A PROOF OF THE KAKEYA MAXIMAL FUNCTION CONJECTURE VIA BIG BUSH ARGUMENT

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Abstract. In this paper we reduce the Kakeya maximal function conjecture to the tube sets of unit measure. We show that the Kakeya maximal function is essentially monotonic. So by adding tubes we can reduce the conjecture to the case of unit measure tube set if we allow the technicality that there are possibly two tubes on the same direction. Then we proof the Kakeya maximal function conjecture from our lemma.

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1. Introduction

We define a line l_i as

$$l_i := \{y \in \mathbf{R}^n | \exists a, x \in \mathbf{R}^n \quad \text{and for all} \quad t \in \mathbf{R} \qquad y = a + xt\}$$

We define the δ-tubes as δ-neighborhoods of lines on $B(0,1)$:

$$T_i^\delta := \{x \in \mathbf{R}^n | |x - y| < \delta, \quad y \in l_i\}.$$

The order of intersection is defined as the number of tubes intersecting in an intersection. We define $A \lesssim B$ to mean that there exists a constant C_n depending only on n such that $A \le C_n B$. We define $A \lessapprox B$ to mean that for any $\epsilon > 0$ there exists a constant C_ϵ depending only on n and ϵ such that $A \le C_\epsilon \delta^\epsilon$. We say that tubes are δ-separated if their angles are δ-separated. Moreover, let $f \in L^1_{loc}(\mathbb{R}^n)$. For each tube in $B(0,1)$ define a as it‘s center of mass. Define the Kakeya maximal function as
$f^*_\delta : S^{n-1} \to \mathbb{R}$ via

$$f^*_\delta(\omega) = \sup_{a \in \mathbb{R}^n} \frac{1}{T^\delta_\omega(a) \cap B(0,1)} \int_{T^\delta_\omega(a) \cap B(0,1)} |f(y)| \mathrm{d}y.$$

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In this paper any constant can depend on dimension n. In study of the Kakeya maximal function conjecture we are aiming at the following bounds

$$||f_\delta^*||_p \leq C_\epsilon \delta^{-n/p+1-\epsilon}||f||_p, \tag{1.1}$$

for all $\epsilon > 0$ and some $n \leq p \leq \infty$. A very important reformulation of the problem by Tao is the following. A bound of the form (1.1) follows from a bound of the form

$$||\sum_{\omega\in\Omega} 1_{B(0,1)}1_{T_\omega(a_\omega)}||_{p/(p-1)} \leq C_\epsilon \delta^{-n/p+1-\epsilon}, \tag{1.2}$$

for all $\epsilon > 0$, and for any set of $N \leq \delta^{1-n}$ δ-separated of δ-tubes. See for example [2] or [1]. It's enough to consider the case $p = n$ and the rest of the cases will follow via interpolation [1,2]. In this paper any constant can depend on dimension n. Our main lemma is the following:

Lemma 1.1. *Let there be a $N \sim \delta^{1-n}$ δ-tubes that are δ-separated. Then we have*

$$||\sum_{\omega\in\Omega} 1_{B(0,1)}1_{T_\omega(a_\omega)}||_{n/(n-1)} \lesssim (\ln N)^{(n-1)/n}||\sum_{\omega'\in\Omega'} 1_{B(0,1)}1_{T_\omega(a_\omega)}||_{n/(n-1)},$$

where Ω' is almost δ-separated with two tubes of the same direction and

$$|\bigcup_{\omega'\in\Omega'} T^\delta_{\omega'}| \sim \delta^{n-1}N. \tag{1.3}$$

We will proof that then we have:

Theorem 1.2. *Let there be a $N \sim \delta^{1-n}$ δ-tubes that are δ-separated. Then we have for $n > 2$ that*

$$||\sum_{\omega\in\Omega} 1_{B(0,1)}1_{T_\omega(a_\omega)}||_{n/(n-1)} \lesssim (\ln N)^{(n-1)/n}.$$

The case $n = 2$ of the Kakeya maximal function conjecture is well know to be true [1]. The case $n = 1$ is trivial.

