Extreme Oscillation Phenomenon of Relativistic Propagator

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Abstract

We study the relativistic Schrödinger equation of a massive point particle in one dimension both with analytical calculations and with numerical computations, and we find that this equation is almost consistent with Special Relativity, with an apparent problem of small amplitude leaking from outside the past light cone. We find a conjecture that is related to an extreme oscillation phenomenon. We find a paradox that is related to Klein-Gordon equation.

Let's start by focusing on a question that if we define a function $f : \mathbb{R} \to \mathbb{C}$ according to the formula

$$f(x) \;=\; e^{ia\sqrt{1+x^2}},$$

where $a \in \mathbb{R}$ is some constant, then what formula could be used to describe its Fourier transform

$$\hat{f}(\xi) \;=\; \int\limits_{-\infty}^{\infty} f(x)e^{-i\xi x}dx \;=\; \int\limits_{-\infty}^{\infty} e^{ia\sqrt{1+x^2}}e^{-i\xi x}dx \;?$$

It turns out that this problem is related to the time evolution of relativistic quantum mechanical wave functions, so this is a significant problem, and it would make sense to spend some time on the purely mathematical side of the question too. If one tries to come up with a formula for an antiderivative that could be used to evaluate the integral, one will find the task to be too difficult. However, it is possible to see from the integral expression that the integral is divergent. So one possible answer to the question is that the Fourier transform doesn't exist, because the integral that is supposed to define it doesn't converge. There exist reasons to believe that this is not the best possible answer. Figures 1 and 2 show what happens, if we replace the integration domain $]-\infty,\infty[$ with a cut domain $[-R, R]$, and then compute estimates of the Fourier integral with a computer.

I propose a conjecture that for $|a| \gtrsim 1$ the Fourier transform $\hat{f}(\xi)$ can reasonably be described by a formula

$$\hat{f}(\xi) \;=\; \begin{cases} \text{extreme oscillation} \;+\; \sqrt{\frac{2\pi i}{a}}e^{ia\sqrt{1-\frac{\xi^2}{a^2}}-\frac{3}{4}\ln\left(1-\frac{\xi^2}{a^2}\right)+\cdots}, & \text{if } |\xi| \le a, \\ \text{extreme oscillation} \;+\; \text{fast convergence to zero}, & \text{if } |\xi| \ge a. \end{cases}$$

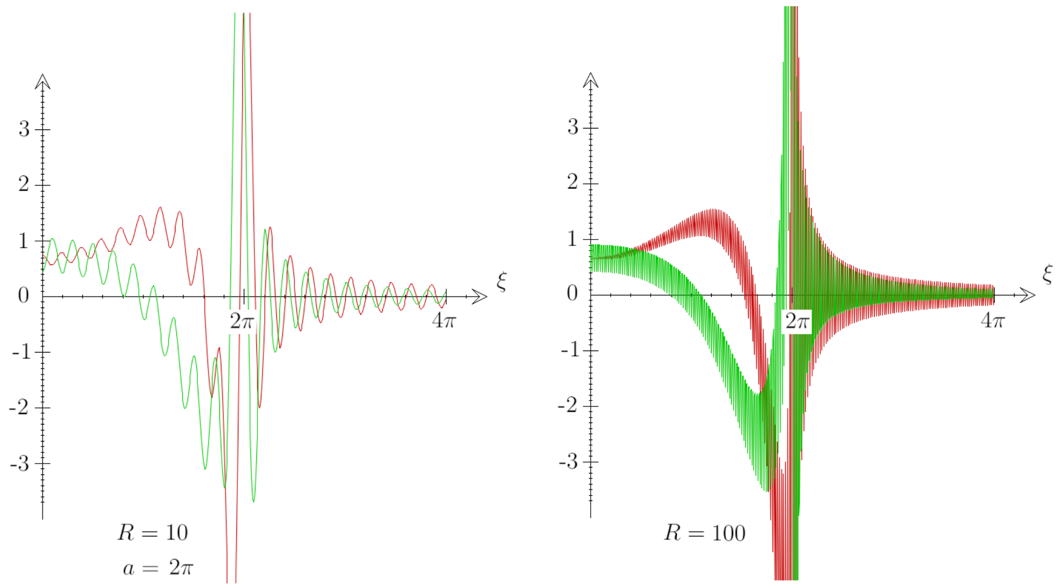

Figure 1: Estimates of the Fourier transform $\hat{f}(\xi)$ with a coefficient value $a = 2\pi$. The red graph is $\mathrm{Re}(\hat{f}(\xi))$, and the green graph is $\mathrm{Im}(\hat{f}(\xi))$. It looks like that these graphs are sums of two components. One component doesn't depend on R, and is not divergent, and the other component does depend on R, and is divergent.

Here we use the choices $\sqrt{\frac{i}{a}} = \frac{1+i}{\sqrt{2a}}$ if $a > 0$, and $\sqrt{\frac{i}{a}} = \frac{1-i}{\sqrt{2|a|}}$ if $a < 0$. Many readers will probably want some clarification on what is meant by "extreme oscillation" here. An example of extreme oscillation could be the quantity $e^{\infty \cdot i\xi}$. If we vary the value of ξ slowly and continuously, the quantity $e^{\infty \cdot i\xi}$ will rotate around the origin of the complex plane with an infinite angular speed, so that is extreme oscillation. The key property of extreme oscillation here is that if we integrate this kind of extremely oscillating quantity over any interval, the result will always be zero. So by extreme oscillation we mean such oscillation where there is an equal amount of positive and negative values on both the real and the imaginary axes. Also, if an extremely oscillating quantity is multiplied pointwisely with some ordinary function, and then the product is integrated over some interval, that will produce zero too.

Suppose we define a bump function b_α as

$$b_\alpha(x) = \begin{cases} \frac{1}{2\alpha\pi}\big(\cos\big(\frac{x}{\alpha}\big) + 1\big), & \text{if} \quad |x| \leq \alpha\pi, \\ 0, & \text{if} \quad |x| \geq \alpha\pi, \end{cases}$$

where $\alpha > 0$ is some small constant. This b_α is differentiable, and it approaches the delta function in the limit $\alpha \to 0$. What happens, if we compute

