A Novel Derivation of the Reissner-Nordstrom and Kerr-Newman Black Hole Entropy from truly Charge Spinning Point Mass Sources

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November 2023

Abstract

Recently we have shown how the Schwarzschild Black Hole Entropy in all dimensions emerges from truly point mass sources at $r = 0$ due to a non-vanishing scalar curvature $\mathcal{R}$ involving the Dirac delta distribution in the computation of the Euclidean Einstein-Hilbert action. As usual, it is required to take the inverse Hawking temperature β as the length of the circle S^1_β obtained from a compactification of the Euclidean time in thermal field theory which results after a Wick rotation, $it = \tau$, to imaginary time. In this work we extend our novel procedure to evaluate both the Reissner-Nordstrom and Kerr-Newman black hole entropy from truly charge spinning point mass sources.

Keywords : General Relativity; Black Holes; Entropy.

PACS : 04.60.-m, 04.65.+e, 11.15.-q, 11.30.Ly

Recently we have shown how the Schwarzschild Black Hole Entropy (in all dimensions) emerges from truly point mass sources at $r = 0$ due to a non-vanishing scalar curvature involving the Dirac delta distribution [8]. It is the density and *anisotropic* pressure components associated with the point mass delta function source at the origin $r = 0$ which furnish the Schwarzschild black hole entropy in all dimensions $D \geq 4$ after evaluating the Euclidean Einstein-Hilbert action. As usual, it is required to take the inverse Hawking temperature β as the length of the circle S^1_β obtained from a compactification of the Euclidean time in thermal field theory which results after a Wick rotation, $it = \tau$, to imaginary time. The appealing and salient result is that there is *no* need to introduce the Gibbons-Hawking-York boundary term [5], [6] in order to arrive at the black hole entropy because in our case one has that $\mathcal{R} \neq 0$. Furthermore, there is no need to introduce a complex integration contour to *avoid* the singularity as shown by Gibbons and Hawking. On the contrary, the source of the black hole entropy stems entirely from the scalar curvature *singularity* at the origin $r = 0$. In this work we show how to generalize our construction in order to derive the Reissner-Nordstrom [10] and Kerr-Newman [13] black hole entropy. The physical implications of this finding warrants further investigation since it suggests a profound connection between the notion of gravitational entropy and spacetime singularities.

We shall use throughout this work the units of $\hbar = c = k_B = 1$. The higher-dimensional extension of the Schwarzschild metric [2], [3] was found by Tangherlini [4] and is given by

$$ds^2 \;=\; -\; f(r)\;(dt)^2 + \frac{(dr)^2}{f(r)} + r^2\;(d\Omega_{D-2})^2, \;\; f(r) \;=\; 1 - \frac{16\pi G_D M}{(D-2)\Omega_{D-2} r^{D-3}} \tag{1}$$

where G_D is the D-dim Newton's constant, M the black hole mass. The solid angle of a $D-2$-dim hypersphere is $\Omega_{D-2} = 2\pi^{\frac{D-1}{2}}/\Gamma(\frac{D-1}{2})$. The horizon radius is determined from the condition $f(r_h) = 0$ giving

$$r_h \;=\; \left(\frac{16\pi G_D M}{(D-2)\;\Omega_{D-2}} \right)^{\frac{1}{D-3}} \tag{2}$$

such that the metric (1) can be rewritten as

$$ds^2 \;=\; -\;[\;1 - (\frac{r_h}{r})^{D-3}\;]\;(dt)^2 \;+\; [\;1 - (\frac{r_h}{r})^{D-3}\;]^{-1}\;(dr)^2 \;+\; r^2\;(d\Omega_{D-2})^2 \tag{3}$$

The Schwarzschild metric leads to a vanishing Ricci tensor and scalar curvature $\mathcal{R} = 0$, hence in order to arrive at a key delta function singularity at the origin one has to replace r for $|r|$ in the metric (1). More precisely, one needs to make the replacement $f(r) \to f(|r|)$ in (3) as follows

$$1 - (\frac{r_h}{r})^{D-3} \;\to\; 1 - (\frac{r_h}{|r|})^{D-3} \;=\; 1 - [(\frac{r_h}{r})(\frac{r}{|r|})]^{D-3} \;=\; 1 - [(\frac{r_h}{r}) sgn(r)]^{D-3} \tag{4}$$

The ratio $\frac{r}{|r|} = \frac{|r| sgn(r)}{|r|} = sgn(r)$ can be expressed in terms of sign function $sgn(r)$, and which is defined by $sgn(r) = 1$, for $r > 0$; $sgn(r) = -1$, for $r < 0$; and $sgn(r = 0) = 0$, the arithmetic mean of $1, -1$, and it will be instrumental in deriving the non-zero scalar curvature. The derivative of the sign function is $\frac{d}{dr} sgn(r) = 2\delta(r)$ [1]. It is the derivatives of the sign function appearing in eq-(4) which will generate the key $\delta(r)$ terms in the scalar curvature. If one wishes to be mathematically rigorous in using distributions in nonlinear theories like

[1]The factor of 2 is due to the jump of 2 from -1 to $+1$

general relativity one needs to recur to the Colombeau's theory of distributions [7] instead of the Dirac delta distributions.

