0	2	0	1	1
89	1	2	0	1	1	4	0
268	0	1	1	4	0	2	0
134	0	4	0	2	0	1	1
67	1	2	0	1	1	4	0
202	0	1	1	4	0	2	0
101	1	4	0	2	0	1	1
304	0	2	0	1	1	4	0
152	0	1	1	4	0	2	0
76	0	4	0	2	0	1	1
38	0	2	0	1	1	4	0
19	1	1	1	4	0	2	0
58	0	4	0	2	0	1	1
29	1	2	0	1	1	4	0
88	0	1	1	4	0	2	0
44	0	4	0	2	0	1	1
22	0	2	0	1	1	4	0
11	1	1	1	4	0	2	0
34	0	4	0	2	0	1	1
17	1	2	0	1	1	4	0
52	0	1	1	4	0	2	0
26	0	4	0	2	0	1	1
13	1	2	0	1	1	4	0

40	0	1	1	4	0	2	0
20	0	4	0	2	0	1	1
10	0	2	0	1	1	4	0
5	1	1	1	4	0	2	0
16	0	4	0	2	0	1	1
8	0	2	0	1	1	4	0
4	0	1	1	4	0	2	0
2	0	4	0	2	0	1	1
1	1	2	0	1	1	4	0
4	0	1	1	4	0	2	0
2	0	4	0	2	0	1	1
1	1	2	0	1	1	4	0