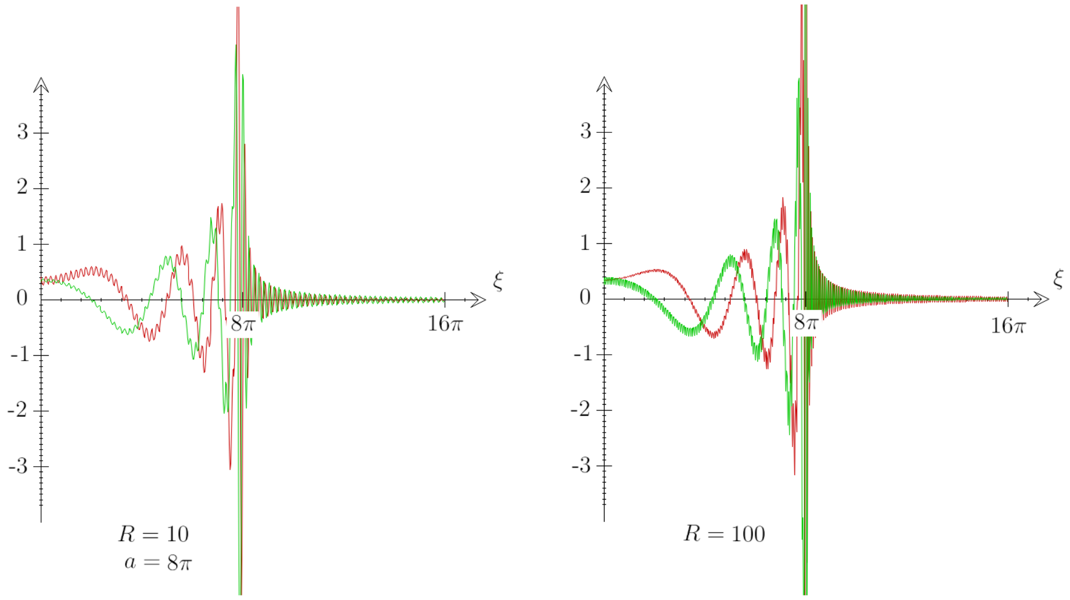

Figure 2: Estimates of the Fourier transform $\hat{f}(\xi)$ with a coefficient value $a = 8\pi$. There seems to be a relation that the amplitude of the divergent component approaches zero as the coefficient a grows. The divergent component will likely still exist even for large a though.

the convolution of $\hat{f}$ and b_α

$$(\hat{f} * b_\alpha)(\xi) = \int_{\xi-\alpha\pi}^{\xi+\alpha\pi} \hat{f}(\tau)\frac{1}{2\alpha\pi}\Big(\cos\Big(\frac{\xi-\tau}{\alpha}\Big)+1\Big)d\tau?$$

If the conjecture is true, the result should be

$$(\hat{f} * b_\alpha)(\xi) \approx \begin{cases} \sqrt{\frac{2\pi i}{a}}e^{ia\sqrt{1-\frac{\xi^2}{a^2}}-\frac{3}{4}\ln\left(1-\frac{\xi^2}{a^2}\right)}, & \text{if} \quad |\xi| \le a, \\ 0, & \text{if} \quad |\xi| \ge a, \end{cases}$$

because the convolution should make the extremely oscillating component vanish.

What happens, if we substitute the Fourier transform integral into the definition of the convolution, and change the order of integrals? We get a result

$$\int\limits_{\xi-\alpha\pi}^{\xi+\alpha\pi}\Big(\int\limits_{-\infty}^{\infty} e^{ia\sqrt{1+x^2}}e^{-i\tau x}dx\Big)\frac{1}{2\alpha\pi}\Big(\cos\Big(\frac{\xi-\tau}{\alpha}\Big)+1\Big)d\tau$$

$$= \int\limits_{-\infty}^{\infty} e^{ia\sqrt{1+x^2}}\frac{1}{2\alpha\pi}\Big(\int\limits_{\xi-\alpha\pi}^{\xi+\alpha\pi} e^{-i\tau x}\Big(\cos\Big(\frac{\xi-\tau}{\alpha}\Big)+1\Big)d\tau\Big)dx$$

$$= \int\limits_{-\infty}^{\infty} e^{ia\sqrt{1+x^2}}e^{-i\xi x}\frac{\sin(\alpha x\pi)}{2\alpha\pi}\Big(\frac{2}{x} - \frac{1}{x+\frac{1}{\alpha}} - \frac{1}{x-\frac{1}{\alpha}}\Big)dx.$$

This quantity can be interpreted to be a Fourier transform of a pointwise product of $e^{ia\sqrt{1+x^2}}$ and a new attenuation factor. The points $x = 0$ and $x = \pm\frac{1}{\alpha}$ do not make the integral diverge, because the poles cancel with the zeros of the sine function. In the limits $x \to \pm\infty$ we have an approximation

$$\frac{2}{x} - \frac{1}{x+\frac{1}{\alpha}} - \frac{1}{x-\frac{1}{\alpha}} = -\frac{2}{\alpha^2 x^3} + O\Big(\frac{1}{x^4}\Big).$$

We see that now we have an integral that converges in an ordinary way, although we still don't have a formula for an antiderivative. Since this attenuated integrand converges to the zero in the limits $x \to \pm\infty$ with a nice asymptotic rate, it is possible to write a computer program that estimates this integral with a finite sum. It would be interesting to see whether the conjecture would appear to be true or false in a light of such computation, so let's have a look at it.

If we substitute $e^{-i\xi x} = \cos(\xi x) - i\sin(\xi x)$ into the integral, the contribution from the antisymmetric term $-i\sin(\xi x)$ vanishes, since it is multiplied pointwisely by something symmetric. After $-i\sin(\xi x)$ has been removed, the whole integrand is symmetric, so we can replace the domain $]-\infty,\infty[$ with a domain $[0,\infty[$. Then the integral that we want to compute is

$$(\hat{f} * b_\alpha)(\xi) = \int\limits_{0}^{\infty} e^{ia\sqrt{1+x^2}}\cos(\xi x)\frac{\sin(\alpha x\pi)}{\alpha\pi}\Big(\frac{2}{x} - \frac{1}{x+\frac{1}{\alpha}} - \frac{1}{x-\frac{1}{\alpha}}\Big)dx.$$

We see that $\hat{f} * b_\alpha$ is symmetric, meaning that $(\hat{f} * b_\alpha)(-\xi) = (\hat{f} * b_\alpha)(\xi)$, so there will be no need to compute values for negative ξ.

We have to decide what values of a to use in our computation. The quantities $\hat{f}(\xi)$ and $(\hat{f} * b_\alpha)(\xi)$ remain unchanged if we take complex conjugates of them and replace a with $-a$, so there will be no need to compute estimates with negative a. When $a > 0$, according to the conjecture $(\hat{f} * b_\alpha)(\xi)$ will rotate around the origin roughly $\frac{a}{2\pi}$ times when ξ traverses from 0 to

a, so it would make sense to adjust a in a such way that it produces some small number of rotations that can be inspected by eye. Let's use values $a = 2\pi, 8\pi, 18\pi$ and 32π. However, although the conjecture dealt with the domain $|a| \gtrsim 1$, it will be interesting to see what happens with smaller a too, so let's use a value $a = \frac{\pi}{10}$ too.

We have to decide that for what values of ξ we compute the quantity $(\hat{f} * b_\alpha)(\xi)$. We want to inspect whether the conjecture appears to be true or not, so it makes sense to use the interval $0 \leq \xi \leq 2a$. This means that for the values $a = \frac{\pi}{10}, 2\pi, 8\pi, 18\pi$ and 32π we'll be using the intervals $0 \leq \xi \leq \frac{2\pi}{10}$, $0 \leq \xi < 4\pi$, $0 \leq \xi \leq 16\pi$, $0 \leq \xi < 36\pi$ and $0 \leq \xi < 64\pi$.

We are going to replace the integration domain $[0, \infty[$ with $[0, R]$, and we'll have to decide a value for the parameter R. Let's say that we want 4 decimals right in our computation. We can use the formula $\int_R^\infty \frac{1}{x^3} dx = \frac{1}{2R^2}$ to estimate how much error will come from the cut of the integration domain. So if we want 4 decimals right, it looks like that the relation $10^{-4} \approx \frac{1}{\pi \alpha^3 R^2}$ should hold. We can solve R to be $R \approx \frac{10^2}{\sqrt{\pi}} \alpha^{-\frac{3}{2}} \approx 56.4 \alpha^{-\frac{3}{2}}$. We'll get the values of R later when we decide the values for α.