Therefore the metric one shall be working with is

$$ds^2 \;=\; -\; f(|r|)\;(dt)^2 \;+\; \frac{(dr)^2}{f(|r|)} \;+\; |r|^2\;(d\Omega_{D-2})^2 \;=$$

$$-\;(1-(\frac{r_h}{|r|})^{D-3})\;(dt)^2 \;+\; (1-(\frac{r_h}{|r|})^{D-3})^{-1}\;(dr)^2 \;+\; |r|^2\;(d\Omega_{D-2})^2 \qquad (5)$$

After a very lengthy and laborious calculation one learns that the scalar curvature associated is given by

$$\mathcal{R} \;=\; \frac{d^2 f}{dr^2} \;+\; \frac{2(D-2)}{r}\;\frac{df}{dr} \;-\; \frac{(D-2)(D-3)}{r^2}\;(1-f) \qquad (6)$$

Taking into account now that $\frac{d|r|}{dr} = sgn(r)$ [2] where $sgn(r)$ is the sign function it leads to the following results

$$\frac{d}{dr}sgn(r) \;=\; 2\;\delta(r), \quad \frac{df}{dr} \;=\; (D-3)\;r_h^{D-3}\;\frac{sgn(r)}{|r|^{D-2}},$$

$$\frac{d^2 f}{dr^2} \;=\; -\;(D-2)\;(D-3)\;r_h^{D-3}\;\frac{1}{|r|^{D-1}} \;+\; 2(D-3)\;r_h^{D-3}\;\frac{\delta(r)}{|r|^{D-2}} \qquad (7)$$

Inserting the results of eq-(7) into eq-(6) and taking into account the *identity* $r = |r| sgn(r)$ which leads to key exact *cancellations*, the scalar curvature in eq-(6) turns out to be

$$\mathcal{R}_D \;=\; 2\;\frac{16\pi G_D M}{(D-2)\Omega_{D-2}}\;(D-3)\;\frac{\delta(r)}{|r|^{D-2}} \;=\; 2\;r_h^{D-3}\;(D-3)\;\frac{\delta(r)}{|r|^{D-2}} \qquad (8)$$

The use of $|r|$ in $f(|r|)$ was instrumental in generating the delta function in (8). Had one used $f(r)$ one would have obtained $\mathcal{R} = 0$.

As usual, it is required to take the inverse Hawking temperature β_H as the length of the circle S^1_β obtained from a compactification of the Euclidean time in thermal field theory which results after a Wick rotation, $it = \tau$, to imaginary time. The Hawking temperature of the D-dim Schwarzschild black hole is $T_D = (D-3)/4\pi r_h \Rightarrow \beta_D = 4\pi r_h/(D-3)$, so that the non-trivial Euclidean Einstein-Hilbert action in D-dim is given by the integral

$$I \;=\; -\;\frac{i}{16\pi G_D}\;\int_0^{\beta_D}\;d\tau\;\int_0^\infty\;\mathcal{R}_D\;\Omega_{D-2}\;r^{D-2}\;dr \qquad (9)$$

Note the presence of an $-i$ factor in the Euclidean action I which results from the measure $\sqrt{-g}$ piece since the determinant $g = det(g_{\mu\nu}) > 0$ is now positive due to the Euclidean signature. The minus sign $-i$ is chosen so that $exp(iS_g) =$

[2]The derivative of $|r|$ is discontinuous at $r = 0$, but because it jumps from -1 to $+1$, one may take their arithmetic mean which is 0 and which agrees with the value of $sgn(r = 0) = 0$