2. The proof of the lemma

We assume that $N \sim \delta^{1-n}$ We also drop the δ-upper index and the center points a_i so we have

$$1 \sim \delta^{n-1}N \sim \sum_{i=1}^N |T_i| = \int \sum_{i=1}^N 1_{T_i}.$$

We define

$$E_{2^k} := \{x \in \mathbf{R}^n | 2^k \leq \sum_{i=1}^N 1_{T_i}(x) \leq 2^{k+1}\}.$$

So we have

$$\int_{E_{2^k}} \sum_{i=1}^N 1_{T_i} \sim \sum_{i=1}^N \int_{E_{2^k}} 1_{T_i} \sim \sum_k 2^k|E_{2^k}|. \tag{2.1}$$

However, we can also calculate

$$\sum_k \int_{E_{2^k}} \sum_{i=1}^N 1_{T_i} \sim \sum_k \sum_{i=1}^N \int_{E_{2^k}} 1_{T_i} \sim \sum_k \sum_{i=1}^N |E_{2^k} \cap T_i| \sim \sum_{i=1}^N |T_i| \sim \delta^{n-1}N. \tag{2.2}$$

We also notice that the number of k is less than $\sim \ln N$. Now, we have from (2.1) and from (2.2) that

$$\delta^{n-1} N \sim \sum_k 2^k |E_{2^k}|. \tag{2.3}$$

Next we use our big bush argument. We consider N δ-tubes that are δ-separated. Moreover, all the center points of the tubes are in the origin. This set $\bigcup_{j=1}^N T_j^\delta(0)$ is the so called big bush. It's clear that

$$|\bigcup_{j=1}^N T_j^\delta(0)| \sim \delta^{n-1} N,$$

because if $N \sim \delta^{1-n}$, the big bush covers the unit ball. However the number of tubes N only doubles if take the union with the original tube set! So we take the union

$$E' := \bigcup_{i=1}^N T_i^\delta(a_i) \cup \bigcup_{j=1}^N T_j^\delta(0),$$

and do another dyadic decomposition. We have then

$$\delta^{n-1} N \sim \sum_m 2^m |E'_{2^m}|. \tag{2.4}$$

Now if $x \in E_{2^k}$ then $x \in \bigcup_{m \geq k} E'_{2^m}$! This is the monotonicity condition. It follows because if some point x belongs to $\sim 2^k$ tubes then after adding more tubes x belongs to at least $\sim 2^k$tubes. So we have the key inequality

$$2^k |E_{2^k}|^{(n-1)/n} \lesssim (\sum_{m \geq k} 2^{mn/(n-1)} |E'_{2^m}|)^{(n-1)/n}. \tag{2.5}$$

It's clear via dyadic decomposition that

$$||\sum_{\omega \in \Omega} 1_{B(0,1)} 1_{T_\omega(a_\omega)}||_{n/(n-1)} \sim (\sum_k 2^{kn/(n-1)} |E_{2^k}|)^{(n-1)/n}.$$

So we have from (3.1) that

(2.6)

$$||\sum_{\omega \in \Omega} 1_{B(0,1)} 1_{T_\omega(a_\omega)}||_{n/(n-1)} \sim (\sum_m 2^{kn/(n-1)} |E_{2^k}|)^{(n-1)/n} \lesssim (\ln N)^{(n-1)/n} \max_k 2^k |E_{2^k}|^{n/(n-1)}$$
$$\lesssim (\ln N)^{(n-1)/n} \sum_{m \geq k} (2^{mn/(n-1)} |E'_{2^m}|)^{(n-1)/n}.$$

So we are done proving our main lemma 1.1, because we can combine the above (2.6) with

$$(\sum_{m \geq k} 2^{mn/(n-1)} |E'_{2^m}|)^{(n-1)/n} \lesssim (\sum_m 2^{mn/(n-1)} |E'_{2^m}|)^{(n-1)/n} \sim ||\sum_{\omega' \in \Omega} 1_{B(0,1)} 1_{T'_\omega(a_{\omega'})}||_{n/(n-1)}.$$