We are going to estimate the integrand with its values on N points, and we'll have to decide a value for the parameter N. It turns out that this decision must be made together with the decision of the value for α. How much resolution do we need to get an approximation of the integral? The factors $\cos(\frac{2\pi}{10}x)$, $\cos(4\pi x)$, $\cos(16\pi x)$, $\cos(36\pi x)$ and $\cos(64\pi x)$ for the five values of a are the fastest oscillating factors in the integrand, so we can simplify the question by asking that how much resolution do we need to get hypothetical integrals of only these factors right. It is known that if a function is integrated over a domain $[r_\mathrm{A}, r_\mathrm{B}]$ by using three points and Simpson's summation rule, there can be an error $\frac{(r_\mathrm{B} - r_\mathrm{A})^5}{2880} \sup_u |f^{(4)}(u)|$ [1]. This implies that if we choose some odd value for N, and integrate a function over an interval $[0, R]$ by using the N points and Simpson's summation rule, there can be an error $\frac{R^5}{180(N-1)^4} \sup_u |f^{(4)}(u)|$. So if we want 4 decimals right, it looks like that relations $10^{-4} \approx \frac{(\frac{2\pi}{10})^4 R^5}{180 N^4}$, $\frac{(4\pi)^4 R^5}{180 N^4}$, $\frac{(16\pi)^4 R^5}{180 N^4}$, $\frac{(36\pi)^4 R^5}{180 N^4}$ and $\frac{(64\pi)^4 R^5}{180 N^4}$ should hold. These relations can be written in the forms $10^{-4} \approx 4.94 \cdot 10^5 N^{-4} \alpha^{-\frac{15}{2}}$, $10^{-4} \approx 7.91 \cdot 10^{10} N^{-4} \alpha^{-\frac{15}{2}}$, $10^{-4} \approx 2.02 \cdot 10^{13} N^{-4} \alpha^{-\frac{15}{2}}$, $10^{-4} \approx 5.19 \cdot 10^{14} N^{-4} \alpha^{-\frac{15}{2}}$ and $10^{-4} \approx 5.18 \cdot 10^{15} N^{-4} \alpha^{-\frac{15}{2}}$. We see that there is a relation that smaller α will require larger N. Let's solve α out of these relations. We get $\alpha \approx 19.6 N^{-\frac{8}{15}}$, $\alpha \approx 96.9 N^{-\frac{8}{15}}$, $\alpha \approx 203 N^{-\frac{8}{15}}$, $\alpha \approx 313 N^{-\frac{8}{15}}$ and $\alpha \approx 425 N^{-\frac{8}{15}}$. Let's say that we want to use a value $N = 10^6 + 1$, because this number of terms can be handled nicely by an ordinary personal computer multiple times in a loop over the values of ξ. This choice implies that we get $\alpha \approx 0.0124$, 0.0611, 0.128, 0.197 and 0.268.

These are nice values for α, and will not blur our graphs unnecessarily.

These values of α imply that the integration domains will be given by the values $R \approx 40800$, 3730, 1230, 645 and 407.

We should check whether it looks like that we could be substituting too large numbers into the trigonometric functions. Double floats have roughly 15 decimals of accuracy. We are going to be substituting double values below the magnitude of 10^5 into the trigonometric functions, so there will still be 10 working decimals on the right side of the decimal dot, and it looks like that we are going to be getting the 4 wanted decimals right.

After these preparations everything is ready for the computation. I wrote a program that estimated the quantity $(\hat{f} * b_\alpha)(\xi)$ using Simpson's summation rule with these parameter values, and the results are shown in Figures 3, 4 and 5.

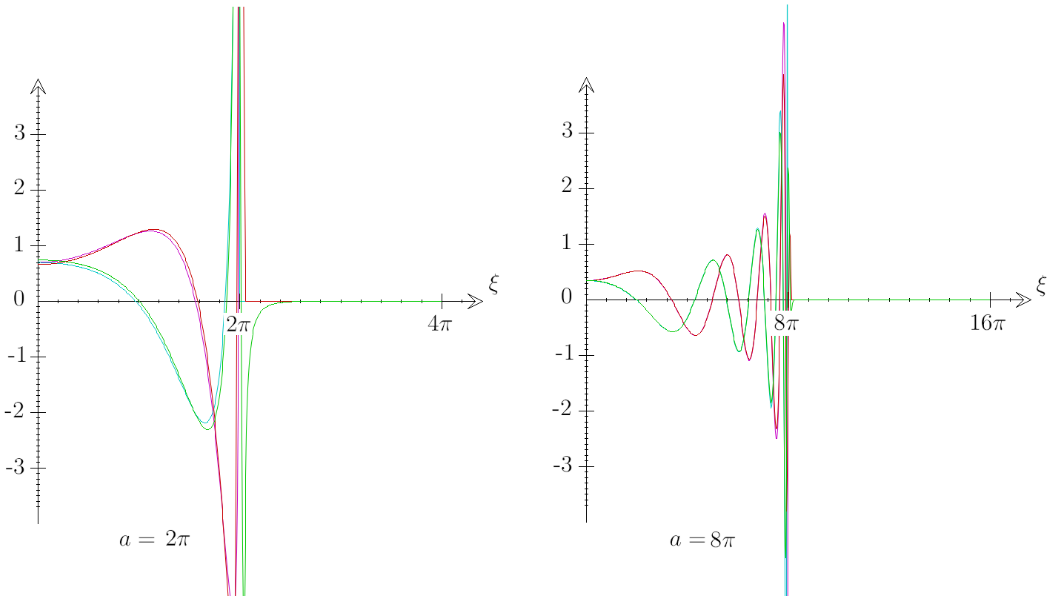

Figure 3: Both the numerically computed $(\hat{f} * b_\alpha)(\xi)$ and the quantity $\sqrt{\frac{2\pi i}{a}}e^{ia\sqrt{1-\frac{\xi^2}{a^2}}-\frac{3}{4}\ln\left(1-\frac{\xi^2}{a^2}\right)}$ mentioned in the conjecture, with constant values $a = 2\pi$ and 8π. The red graph is $\mathrm{Re}((\hat{f} * b_\alpha)(\xi))$, the green graph is $\mathrm{Im}((\hat{f} * b_\alpha)(\xi))$, the purple graph is $\mathrm{Re}\Big(\sqrt{\frac{2\pi i}{a}}e^{ia\sqrt{1-\frac{\xi^2}{a^2}}-\frac{3}{4}\ln\left(1-\frac{\xi^2}{a^2}\right)}\Big)$, and the cyan graph is $\mathrm{Im}\Big(\sqrt{\frac{2\pi i}{a}}e^{ia\sqrt{1-\frac{\xi^2}{a^2}}-\frac{3}{4}\ln\left(1-\frac{\xi^2}{a^2}\right)}\Big)$. On left we see how the purple graph follows the red one closely, and the cyan graph follows the green one closely. On right the purple and the cyan graphs initially get hidden under the red and the green, but become visible close to the limit $a \to 8\pi$.