$exp(-I)$ in the gravitational path integral ($I = -iS_g$). In the region where $r \geq 0$ one can replace $|r|^{D-2}$ for r^{D-2}, and after taking into account that the radial integral (9) is symmetric in r due to $\delta(-r) = \delta(r)$, one has to extend the radial domain of integration as follows

$$\int_0^\infty \delta(r)\ dr \ = \ \frac{1}{2} \int_{-\infty}^\infty \delta(r)\ dr \ = \ \frac{1}{2} \tag{10}$$

in order to fully integrate the delta function. Upon setting $\beta_D = 4\pi r_h/(D-3)$, and inserting the expression (8) for $\mathcal{R}_D$ into (9), one arrives finally at

$$|I| \ = \ \frac{\Omega_{D-2}\ r_h^{D-2}}{4G_D} \ = \ \frac{\Omega_{D-2}}{4G_D} \left(\frac{16\pi G_D M}{(D-2)\ \Omega_{D-2}} \right)^{\frac{D-2}{D-3}} \tag{11}$$

which is the Schwarzschild black hole entropy in D-dimensions. When $D = 4$ one arrives at $4\pi(2GM)^2/4G = 4\pi GM^2$ as expected.

Next we shall find the expressions for the density and pressure of the point-matter source leading to a non-vanishing scalar curvature and which furnishes the higher dimensional black hole entropy. Given the trace of the stress energy tensor $\mathcal{T}_D = T^\mu_\mu$, the *trace* of the Einstein tensor $G_{\mu\nu} = \mathcal{R}_{\mu\nu} - \frac{1}{2} g_{\mu\nu}\mathcal{R}$ obeys the following relation stemming from the field equations

$$-\ \mathcal{R}_D\ \frac{(D-2)}{2} \ = \ 8\pi G_D\ \mathcal{T}_D \ = \ -\ (8\pi G_D) \left(2\ (D-3)\ \frac{M}{\Omega_{D-2}}\ \frac{\delta(r)}{|r|^{D-2}} \right) \tag{12}$$

since the spherically symmetric energy-mass density ρ in D-dim for a point mass source is given by [3]

$$\rho \ = \ \frac{2M}{\Omega_{D-2}\ |r|^{D-2}}\ \delta(r) \ \Rightarrow \ \int_0^\infty \rho\ \Omega_{D-2}\ r^{D-2}\ dr \ = \ 2M \int_0^\infty \delta(r)\ dr \ = \ M \tag{13}$$

one finds that the trace of the stress energy tensor is

$$\mathcal{T}_D \ = \ -\ (D-3)\ [\ \frac{2M}{\Omega_{D-2}\ |r|^{D-2}}\ \delta(r)\ ] \ = \ -\ (D-3)\ \rho \tag{14}$$

Due to the (hyper) spherical symmetry, the $D-2$ transverse pressure components $p_\perp$ to the radial direction are all equal, then the expression in (14) leads to

$$\mathcal{T}_D \ = \ -\rho \ + \ p_r \ + \ (D-2)\ p_\perp \ = \ -\ (D-3)\ \rho \tag{15}$$

One must supplement eq-(15) with the Einstein field equations in order to determine ρ, p_r and the $D-2$ transverse pressure components $p_\perp = p_{\theta_i}, i = 1, 2, \cdots, D-2$,

[3]Note the key extra factor of 2 in eq-(13) that is required to evaluate the integral of $\delta(r)$

$$\mathcal{R}^t_t - \frac{1}{2}\, \delta^t_t\, \mathcal{R} \;=\; 8\pi G_D T^t_t \;=\; -\, 8\pi G_D\, \rho, \;\; \mathcal{R}^r_r - \frac{1}{2}\, \delta^r_r\, \mathcal{R} \;=\; 8\pi G_D T^r_r \;=\; 8\pi G_D\, p_r \tag{21}$$

$$\mathcal{R}^\perp_\perp \;-\; \frac{1}{2}\, \delta^\perp_\perp\, \mathcal{R} \;= 8\pi G_D T^\perp_\perp \;=\; 8\pi G_D\, p_\perp \tag{16}$$

After a lengthy but straightforward algebra one finds that the density and pressure components are

$$\rho \;=\; \frac{2M}{\Omega_{D-2}\; |r|^{D-2}}\; \delta(r), \quad p_r \;=\; -\; \frac{2(D-3)}{(D-2)}\; \rho,$$

$$p_\perp \;=\; \left(\frac{(4-D)(D-2) + 2(D-3)}{(D-2)^2} \right)\; \rho \;\Rightarrow\; -\rho + p_r + (D{-}2)p_\perp \;=\; -(D{-}3)\rho \tag{17}$$

The solutions (17) satisfy the *strong* energy conditions $\rho + \sum p_i \geq 0$ but not the weak energy conditions $\rho + p_i \geq 0$ for all $i = 1, 2, \cdots, D-1$.