3. The proof of the theorem

Next we use the lemma to proof 1.2. We assume

$$1 \sim \delta^{n-1} N$$

and that the big bush condition (1.3) is fulfilled. We also assume that $n > 2$. Suppose we have

$$||\sum_{i=1}^{N} 1_{B(0,1)} 1_{T_i(a_i)}||_{n/(n-1)} \lesssim 1, \tag{3.1}$$

which is by Hölder equivalent to

$$\int C(\sum_{i=1}^{N} 1_{B(0,1)} 1_{T_i(a_i)})^2 = ||F||_n || \sum_{i=1}^{N} 1_{B(0,1)} 1_{T_i(a_i)}||_{n/(n-1)} \lesssim ||F||_n, \tag{3.2}$$

for F linearly dependent of $\sum_{i=1}^{N} 1_{B(0,1)} 1_{T_i(a_i)}$. Thus, F is $\sum_{i=1}^{N} 1_{B(0,1)} 1_{T_i(a_i)}$ times a constant C. So the above (3.2) is equivalent with

$$||F||_2^2 \lesssim ||F||_n. \tag{3.3}$$

We have via dyadic decomposition

$$||F||_n^n \sim \sum_m 2^{nm} |E'_{2^m}|.$$

Because the set E' contains the big bush we have

$$|\bigcup_{m \geq N} E'_{2^m}| \gtrsim B_n(0, \delta) \sim \delta^n.$$

So we have

$$||F||_n^n \sim \sum_m 2^{nm} |E'_{2^m}| \gtrsim N^n \delta^n.$$

So

$$||F||_n \gtrsim N\delta \sim \delta^{2-n}.$$

However, it is well known that

$$||F||_2^2 \lessapprox \delta^{2-n}. \tag{3.4}$$

Two proofs of (3.4) can be found in [1]. So (3.3) essentially holds. However, we show a proof of

$$||F||_2^2 \lesssim \delta^{2-n}, \tag{3.5}$$

for $n > 2$ in the next section.

4. A proof of the well known inequality

We will use the following well known bound for the pairwise intersections of δ-tubes:

Lemma 4.1 (Corbòda)**.** *For any pair of directions $\omega_i, \omega_j \in S^{n-1}$ and any pair of points $a, b \in \mathbb{R}^n \cap B(0,1)$, we have*

$$|T^\delta_{\omega_i}(a) \cap T^\delta_{\omega_j}(b)| \lesssim \frac{\delta^n}{|\omega_i - \omega_j|}.$$

A proof can be found for example in [1]. For any (spherical) cap $\Omega \subset S^{n-1}, |\Omega| \gtrsim \delta^{n-1}, \delta > 0$, define its δ-entropy $N_\delta(\Omega)$ as the maximum possible cardinality for an δ-separated subset of Ω.

Lemma 4.2. *In the notation just defined*

$$N_\delta(\Omega) \sim \frac{|\Omega|}{\delta^{n-1}}.$$

Again, a proof can essentially be found in [1].

We show that

$$||\sum_{i=1}^{N} 1_{B(0,1)} 1_{T_i(a_i)}||_2 \lesssim \delta^{(2-n)/2}. \tag{4.1}$$

We have

$$||\sum_{i=1}^{N} 1_{B(0,1)} 1_{T_i(a_i)}||_2^2 \sim \sum_{j=1}^{N}\sum_{i=1}^{N} \int 1_{B(0,1)} 1_{T_j} 1_{T_i} \lesssim N \sum_{j=1}^{N} |T_j \cap T_i \cap B(0,1)|. \tag{4.2}$$

We do a dyadic decomposition with respect to angle $\phi(T_i, T_j)$ between T_i and T_j and obtain

(4.3)

$$N \sum_{j=1}^{N} |T_j \cap T_i \cap B(0,1)| \sim N \sum_k \sum_{\phi(T_j,T_i)\sim 2^{-k}} |T_j \cap T_i \cap B(0,1)|$$

$$\lesssim N \sum_k \sum_{\phi(T_j,T_i)\lesssim 2^{-k}} |T_j \cap T_i \cap B(0,1)| \lesssim N \sum_k 2^{-k(n-1)} \delta^{1-n} \delta^n 2^k \lesssim N\delta \lesssim \delta^{2-n},$$

where the third to last inequality follows from the lemmas 4.1 and 4.2 and second to last from that $n > 2$. So the claim (4.1) follows from (4.2) and (4.3).

References

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