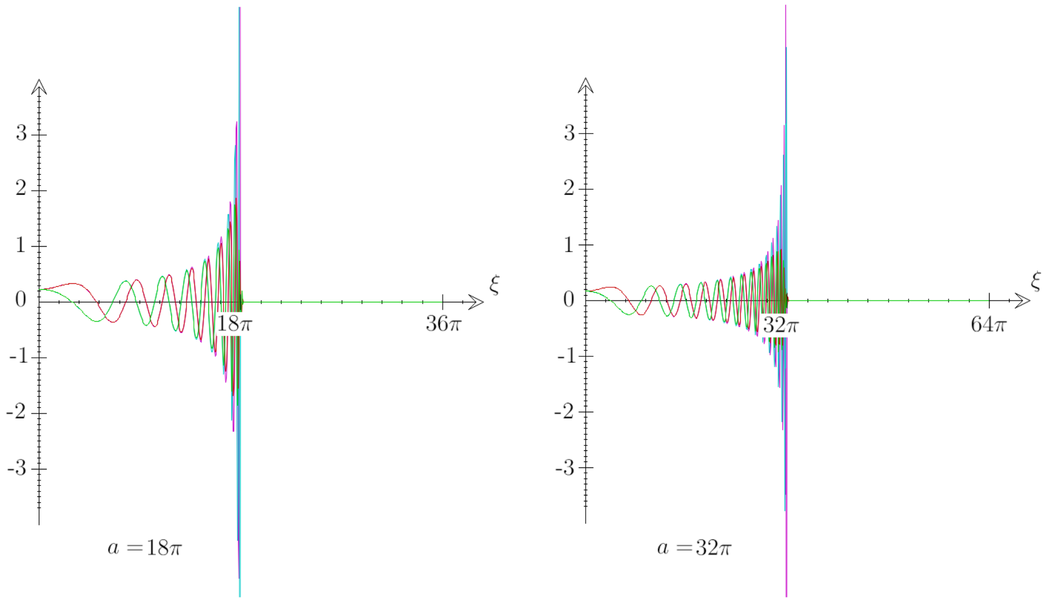

Figure 4: Same thing as in Figure 3 but with larger constant values $a = 18\pi$ and 32π. The red and the green graphs don't diverge as $\xi \to a$ in the same way as the purple and the cyan.

Inspection of Figures 3 and 4 supports the hypothesis that the conjecture is probably true. In the region $\xi < a$ the computed quantity $(\hat{f} * b_\alpha)(\xi)$ follows closely the quantity $\sqrt{\frac{2\pi i}{a}} e^{ia\sqrt{1-\frac{\xi^2}{a^2}} - \frac{3}{4}\ln\left(1-\frac{\xi^2}{a^2}\right)}$ mentioned in the conjecture, and then converges to zero as ξ passes onto the right side of the point a.

With value $a = 2\pi$ the imaginary part of $(\hat{f} * b_\alpha)(\xi)$ seems to converge to zero rather slowly on the right side of the point $\xi = a$. There seems to be a relation that the convergence to zero on the right side of the point $\xi = a$ is relatively faster for larger a.

The formula in the conjecture can be derived as follows: We define a function

$$g(x) \;=\; ia\sqrt{1+x^2} \;-\; i\xi x.$$

We calculate its derivatives

$$g'(x) \;=\; \frac{iax}{\sqrt{1+x^2}} \;-\; i\xi,$$

$$g''(x) \;=\; \frac{ia}{(1+x^2)^{\frac{3}{2}}},$$

$$g'''(x) \;=\; -\frac{3iax}{(1+x^2)^{\frac{5}{2}}},$$

.... We ask that does there exist x such that $g'(x) = 0$? The answer is that if $|\xi| < |a|$, then yes, and the x is

$$x = \frac{\xi}{a}\frac{1}{\sqrt{1-\frac{\xi^2}{a^2}}}.$$

If $|\xi| \geq |a|$, then no $x \in \mathbb{R}$ exists such that $g'(x) = 0$. If $|\xi| < a$, the values of g and its derivatives at the location where $g'(x) = 0$ are

$$g\Big(\frac{\xi}{a}\frac{1}{\sqrt{1-\frac{\xi^2}{a^2}}}\Big) = ia\sqrt{1-\frac{\xi^2}{a^2}},$$

$$g''\Big(\frac{\xi}{a}\frac{1}{\sqrt{1-\frac{\xi^2}{a^2}}}\Big) = ia\Big(1-\frac{\xi^2}{a^2}\Big)^{\frac{3}{2}},$$

$$g'''\Big(\frac{\xi}{a}\frac{1}{\sqrt{1-\frac{\xi^2}{a^2}}}\Big) = -3i\xi\Big(1-\frac{\xi^2}{a^2}\Big)^2,$$

.... So in the region $x \approx \frac{\xi}{a}\frac{1}{\sqrt{1-\frac{\xi^2}{a^2}}}$ we can approximate $g(x)$ with the series

$$\begin{aligned} g(x) &= ia\sqrt{1-\frac{\xi^2}{a^2}} + \frac{ia}{2}\Big(1-\frac{\xi^2}{a^2}\Big)^{\frac{3}{2}}\Big(x - \frac{\xi}{a}\frac{1}{\sqrt{1-\frac{\xi^2}{a^2}}}\Big)^2 \\ &\quad - \frac{i\xi}{2}\Big(1-\frac{\xi^2}{a^2}\Big)^2\Big(x - \frac{\xi}{a}\frac{1}{\sqrt{1-\frac{\xi^2}{a^2}}}\Big)^3 + \cdots. \end{aligned}$$

We can then use the Gaussian integral formula to calculate the approximation

$$\begin{aligned} \hat{f}(\xi) &= \int_{-\infty}^{\infty} e^{ia\sqrt{1+x^2}} e^{-i\xi x} dx \approx e^{ia\sqrt{1-\frac{\xi^2}{a^2}}} \int_{-\infty}^{\infty} e^{\frac{ia}{2}\left(1-\frac{\xi^2}{a^2}\right)^{\frac{3}{2}}\left(x-\frac{\xi}{a}\frac{1}{\sqrt{1-\frac{\xi^2}{a^2}}}\right)^2} dx \\ &= \sqrt{\frac{2\pi i}{a}} e^{ia\sqrt{1-\frac{\xi^2}{a^2}}}\Big(1-\frac{\xi^2}{a^2}\Big)^{-\frac{3}{4}}. \end{aligned}$$

Some people might feel that this approximation is nonsense. The original integral is divergent, but after the approximation we get something convergent, so there could be no way this approximation would be working. Those who have spent some time with the conjecture studied above can see this differently. The divergent integral apparently has two components: One is extremely oscillating, and the other one is an ordinary function. For some reason the Gaussian integral approximation is most apparently producing an approximation of the ordinary component.

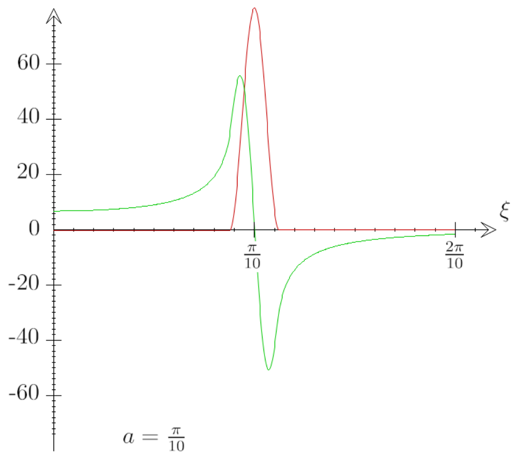

Figure 5: The graphs of $\mathrm{Re}((\hat{f} * b_\alpha)(\xi))$ and $\mathrm{Im}((\hat{f} * b_\alpha)(\xi))$ with a smaller value $a = \frac{\pi}{10}$.