One may object to the above expressions (17) because the angular coordinates are not well defined at $r = 0$. This is not a problem because one can simply perform a coordinate change of the stress energy tensor $T_{\mu\nu}$ to Cartesian coordinates which are well defined at $r = 0$ [4]. The solutions (17) are consistent with the conservation equation of the stress energy tensor $\nabla_\mu T^{\mu\nu} = 0$. It can be more easily verified in $D = 4$ where one arrives at

$$\rho \;=\; -\; p_r \;=\; \frac{2M}{4\pi r^2}\; \delta(r), \quad p_\perp \;=\; \frac{1}{2}\; \rho \;=\; \frac{M}{4\pi r^2}\; \delta(r) \tag{18}$$

One can check that the expressions (18) are consistent with the conservation equation

$$\nabla_\mu T^{\mu\nu} = 0 \;\Rightarrow\; p_\perp \;+\; \rho \;+\; \frac{r}{2}\; \frac{d\rho}{dr} \;=\; 0 \tag{19}$$

and which can be verified explicitly after using the identities $r\frac{d}{dr}(\delta(r)) = -\delta(r)$; $r^n \frac{d^n}{dr^n}(\delta(r)) = (-1)^n n! \delta(r)$. Similar results as those found in eqs-(18,19) were obtained in [9] by choosing a mass density given by a Gaussian $M(\sigma)^{-3/2} exp(-r^2/\sigma)$ where the Gaussian width $\sqrt{\sigma}$ was related to the noncommutativity parameter associated with the noncommutative spacetime coordinates $[\hat{x}^\mu, \hat{x}^\nu] \;=\; i\Theta^{\mu\nu}\mathbf{1}$ after equating the norm to $\sigma :\; \sqrt{\Theta_{\mu\nu}\Theta^{\mu\nu}} = \sigma$.

After this discussion one concludes that the expressions (17) are the density and *anisotropic* pressure components associated with the point mass delta function source at the origin $r = 0$ and which furnish the Schwarzschild black hole entropy (up to a factor of $-i$) in all dimensions $D \geq 4$ by a direct evaluation of the Euclidean Einstein-Hilbert action.

[4] In Cartesian coordinates the stress energy tensor will have off-diagonal components

Let us derive now the black hole entropy for the four-dim Reissner-Nordstrom charged black hole of mass M and charge q [10]. After replacing $r \rightarrow |r|$ it yields

$$ds^2 \;=\; -\,(1 - \frac{2GM}{|r|} + \frac{q^2}{r^2})\;(dt)^2 + (1 - \frac{2GM}{|r|} + \frac{q^2}{r^2})^{-1}\;(dr)^2 + r^2\;(d\Omega)^2,\;\; r^2 = |r|^2 \tag{20}$$

where the solid angle infinitesimal element is $(d\Omega)^2 = (d\theta)^2 + sin^2(\theta)(d\phi)^2$.

In $D = 4$ the Maxwell action is conformally invariant and as a result the electromagnetic stress energy tensor is traceless since under infinitesimal conformal scalings of the metric one has $\delta g^{\mu\nu} = \lambda g^{\mu\nu}$ so that $\delta \mathcal{L}_{EM} = (\frac{\delta \mathcal{L}_{EM}}{\delta g^{\mu\nu}})\delta g^{\mu\nu} = -\lambda \frac{\sqrt{-g}}{2} T^{(EM)}_{\mu\nu} g^{\mu\nu} = -\lambda \frac{\sqrt{-g}}{2} T^{(EM)} = 0$, hence one finds that the trace $T^{(EM)} = 0$. Therefore there is no contribution to the scalar curvature scalar $\mathcal{R}$ from the EM field, so the value of $\mathcal{R}$ is due entirely to the point-mass delta function source and given by $\mathcal{R} = 4GM \frac{\delta(r)}{r^2}$. The inverse Hawking temperature for the Reissner-Nordstrom black hole is given in terms of the outer horizon radius r_+ as [12]

$$\beta \;=\; \frac{2\pi (r_+)^3}{G(Mr_+ - q^2)},\;\; r_+ \;=\; GM \;+\; \sqrt{(GM)^2 \;-\; q^2 G} \tag{21}$$

therefore, the Euclidean Einstein-Hilbert action becomes

$$I \;=\; -\;\frac{i}{16\pi G}\int_0^\beta d\tau \int_0^\infty \mathcal{R}\; 4\pi r^2\; dr \;=\; -\;\frac{i}{2}\;\beta\; M \;=\; -i\;\frac{\pi (r_+)^3 M}{G(Mr_+ - q^2)} \tag{22}$$