This Gaussian approximation seems to become accurate only in the region $|a| \gg 1$. This raises a new question that what happens when $|a| < 1$? At the time of writing this article I don't know what kind of analytical formulas could be used to approximate the Fourier integral for $|a| < 1$, but I did compute numerically an estimate of $(\hat{f} * b_\alpha)(\xi)$ with the small value $a = \frac{\pi}{10}$, and the result is shown in Figure 5. Based on the graphs in Figure 5 it seems that approximations

$$\mathrm{Re}((\hat{f} * b_\alpha)(\xi)) \;\approx\; \pi\big(\delta(\xi - a) \;+\; \delta(\xi + a)\big) \qquad \text{and}$$
$$\mathrm{Im}((\hat{f} * b_\alpha)(\xi)) \;\propto\; \delta'(\xi - a) \;-\; \delta'(\xi + a)$$

become accurate in the limit $a \to 0$. Here δ' means the derivative of Dirac delta function. So in some sense $\mathrm{Re}((\hat{f} * b_\alpha)(\xi))$ does not directly form a delta function at the origin $\xi = 0$ in the limit $a \to 0$, but instead first at the locations $\xi = \pm a$. These two delta functions then merge into one delta function at the origin in the limit $a \to 0$. Similarly $\mathrm{Im}((\hat{f} * b_\alpha)(\xi))$ does not uniformly converge directly to zero, but instead it forms two derivatives of delta functions at the locations $\xi = \pm a$. These derivatives of delta functions then cancel in the limit $a \to 0$.

Let's continue onto the topic of theoretical physics. Suppose we want to quantize a one dimensional system described by a Hamiltonian

$$H(x, p) \;=\; \sqrt{(mc^2)^2 + c^2 p^2}.$$

Here $x \in \mathbb{R}$ is a spatial coordinate on which the Hamiltonian does not depend on, and $p \in \mathbb{R}$ is the canonical momentum. In this model the canonical momentum is equal to the physical momentum. The parameter $m > 0$ describes the mass of a point particle, and c is the speed of light. The Schrödinger equation of this system should look like

$$i\hbar\partial_t\psi(t,x) \;=\; \sqrt{(mc^2)^2 - c^2\hbar^2\partial_x^2}\psi(t,x),$$

where $\hbar$ is the reduced Planck's constant. This equation can be called *the relativistic Schrödinger equation of a massive point particle in one dimension.* There is a problem that it is not obvious how this kind of Hamiltonian operator should be interpreted. Some people [2] believe that we could apply the Taylor series

$$\sqrt{1+z} \;=\; 1 \;+\; \frac{1}{2}z \;+\; \sum_{k=2}^{\infty} \frac{(-1)^{k+1}\cdot(2k-3)!!}{k!\cdot 2^k} z^k$$

where $z \in \mathbb{C}$, and obtain a series representation

$$\sqrt{(mc^2)^2 - c^2\hbar^2\partial_x^2} \;=\; mc^2 \;-\; \frac{\hbar^2}{2m}\partial_x^2 \;-\; \sum_{k=2}^{\infty} \frac{(2k-3)!!}{k!\cdot 2^k} \frac{\hbar^{2k}}{m^{2k-1}c^{2k-2}}\partial_x^{2k}.$$

The problem with this attempt is that the Taylor series converges only for $|z| < 1$. This means that if we want to substitute some operator in the place of z, it would be desirable that the operator belonged to some normed algebra and had a norm less than 1. The differential operator ∂_x is an unbounded operator, so it is nowhere near having the needed norm.

An approach that seems to produce sensical ouput is that we first postulate that the effect of this operator on plane waves is given by the formula

$$\sqrt{(mc^2)^2 - c^2\hbar^2\partial_x^2}e^{\frac{i}{\hbar}px} \;=\; \sqrt{(mc^2)^2 + c^2p^2}e^{\frac{i}{\hbar}px},$$

and then also postulate that this operator is linear. It follows that the effect of the operator will be uniquely determined on any wave function that can be written as a linear combination of these plane waves. Let's define a Fourier transform with a such convention that $\psi(x)$ and $\hat{\psi}(p)$ are related according to the formulas

$$\hat{\psi}(p) \;=\; \int_{-\infty}^{\infty} e^{-\frac{i}{\hbar}px}\psi(x)dx \qquad \text{and} \qquad \psi(x) \;=\; \frac{1}{2\pi\hbar}\int_{-\infty}^{\infty} e^{\frac{i}{\hbar}px}\hat{\psi}(p)dp.$$

We can then solve that the effect of the operator $\sqrt{(mc^2)^2 - c^2\hbar^2\partial_x^2}$ on a wave function $\psi(x)$ is

$$\sqrt{(mc^2)^2 - c^2\hbar^2\partial_x^2}\;\psi(x)$$
$$= \frac{1}{2\pi\hbar}\int_{-\infty}^{\infty} \sqrt{(mc^2)^2 + c^2p^2}e^{\frac{i}{\hbar}px}\Big(\int_{-\infty}^{\infty} e^{-\frac{i}{\hbar}px'}\psi(x')dx'\Big)dp.$$

Operators defined in this way are called pseudo-differential operators. [3]

We can interpret the operator $\sqrt{(mc^2)^2 - c^2\hbar^2\partial_x^2}$ in a such way that it is just a spatial representation of some abstract operator. Then the Fourier space representation of the same abstract operator is simply the multiplication operator

$$\hat{\psi} \mapsto M\hat{\psi}, \qquad (M\hat{\psi})(p) \;=\; \sqrt{(mc^2)^2 + c^2p^2}\hat{\psi}(p).$$

The Schrödinger equation in the Fourier space is

$$i\hbar\partial_t\hat{\psi}(t,p) \;=\; \sqrt{(mc^2)^2 + c^2p^2}\hat{\psi}(t,p).$$

If some initial value $\hat{\psi}(0,p)$ is fixed, the solution to the Schrödinger equation for $t > 0$ is given by the formula

$$\hat{\psi}(t,p) \;=\; e^{-\frac{it}{\hbar}\sqrt{(mc^2)^2+c^2p^2}}\hat{\psi}(0,p).$$

Suppose some initial value $\psi(0,x)$ is fixed in the spatial representation. How could we write a solution $\psi(t,x)$ to the Schrödinger equation for $t > 0$? One answer is obtained by first calculating the Fourier transform $\hat{\psi}(0,p)$ of the initial value $\psi(0,x)$, then writing the time evolution $\hat{\psi}(t,p)$ in Fourier space, and then for any fixed $t > 0$ calculating $\psi(t,x)$ as the inverse Fourier transform of $\hat{\psi}(t,p)$. This means that we get a solution formula

$$\psi(t,x) \;=\; \frac{1}{2\pi\hbar}\int_{-\infty}^{\infty} e^{-\frac{it}{\hbar}\sqrt{(mc^2)^2+c^2p^2}} e^{\frac{i}{\hbar}px}\Big(\int_{-\infty}^{\infty} e^{-\frac{i}{\hbar}px'}\psi(0,x')dx'\Big)dp.$$

If we change the order of the integrals, we get a solution formula

$$\psi(t,x) \;=\; \int_{-\infty}^{\infty} P(t,x-x')\psi(0,x')dx',$$

where

$$P(t,x-x') \;=\; \frac{1}{2\pi\hbar}\int_{-\infty}^{\infty} e^{-\frac{it}{\hbar}\sqrt{(mc^2)^2+c^2p^2}} e^{\frac{i}{\hbar}p(x-x')}dp.$$

The quantity $P(t,x-x')$ can be called *the propagator of the relativistic Schrödinger equation of a massive point particle in one dimension.* This is a very interesting time evolution formula, because from here we can see that $\psi(t,x)$ can be considered to have been formed as a linear combination of the past values $\psi(0,x')$ where $x' \in \mathbb{R}$. The values $P(t,x-x')$ are the propagation amplitudes that tell how to weight the past values $\psi(0,x')$.