One must add now the EM contribution to the Euclidean action. The canonical action is $S_{EM} = -\frac{1}{4}\int d^4x \sqrt{-g} F^2$. However, when one combines the gravitational action S_g with the EM action one must take into account a multiplicative factor α such that the variation of the combined system is $\delta[\frac{1}{16\pi G} S_g + \alpha S_{EM}] = 0$ and is consistent with the Einstein field equations $\mathcal{R}_{\mu\nu} - \frac{1}{2} g_{\mu\nu}\mathcal{R} = 8\pi G T_{\mu\nu}$. The multiplicative factor α which is consistent with the following expression for the EM stress energy tensor

$$T^{(EM)}_{\mu\nu} \;=\; \frac{1}{4\pi}\;[\; F_{\mu\sigma} F^{\sigma}_{\;\;\nu} \;-\; \frac{1}{4}\; g_{\mu\nu}\; F_{\alpha\beta} F^{\alpha\beta}\;] \tag{23}$$

turns out to be $\alpha = \frac{1}{4\pi}$. For further details of the need to introduce a multiplicative factor α see [12] (Appendix E) . Therefore one must evaluate the integral after the Wick rotation to Euclidean time,

$$I_{EM} \;=\; (-i)\;\alpha\; S_{EM} \;=\; \frac{i}{16\pi}\int d^4x\; \sqrt{-g}\; F_{\alpha\beta} F^{\alpha\beta} \tag{24}$$

The solution for the components of A_μ corresponding to the Einstein-Maxwell system is $A_\mu = (\frac{q}{|r|}, 0, 0, 0)$ (one does not include the magnetic monopole solution) and the only non-vanishing component of $F_{\mu\nu}$ is $F_{rt} = \nabla_r A_t - \nabla_t A_r = \partial_r A_t - \partial_t A_r = \frac{-q sgn(r)}{r^2}$. Substituting F_{rt} into (24) and taking into account that for the Euclidean metric one has $F_{rt}F^{rt} = F_{rt}F_{rt} g^{rr} g^{tt} = (F_{rt})^2$, it gives

$$I_{EM} \ = \ i \ \beta \ \left(\frac{1}{4} \int_0^\infty \frac{q^2}{r^2} \ dr \right) \ = \ - \ i \ \beta \ \frac{1}{4} \ [\frac{q^2}{r}]_0^\infty \tag{25}$$

The I_{EM} diverges as expected due to the singularity at $r = 0$. We are going to introduce an ultraviolet cutoff ϵ and split the integral domain into $[\epsilon, r_o]$ and $[r_o, \infty]$, where r_o is given by $r_o = \frac{r_+}{2}$ (inside the outer horizon). In doing so one has

$$[\frac{q^2}{r}]_\epsilon^\infty \ = \ [\frac{q^2}{r}]_\epsilon^{r_o} \ + \ [\frac{q^2}{r}]_{r_o}^\infty \ = \ (\frac{2q^2}{r_+} \ - \ \frac{q^2}{\epsilon}) \ + \ (0 - \frac{2q^2}{r_+}) \ = \ C \ - \ \frac{2q^2}{r_+} \tag{26a}$$

with

$$C \ \equiv \ \frac{2q^2}{r_+} \ - \ \frac{q^2}{\epsilon} \ = \ \frac{2q^2}{r_+}(1 - \frac{r_+}{2\epsilon}) \tag{26b}$$

such that $lim_{\epsilon \to 0} \ C \to -\infty$. After introducing the cutoff one arrives at

$$I_{EM} \ = \ - \ i \ [\ \frac{\beta}{2} \ (-\frac{q^2}{r_+}) \ + \ \frac{\beta C}{4}] \tag{27}$$

Upon substituting the value of β given by eq-(21) into eq-(27), the net contribution $I = I_g + I_{EM}$ becomes

$$I \ = \ - \ i \ [\ \frac{\beta}{2} \ (M \ - \ \frac{q^2}{r_+}) \ + \ \frac{\beta C}{4}] \ = \ - \ i \ [\ \frac{4\pi (r_+)^2}{4G} \ + \frac{\beta C}{4} \ ] \tag{28}$$