One relevant question is that does the relativistic Schrödinger equation keep the value of the quantity

$$\int_{-\infty}^{\infty} |\psi(t,x)|^2 dx$$

unchanged as t grows. The operator $\sqrt{(mc^2)^2 - c^2\hbar^2\partial_x^2}$ is an example of a Hermitian operator H, and with any Hermitian operator H we can always calculate that

$$\langle\psi(t)|\psi(t)\rangle \;=\; \langle\psi(0)|\underbrace{e^{\frac{it}{\hbar}H^\dagger}e^{-\frac{it}{\hbar}H}}_{=\mathrm{id}}|\psi(0)\rangle \;=\; \langle\psi(0)|\psi(0)\rangle,$$

so the answer to the question is yes. Whether there exists a probability density current or not is another matter that we omit in this article though.

There is a problem that the integral that defines the propagation amplitudes diverges. This maybe means that we should not have changed the order of the integrals. However, this doesn't necessarily mean that the propagator would be nonsense; we just have to interpret it somehow. One possible interpretation is that we first define a regularized propagator

$$P(t,x-x',R) \;=\; \frac{1}{2\pi\hbar}\int_{-R}^{R} e^{-\frac{it}{\hbar}\sqrt{(mc^2)^2+c^2p^2}}e^{\frac{i}{\hbar}p(x-x')}dp,$$

and then define the time evolution of the wave function with a formula

$$\psi(t,x) \;=\; \lim_{R\to\infty}\int_{-\infty}^{\infty} P(t,x-x',R)\psi(0,x')dx'.$$

Another option is that we first define a regularized propagator

$$P(t,x-x',\varepsilon) \;=\; \frac{1}{2\pi\hbar}\int_{-\infty}^{\infty} e^{-\varepsilon p^2}e^{-\frac{it}{\hbar}\sqrt{(mc^2)^2+c^2p^2}}e^{\frac{i}{\hbar}p(x-x')}dp,$$

where $\varepsilon > 0$, and then define the time evolution of the wave function with a formula

$$\psi(t,x) \;=\; \lim_{\varepsilon\to 0}\int_{-\infty}^{\infty} P(t,x-x',\varepsilon)\psi(0,x')dx'.$$

Actually it could be that even

$$\psi(t,x) \;=\; \int_{-\infty}^{\infty}\lim_{\varepsilon\to 0} P(t,x-x',\varepsilon)\psi(0,x')dx'$$

works with this regularization. So the divergent integral in the propagator formula can be interpreted to be a symbol that means something like this. Anyway, we should recognize the fact that the relation "$P(t, x-x') \in \mathbb{C}$" is not really true as such.

By calculating with delta functions we can see that the propagator has the following two interesting properties:

$$P(0, x-x') \;=\; \delta(x-x') \tag{1}$$

and

$$\int_{-\infty}^{\infty} P(t_\mathrm{C}-t_\mathrm{B}, x-x')P(t_\mathrm{B}-t_\mathrm{A}, x'-x'')dx' \;=\; P(t_\mathrm{C}-t_\mathrm{A}, x-x''). \tag{2}$$

These are obviously very important properties. Equation (1) means that when we use the propagator to generate a solution $\psi(t,x)$ for $t>0$ out of some initial value $\psi(0,x)$, the generated solution will have the right initial value, meaning:

$$\lim_{t\to 0}\psi(t,x) \;=\; \psi(0,x).$$

The property described by Equation (1) can be called the initial value property of the propagator. If the propagator did not have this initial value property, it would maybe be generating some solutions to the Schrödiger equation, but not the wanted ones.

Equation (2) means that if we have three time values $t_\mathrm{A} < t_\mathrm{B} < t_\mathrm{C}$ and some initial value $\psi(t_\mathrm{A}, x)$, it will make no difference whether we first use the propagator to generate $\psi(t_\mathrm{B}, x)$ out of $\psi(t_\mathrm{A}, x)$, and then generate $\psi(t_\mathrm{C}, x)$ out of $\psi(t_\mathrm{B}, x)$, or we use the propagator to generate $\psi(t_\mathrm{C}, x)$ directly out of $\psi(t_\mathrm{A}, x)$. The property described by Equation (2) can be called the associativity property of the propagator. If the propagator did not have this associativity property, the generated time evolution would depend on how the time axis would get sliced. There does not exist a one correct slicing of the time axis, so we wouldn't have a well defined time evolution.

If one attempts to study relativistic Quantum Mechanics from mainstream sources, one will learn about relativistic propagators that do not have the properties (1) and (2). We can wonder that what's the meaning of those type of propagators.

One relevant question is that is the relativistic Schrödinger equation Lorentz invariant? The answer is yes; if we assume that the wave function is pointwisely scalar. Since this result is not well known, we can prove it here. If we let $\psi(t,x)$ be some solution to the relativistic Schrödinger equation, we can write it in a form

$$\psi(t,x) \;=\; \frac{1}{2\pi\hbar}\int_{-\infty}^{\infty} e^{-\frac{it}{\hbar}\sqrt{(mc^2)^2+c^2p^2}}e^{\frac{i}{\hbar}px}\hat{\psi}(0,p)dp.$$

One way of seeing that this is a solution to the relativistic Schrödinger equation is that if we multiply the expression with the operator $i\hbar\partial_t - \sqrt{(mc^2)^2 - c^2\hbar^2\partial_x^2}$, and change the order of the operators and the integral, then the integrand vanishes.

Suppose we define new coordinates $\overline{t}$ and $\overline{x}$ so that they are related to t and x according to the relation

$$\begin{pmatrix} c\overline{t} \\ \overline{x} \end{pmatrix} = \begin{pmatrix} \cosh(\eta) & -\sinh(\eta) \\ -\sinh(\eta) & \cosh(\eta) \end{pmatrix} \begin{pmatrix} ct \\ x \end{pmatrix},$$

where $\eta \in \mathbb{R}$ is some rapidity. There will be another wave function $\overline{\psi}$ that describes the same abstract object as ψ, but in the new coordinate set. This means that the relation $\overline{\psi}(\overline{t}, \overline{x}) = \psi(t, x)$ holds. Now the question that we want to answer is that does $\overline{\psi}$ satisfy the relativistic Schrödinger equation too. The values of $\overline{\psi}(\overline{t}, \overline{x})$ come from the formula

$$\begin{aligned}
\overline{\psi}(\overline{t}, \overline{x}) &= \frac{1}{2\pi\hbar} \int_{-\infty}^{\infty} e^{-\frac{i}{\hbar}\big(\cosh(\eta)\overline{t} + \sinh(\eta)\frac{1}{c}\overline{x}\big)\sqrt{(mc^2)^2 + c^2p^2}} \\
&\qquad e^{\frac{i}{\hbar}p\big(\sinh(\eta)c\overline{t} + \cosh(\eta)\overline{x}\big)} \hat{\psi}(0, p)dp \\
&= \frac{1}{2\pi\hbar} \int_{-\infty}^{\infty} e^{-\frac{i}{\hbar}\big(\cosh(\eta)\sqrt{(mc^2)^2 + c^2p^2} - \sinh(\eta)cp\big)\overline{t}} \\
&\qquad e^{\frac{i}{\hbar}\big(-\sinh(\eta)\frac{1}{c}\sqrt{(mc^2)^2 + c^2p^2} + \cosh(\eta)p\big)\overline{x}} \hat{\psi}(0, p)dp = \cdots
\end{aligned}$$