Therefore the magnitude turns out to be

$$|I| \ = \ \frac{4\pi (r_+)^2}{4G} \ + \ \frac{\beta C}{4} \ = \ \frac{A(r_+)}{4L_P^2} \ + \ \frac{\beta C}{4} \tag{29}$$

where the area of the outer horizon is $4\pi (r_+)^2$ and $G = L_P^2$ (L_P is the Planck length in four-dim). The end result is that $|I|$ given by eq-(29) agrees with the Reissner-Nordstrom black hole entropy up to an *additive* constant, which diverges in the $\epsilon = 0$ limit. For a detailed discussion of the relevance of an additive constant in the evaluation of entropy see [19]. One is then forced to perform a *subtraction* in order to remove the divergent piece of (29) and arrive at the finite value for the Reissner-Nordstrom black hole entropy $A(r_+)/4G$. This was *not* necessary to do so in the Schwarzschild black hole entropy case as shown in eq-(11). One should emphasize that the classical divergence of the EM field at $r = 0$ is responsible for the divergence of $|I|$ in eq-(29), whereas the ultraviolet divergences in the entanglement entropy between two spacetime regions are due to non-local correlations in QFT, see [17] for a very recent discussion.

Gibbons and Hawking [6] followed a very *different* procedure than the one taken in this work. In order to overcome the singularities that black hole metrics have they complexified the metric and evaluated the action on a contour which avoids the singularities. In particular, they also were required to perform a gauge transformation in order to obtain a regular potential at the horizon, and arrived at $|I| = \frac{\beta}{2}(M - \frac{q^2}{r_+})$ which also agrees with the magnitude of the finite part of eq-(28).

Let us analyze the behavior of the additive constant $\frac{\beta C}{4}$ as $M \to 0, q^2 \to 0$ due to a Hawking evaporation process and verify that the entropy increases from a very large initial negative value ($-\infty$ in the $\epsilon \to 0$ limit) to a *zero* final value. Since the area $A(r_+)$ also shrinks to zero at the end of the evaporation, the final entropy (29) reaches zero and no violation of the second law takes place since $\Delta S > 0$. One has that

$$\frac{\beta C}{4} \ = \ \frac{1}{4} \ \frac{2\pi (r_+)^3}{G(Mr_+ - q^2)} \ \frac{2q^2}{r_+}(1 - \frac{r_+}{2\epsilon}) \tag{30}$$

We shall take the limits in the following form

$$M \to 0, \ \ q^2 \to 0, \ \ r_+ \to 0, \ \ \epsilon \to 0; \ \ \frac{q^2}{Mr_+} \to \frac{1}{2}, \ \ \frac{r_+}{2\epsilon} = \frac{r_o}{\epsilon} \to 1 \tag{31}$$

so that the final value of the additive constant is zero as expected

$$\frac{\beta C}{4} \ \to \ \frac{4\pi (r_+)^2}{L_P^2} \ (1 - \frac{r_+}{2\epsilon}) \ \to \ 0 \tag{32}$$

Most recently, a plethora of activity has been centered concerning the relation between *generalized* entropy $S_{gen} = \frac{A}{4G} + S_{ext}$ and von Neumann entropy such that the second law $\Delta S_{gen} \geq 0$ is obeyed at all times [16], even after Hawking evaporation takes place where the area A decreases since the thermal radiation's contribution compensates for the decrease in area. After reinstating the physical constants that were set to unity one has $S_{gen} = \frac{k_B c^3 A}{4G\hbar} + S_{ext}$. While the individual terms in S_{gen} are ill-defined in the semi-classical limit, their sum is well-defined if one takes into account perturbative quantum gravitational effects [18]. For a detailed discussion of von Neumann algebras, and generalized entropy see [18], [19], [15].

To finalize let us discuss the charged and rotating massive Kerr-Newman black hole whose fundamental parameters are the mass M, charge Q and angular momentum J. Gibbons and Hawking [6] extended their procedure to evaluate the Euclidean action integrals via complex contour integrals in other spacetimes which do not necessarily have a real Euclidean section like the Kerr-Newman metric solution and arrived at the expression for the black hole entropy. In our case, the EM action $-\frac{1}{16\pi}\int d^4x\sqrt{-g}F^2$ is divergent so a cut-off r_o directly related to the outer horizon radius r_+ would be needed in order to extract the finite part. The components of A_μ and $F_{\mu\nu}$ in Boyer-Lindquist coordinates are, respectively,

$$A_\mu \ = \ \left(\frac{r\ Q\ \sqrt{G}}{r^2 + a^2 cos^2\theta}, \ 0, \ 0, \ \ - \frac{a\ r\ Q\ \sqrt{G}\ sin^2\theta}{r^2 + a^2 cos^2\theta} \right), \ \ a \equiv \frac{J}{M} \tag{33a}$$

$$F_{rt} = \partial_r A_t, \quad F_{\theta t} = \partial_\theta A_t, \quad F_{r\phi} = \partial_r A_\phi, \quad F_{\theta\phi} = \partial_\theta A_\phi \qquad (33b)$$