At this point we have to do some intermediate calculations before we can see where the calculation of $\overline{\psi}(\overline{t}, \overline{x})$ goes to. We can check that a derivative formula

$$\begin{aligned}
&D_p\Big(-\sinh(\eta)\frac{1}{c}\sqrt{(mc^2)^2 + c^2p^2} + \cosh(\eta)p\Big) \\
&= -\sinh(\eta)\frac{cp}{\sqrt{(mc^2)^2 + c^2p^2}} + \cosh(\eta)
\end{aligned}$$

is true. Since the relations $|\sinh(\eta)| < \cosh(\eta)$ and $|cp| < \sqrt{(mc^2)^2 + c^2p^2}$ are true, we see that the derivative quantity is always positive. The Taylor series of the square root can be used to show that there are also the limits

$$\lim_{p \to \pm\infty} \Big(-\sinh(\eta)\frac{1}{c}\sqrt{(mc^2)^2 + c^2p^2} + \cosh(\eta)p\Big) = \pm\infty.$$

These facts mean that we can define a new integration variable $\overline{p}$ with the relation

$$\overline{p} = -\sinh(\eta)\frac{1}{c}\sqrt{(mc^2)^2 + c^2p^2} + \cosh(\eta)p.$$

Now if p traverses through the interval $]-\infty,\infty[$, also $\overline{p}$ traverses through the interval $]-\infty,\infty[$ monotonously. The reason for why we are interested in this change of variable is that this $\overline{p}$ quantity is what gets multiplied by $\overline{x}$ in the exponent in the calculation of $\overline{\psi}(\overline{t},\overline{x})$. We can calculate that

$$\begin{aligned}(mc^2)^2 + c^2\overline{p}^2 &= (\cosh(\eta))^2(mc^2)^2 \;+\; \big((\cosh(\eta))^2 + (\sinh(\eta))^2\big)c^2p^2 \\ &\quad -\; 2\cosh(\eta)\sinh(\eta)cp\sqrt{(mc^2)^2 + c^2p^2} \\ &= \Big(\cosh(\eta)\sqrt{(mc^2)^2 + c^2p^2} \;-\; \sinh(\eta)cp\Big)^2.\end{aligned}$$

The second equation can be difficult to discover, but once it has been seen, checking it is straightforward. We see that the quantity that gets multiplied by $\overline{t}$ in the exponent is

$$\cosh(\eta)\sqrt{(mc^2)^2 + c^2p^2} \;-\; \sinh(\eta)cp \;=\; \sqrt{(mc^2)^2 + c^2\overline{p}^2}.$$

We can solve p to be

$$p \;=\; \sinh(\eta)\frac{1}{c}\sqrt{(mc^2)^2 + c^2\overline{p}^2} \;+\; \cosh(\eta)\overline{p},$$

and then calculate that

$$\begin{aligned}(mc^2)^2 + c^2p^2 &= (\cosh(\eta))^2(mc^2)^2 \;+\; \big((\cosh(\eta))^2 + (\sinh(\eta))^2\big)c^2\overline{p}^2 \\ &\quad +\; 2\cosh(\eta)\sinh(\eta)c\overline{p}\sqrt{(mc^2)^2 + c^2\overline{p}^2} \\ &= \Big(\sinh(\eta)c\overline{p} \;+\; \cosh(\eta)\sqrt{(mc^2)^2 + c^2\overline{p}^2}\Big)^2.\end{aligned}$$

From here we see that

$$\cosh(\eta)\sqrt{(mc^2)^2 + c^2\overline{p}^2} \;+\; \sinh(\eta)c\overline{p} \;=\; \sqrt{(mc^2)^2 + c^2p^2},$$

which also turns out useful. Now the Jacobian related to the change of variable can be written in the form

$$\begin{aligned}\frac{dp}{d\overline{p}} &= \sinh(\eta)\frac{c\overline{p}}{\sqrt{(mc^2)^2 + c^2\overline{p}^2}} \;+\; \cosh(\eta) \\ &= \frac{\cosh(\eta)\sqrt{(mc^2)^2 + c^2\overline{p}^2} + \sinh(\eta)c\overline{p}}{\sqrt{(mc^2)^2 + c^2\overline{p}^2}} \;=\; \frac{\sqrt{(mc^2)^2 + c^2p^2}}{\sqrt{(mc^2)^2 + c^2\overline{p}^2}}.\end{aligned}$$

We can put these pieces together, and complete the calculation of $\overline{\psi}(\overline{t},\overline{x})$:

$$\cdots \;=\; \frac{1}{2\pi\hbar}\int_{-\infty}^{\infty} e^{-\frac{i\overline{t}}{\hbar}\sqrt{(mc^2)^2 + c^2\overline{p}^2}}e^{\frac{i}{\hbar}\overline{p}\overline{x}}\hat{\psi}(0,p(\overline{p}))\frac{\sqrt{(mc^2)^2 + c^2(p(\overline{p}))^2}}{\sqrt{(mc^2)^2 + c^2\overline{p}^2}}d\overline{p}.$$

This expression has the property that if we multiply it with the operator $i\hbar\partial_{\overline{t}} - \sqrt{(mc^2)^2 - c^2\hbar^2\partial_{\overline{x}}^2}$, and change the order of the operators and the integral, then the integrand vanishes. So when we transform a scalar solution of the relativistic Schrödinger equation with a Lorentz boost, it remains as a solution of the relativistic Schrödinger equation. This means that we can say that the relativistic Schrödinger equation is Lorentz invariant.

There is also another way of seeing that the relativistic Schrödinger equation is Lorentz invariant, which is slightly more intuitive, although less rigorous: One fact is that a solution of the relativistic Schrödinger equation is also a solution of Klein-Gordon equation, and another fact is that Klein-Gordon equation is Lorentz invariant, since it can be written in the covariant form $c^2\hbar^2\partial_\mu\partial^\mu\psi + (mc^2)^2\psi = 0$. These two facts do not yet directly imply that the relativistic Schrödinger equation would be Lorentz invariant. It turns out that solutions of Klein-Gordon equation can be written as linear combinations of solutions of the two equations $i\hbar\partial_t\psi = \pm\sqrt{(mc^2)^2 - c^2\hbar^2\partial_x^2}\psi$. We probably believe it when we get informed that the solutions of the two equations $i\hbar\partial_t\psi = \pm\sqrt{(mc^2)^2 - c^2\hbar^2\partial_x^2}\psi$ don't mix in Lorentz boosts. From there we can see that solutions of the relativistic Schrödinger equation must remain as solutions of the relativistic Schrödinger equation in Lorentz boosts.