The angular rotation frequency Ω of the black hole at the horizon, and the black hole's electric potential Φ, given by the line integral of the black hole's electric field from infinity to any location on the horizon, are as follows [13]

$$\Omega \;=\; \frac{J}{M}\,\frac{1}{r_+^2 + (J/M)^2}, \quad \Phi \;=\; Q\,\frac{r_+}{r_+^2 + (J/M)^2} \qquad (34)$$

with the outer and inner horizon radius given by

$$r_\pm \;=\; (GM) \;\pm\; \sqrt{(GM)^2 \;-\; GQ^2 \;-\; (J/M)^2} \qquad (35)$$

After choosing a judicious cut-off r_o proportional to r_+, the finite part of the Euclidean EM action $-\frac{1}{16\pi}\int d^4x\sqrt{-g}F^2$ turns out to be $i\frac{\beta}{2}\Phi Q$ with Φ given by the Kerr-Newman black hole's electric potential in eq-(34).

Proceeding, from eq-(35) one infers that

$$GM \;=\; \frac{1}{2}(r_+ \;+\; r_-), \quad \sqrt{(GM)^2 \;-\; GQ^2 \;-\; (J/M)^2} \;=\; \frac{1}{2}(r_+ \;-\; r_-) \qquad (36)$$

and

$$-GQ^2 \;=\; (\frac{r_+ \;-\; r_-}{2})^2 \;-\; (\frac{r_+ \;+\; r_-}{2})^2 \;+a^2, \quad a \equiv \frac{J}{M} \qquad (37)$$

The relations (36,37) are crucial in what follows. The value of the inverse Hawking temperature β is

$$\beta \;=\; \frac{1}{T} \;=\; 2\pi\,\frac{r_+^2 + (J/M)^2}{r_+ - GM} \qquad (38)$$

Defining $a \equiv J/M$, and taking into account that the mass of the black hole M_H and the mass parameter M obey the relation $M = M_H + 2\Omega J \Rightarrow M_H = M - 2\Omega J$ [6], since the rotational energy contributes to the total mass, then the total Euclidean action $I = I_g + I_{EM}$ has the *same* functional form as the expression in eq-(28) for the finite part of the Reissner-Nordstrom case, and the magnitude $|I|$ ends up being

$$|I| \;=\; \frac{\beta}{2}\,(M_H \;-\; \Phi Q) \;=\; \frac{\beta}{2}\,(M \;-\; 2\Omega J \;-\; \Phi Q) \;=$$
$$\frac{\pi(r_+^2 + a^2)}{G}\,\left(\frac{GM(r_+^2 + a^2) \;-\; GQ^2 r_+ \;-\; 2GMa^2}{(r_+ - GM)\;(r_+^2 \;+\; a^2)}\right) \qquad (39)$$

after substituting the expressions in eqs-(34,38) for Ω, Φ and β. Using the key relations (36,37) one can show after some algebra that the quantity in the brackets in eq-(39) is precisely *unity*. Both the numerator and denominator are *equal* to

$$\frac{1}{2}\,(\; r_+^3 \;-\; r_+^2\, r_- \;+\; a^2\, r_+ \;-\; a^2\, r_- \;) \qquad (40)$$

Therefore, one ends up with the final expression

$$|I| \;=\; \frac{4\pi(r_+^2 \;+\; a^2)}{4G} \;=\; \frac{4\pi(r_+^2 \;+\; a^2)}{4L_P^2}, \quad a = \frac{J}{M} \tag{41}$$

which is precisely the Kerr-Newman black hole entropy where the area of the horizon is $A = \int d\theta \int d\phi \sqrt{g_{\theta\theta} g_{\phi\phi}} = 4\pi(r_+^2 + a^2)$.

Another way of explaining how this result (41) originates is to recall how Newman and Janis [13] showed that the Kerr metric could be obtained from the Schwarzschild metric by means of a coordinate transformation and allowing the radial coordinate to take on *complex* values. The Newman-Janis algorithm is based on making the replacement $r \to r + ia$. Originally, no clear reason for why the algorithm works was known and many physicists considered it to be an ad hoc procedure or a "fluke" not worthy of further investigation until Drake and Szekeres [14] gave a detailed explanation of the success of the algorithm and proved the uniqueness of certain solutions. In particular, the Kerr–Newman metric associated to a charged-rotating black hole can be obtained from the Reissner-Nordstrom metric by means of a coordinate transformation and allowing the radial coordinate to take on *complex* values. Consequently, by replacing $r_+^2 \to (r_+ + ia)(r_+ - ia) = r_+^2 + a^2$ in the quantity $4\pi r_+^2$ one recovers the Kerr-Newman black hole entropy in a straightforward fashion.