Some people believe that they could prove that the relativistic Schrödinger equation would not be Lorentz invariant with a reasoning like this: "If an equation looks like $\partial_\mu\partial^\mu\psi = 0$ or $\partial_\mu F^{\mu\nu} = J^\nu$, then it is Lorentz invariant. When we write $\sqrt{(mc^2)^2 - c^2\hbar^2\partial_x^2}$ using the Taylor series, we see that the relativistic Schrödinger equation does not look like $\partial_\mu\partial^\mu\psi = 0$ or $\partial_\mu F^{\mu\nu} = J^\nu$. Therefore the relativistic Schrödinger equation is not Lorentz invariant." There are two problems with this argument: Firstly, we are not supposed to use the Taylor series to describe the pseudo-differential operator, and secondly, there is a major logical blunder. It is true that if an equation looks like $\partial_\mu\partial^\mu\psi = 0$ or $\partial_\mu F^{\mu\nu} = J^\nu$, then it is Lorentz invariant. However, this does not mean that if an equation does not look like $\partial_\mu\partial^\mu\psi = 0$ or $\partial_\mu F^{\mu\nu} = J^\nu$, then it would not be Lorentz invariant. It could be that an equation is Lorentz invariant for some other reason. As we just learned above, this is what happens with the relativistic Schrödinger equation. We can say that the relativistic Schrödinger equation is not Lorentz covariant, though; that is true. There is no reason to assume that the relevant time evolution equations in general should have a covariant form.

There is a one big problem with the relativistic Schrödinger equation. It is that if this equation is supposed to be consistent with Special Relativity, then why does it look like that $\psi(t, x)$ can be written as a linear combination of $\psi(0, x')$, where x' traverses through the whole space $\mathbb{R}$, and not as a linear combination of $\psi(0, x')$ where $|x - x'| \leq ct$? If somebody has an opinion that the relativistic Schrödinger equation is not acceptable, because according to it $\psi(t, x)$ depends on values $\psi(0, x')$ outside the past light cone $|x - x'| \leq ct$, we can consider that to be a legitimate opinion. It is true that it is little

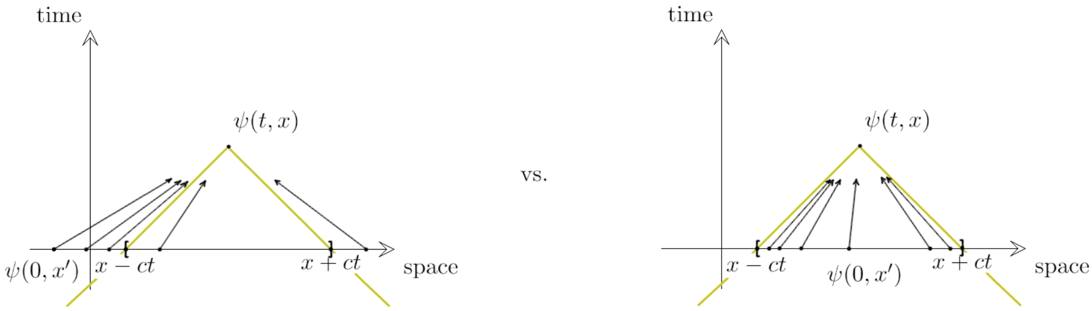

Figure 6: The big question is that when $\psi(t,x)$ is written as a linear combination of the past values $\psi(0,x')$, on which past values does it really depend. On $\psi(0,x')$ where $x' \in \mathbb{R}$, or on $\psi(0,x')$ where $x-ct < x' < x+ct$?

strange that $\psi(t,x)$ seems to be not uniquely determined by the values $\psi(0,x')$ inside the light cone $|x-x'| \le ct$ only. Those who have studied the conjecture discussed above have some understanding of what is going on. The divergent integral that defines the propagator $P(t,x-x')$ is the same divergent integral that defines the Fourier transform $\hat{f}$ that was the subject of the conjecture. The propagator can be written in the form

$$P(t,x-x') \;=\; \frac{mc}{2\pi\hbar}\hat{f}\Big(\frac{mc(x-x')}{\hbar}\Big), \qquad \text{where} \quad a = -\frac{tmc^2}{\hbar}.$$

We now know from the behaviour of this conjectured and numerically studied quantity that when it is used as an integration kernel, the main contribution comes from the region

$$\Big|\frac{mc(x-x')}{\hbar}\Big| \;\le\; \frac{tmc^2}{\hbar} \qquad\Longleftrightarrow\qquad |x-x'| \;\le\; ct,$$

and outside this region the contribution vanishes very fast. So the contradiction with Special Relativity is not extreme.

We should recognize that this is overall a difficult subject, and we should try to not jump to quick conclusions. If we decide, after the myths about the relativistic Schrödinger equation have been debunked, that let's get serious about the question that is there a problem with Special Relativity or not, the full truth seems to be that yes there is a problem in combining the relativistic Schrödinger equation and Special Relativity. The ordinary component of $\hat{f}(\xi)$ does not go to zero immediately when ξ continuously passes from the region $|\xi| < |a|$ to the region $|\xi| > |a|$, but only very fast. The problem in combining the relativistic Schrödinger equation and Special Relativity is not as extreme and blatant as some people believe, but the problem does exist. We can say that the relativistic Schrödinger equation implies some small amplitude leaking from outside the past light cone.

The facts that we have now learned produce a new paradox: If it is true that the relativistic Schrödinger equation is not consistent with Special Relativity because of the small amplitude leaking from outside the light cone, then how is it possible that solutions of the relativistic Schrödinger equation are also solutions of Klein-Gordon equation, and Klein-Gordon equation is fully consistent with Special Relativity? Are these solutions simultaneously not consistent and consistent with Special Relativity? One thing that maybe is relevant for a solution to this paradox is that if we want to construct an initial value $\partial_t\psi(0, x')$ in the region $|x - x'| < ct$ inside the light cone for a solution of Klein-Gordon equation, for the purpose of generating a solution to the relativistic Schrödinger equation, it could be that for that it is not sufficient to know an initial value $\psi(0, x')$ in the region $|x - x'| < ct$ inside the light cone. It seems that we need to know the values $\psi(0, x')$ in the region $|x - x'| > ct$ outside the light cone, to construct the initial value $\partial_t\psi(0, x')$ in the region $|x - x'| < ct$ inside the light cone. This need arises from the non-local nature of pseudo-differential operators. I'm not sure if this fully solves the paradox though; it still feels like depending on how you look at it.

It is not the intention here to claim that the small amplitude leaking from outside the past light cone would be a novel result. For example, Sidney Coleman discusses this same phenomenon in Chapter 1.3 *Determination of the position operator X* of book [4] that is based on lecture notes from 1970's or maybe 1960's. Coleman even shows how to use analytic techniques to estimate the rate at which the propagation amplitudes converge to zero outside the light cone, which is interesting. However, Coleman neglects the presence of divergence and extreme oscillation in his calculations, so the calculations can be criticized for being not quite rigorous. At this moment in time it would be advisable for people interested in this topic to try to combine information from all available sources, including this article of ours here. Peskin & Schroeder mention few words on this topic too [5].

At the time of writing this article mainstream physicists believe that the relativistic Schrödinger equation has already been rejected as a wrong equation for relativistic Quantum Mechanics. We could ask the mainstream people that if the relativistic Schrödinger equation is wrong, then what equation is the right one? The mainstream people have not been able to answer this question, and there seems to be a concensus that the question should be avoided. A typical way to distract the discussion away from this topic is to point out that modern relativistic quantum theories are multiparticle theories [4]. That claim is true, but shouldn't we have some way of handling the spatial representations of the wave functions in the multiparticle theories too?

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