To conclude, the *crux* of all of these derivations of the back hole entropies relies in the key fact that the scalar curvature $\mathcal{R}$ is *no* longer zero. And due to the contribution of the delta function $\delta(r)$ point mass source yields a non-trivial Euclidean Einstein-Hilbert action given by $\frac{1}{2}\beta M_H$. Since the scalar curvature involves two derivatives, by replacing r for $|r|$, one will generate the singular $\delta(r)$ terms but whose integration will be finite. Whereas the EM contribution leads to a divergence because the field strengths are given in terms of first derivatives $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$, leading to the sign function $sgn(r)$. Had there been a second derivative one would have had a delta function. For this reason one will end up with an infinite additive constant if one integrates the EM action all the way to the origin. An ultraviolet cut-off r_0 (proportional to r_+) has to be introduced. Whereas Gibbons and Hawking avoided singularities via a complex contour integration procedure [6].

Acknowledgments

We are indebted to M. Bowers for assistance.

References

[1] A. Einstein, "On the General Theory of Relativity", Sitzungsber Preuss Akad Berlin **II** (1915) 778.

[2] K. Schwarzschild, "On the gravitational field of a mass point according to Einstein's theory", Sitzungsber Preuss Akad Berlin **I** (1916) 189.

[3] D. Hilbert, "Die Grundlagen der Physik", Nachr. Ges. Wiss Gottingen Math. Phys **K1**, (1917) 53.

[4] F. R. Tangherlini, "Schwarzschild field in n dimensions and the dimensionality of space problem", Nuovo Cimento **27** (1963) 636.

[5] J. York, "Role of conformal three-geometry in the dynamics of gravitation". Physical Review Letters. **28** (16) (1972) 1082.

[6] G. Gibbons and S. Hawking," Action integrals and partition functions in quantum gravity", Phys. Rev. **D 15** (1977) 2752.

[7] J. Colombeau, *New Generalized Functions and Multiplication of Distributions* (North Holland, Amsterdam 1984).

[8] C. Castro Perelman, "A Novel derivation of Black Hole entropy in all dimensions from truly point mass sources", (submitted to Physics Lett B).

[9] P. Nicolini, A. Smailagic and E. Spallucci, "Noncommutative geometry inspired Schwarzschild black hole", Phys. Lett. **B 632** (2006) 547.

[10] H. Reissner, "Uber die Eigengravitation des elektrischen Feldes nach der Einsteinschen Theorie", Annalen der Physik **50** (9) (1916) 106.

G. Nordstrom, "On the Energy of the Gravitational Field in Einstein's Theory", Verhandl. Koninkl. Ned. Akad. Wetenschap., Afdel. Natuurk., Amsterdam. **26** (1918) 1201.

H. Weyl, "Zur Gravitationstheorie", Annalen der Physik **54** (18) (1917) 117.

[11] R.P Kerr, "Gravitational field of a spinning mass as an example of algebraically special metrics" Physical Review Letters. **11** (5) (1963) 237.

[12] R. M. Wald, *General Relativity* (University of Chicago Press, 1984).

[13] E. Newman, and A. Janis, "Note on the Kerr Spinning Particle Metric", Journal of Mathematical Physics. **6** (1965) 4.

E. Newman, E. Couch, K. Chinnapared, A. Exton, A. Prakash, R. Torrence, "Metric of a Rotating, Charged Mass". Journal of Mathematical Physics. **6** (6) (1965) 918.

[14] S. Drake and P. Szekeres , "An Explanation of the Newman–Janis Algorithm", arXiv:gr-qc/9807001.

[15] R. Bousso, X. Dong, N. Engelhardt, T. Faulkner, T. Hartman, S. Shenker, and D. Stanford, "Snowmass White Paper: Quantum Aspects of Black Holes and the Emergence of Spacetime", arXiv.org : 2201.03096.

[16] A. Wall, "A Survey of Black Hole Thermodynamics" arXiv : 1804.10610

[17] A. Perez, “Black Hole Entropy and Planckian Discreteness” arXiv : 2310.19389.

[18] J. Kudler-Flam, S. Leutheusser and G. Satishchandran, “Generalized Black Hole Entropy is von Neumann Entropy”, arXiv : 2309.15897.

[19] E. Witten, “Physics with Ed”, 22 Lectures at Princeton University, Fall 2022 :

https://www.youtube.com/playlist?list=PLwWlDvl8lCT4ntvF7TKWMyBfxtVGWDuVQ.