A Power Series Divergent at Half Pi Yields a Sequence Briskly Convergent to Pi

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Abstract The tangent function's Taylor expansion about zero is a series of odd powers with positive coefficients; it converges when the absolute value of its argument is less than half pi, and diverges to positive infinity when its argument equals half pi, so with the aid of the ratio test it is seen that twice the square root of the ratio of its successive coefficients is a sequence which converges to pi. The odd derivatives of the tangent function are polynomials in powers of its square with positive integer coefficients, so a recursion of positive integers can be found from which the coefficients of the series described above may be successively obtained. Its related sequence which converges to pi does so monotonically from above, and appears to refine its approximation to pi by about one significant decimal figure per successive term.

1. Interesting ramifications of the Taylor expansion of the tangent function about zero

The function $\tan\theta = (\sin\theta/\cos\theta)$ is odd in θ, positive and increasing without bound when $0 < \theta < \pi/2$, and analytic in the complex θ-plane except for an infinite number of simple poles at $\theta = \pm(2n-1)(\pi/2)$, $n = 1, 2, \ldots$. In light of the analytic structure of $\tan\theta$ in the complex θ-plane, the Taylor expansion of $\tan\theta$ about $\theta = 0$ converges to $\tan\theta$ for all values of θ which satisfy $|\theta| < \pi/2$, but since $\tan\theta$ diverges to $+\infty$ as $\theta \to \pi/2\,-$, the Taylor expansion of $\tan\theta$ about $\theta = 0$ likewise diverges to $+\infty$ when $\theta = \pi/2$. Since $\tan\theta$ is odd in θ, the coefficients of the even powers of θ in the Taylor expansion of $\tan\theta$ about $\theta = 0$ must all vanish. Therefore the coefficients of the powers of θ in the Taylor expansion of $\tan\theta$ about $\theta = 0$ satisfy,

$$\sum_{n=1}^{\infty}\Big(\big[(D_\theta)^{2n-1}\tan\theta\big]_{\theta=0}\Big/(2n-1)!\Big)\,\theta^{2n-1} = \tan\theta \quad \text{when} \quad |\theta| < \pi/2, \tag{1.1a}$$

and,

$$\sum_{n=1}^{N}\Big(\big[(D_\theta)^{2n-1}\tan\theta\big]_{\theta=0}\Big/(2n-1)!\Big)\,(\pi/2)^{2n-1} \to +\infty \quad \text{when} \quad N \to \infty. \tag{1.1b}$$

In the next section we note that for $n = 1, 2, \ldots$, $(D_\theta)^{2n-1}\tan\theta$ is a polynomial of order n in powers of $(\tan\theta)^2$, *whose coefficients, which include* $\big[(D_\theta)^{2n-1}\tan\theta\big]_{\theta=0}$, *are positive integers* (for which we obtain a recursion). Therefore the power series in Eqs. (1.1a) and (1.1b) *have alternating positive and zero coefficients.* We next *restrict* these power series to values of θ which satisfy $\theta = ((\pi/2) - \epsilon)$, where $0 \le \epsilon \ll 1$, *upon which they have alternating positive and zero terms,* and can be regarded as *the ϵ-parameterized family of series,*

$$\sum_{n=1}^{\infty}\Big(\big[(D_\theta)^{2n-1}\tan\theta\big]_{\theta=0}\Big/(2n-1)!\Big)\,((\pi/2)-\epsilon)^{2n-1},$$

where each series of this family which has $\epsilon > 0$ *converges,* but the series which has $\epsilon = 0$ *diverges.* Since the series of this family *have no negative terms,* their convergence or divergence *is also ascertained by the ratio test.* If we denote as q_{n+1} *the quotient of the coefficient of the power* $((\pi/2)-\epsilon)^{2n+1}$ *of such an ϵ-family series with its coefficient of the power* $((\pi/2)-\epsilon)^{2n-1}$, *then the ratio test must necessarily yield the results,*

$$\lim_{n\to\infty} q_{n+1}((\pi/2)-\epsilon)^2 < 1 \text{ when } \epsilon > 0 \quad \text{and} \quad \lim_{n\to\infty} q_{n+1}(\pi/2)^2 \ge 1 \implies \lim_{n\to\infty} q_{n+1}(\pi/2)^2 = 1. \tag{1.2a}$$

Therefore $\pi = 2\lim_{n\to\infty}\sqrt{1/q_{n+1}}$, i.e.,

$$\pi = 2\lim_{n\to\infty}\sqrt{(2n+1)(2n)\big[(D_\theta)^{2n-1}\tan\theta\big]_{\theta=0}\Big/\big[(D_\theta)^{2n+1}\tan\theta\big]_{\theta=0}}\;. \tag{1.2b}$$

2. A recursion for numerical calculation of successive odd derivatives of the tangent function

The Eq. (1.2b) sequence for π requires $\big[(D_\theta)^{2n-1}\tan\theta\big]_{\theta=0}$ for $n = 1, 2, \ldots$. We first note that,

$$D_\theta\tan\theta = D_\theta(\sin\theta/\cos\theta) = (\cos\theta/\cos\theta) + (\sin\theta)(-1/(\cos\theta)^2)(-\sin\theta) = 1 + (\tan\theta)^2. \tag{2.1a}$$

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We next proceed to $(D_\theta)^3 \tan\theta$,

$$(D_\theta)^3 \tan\theta = (D_\theta)^2[D_\theta \tan\theta] = D_\theta[D_\theta(1 + (\tan\theta)^2)] = D_\theta[2\tan\theta(1 + (\tan\theta)^2)] =$$
$$D_\theta[2\tan\theta + 2(\tan\theta)^3] = [2 + 6(\tan\theta)^2](1 + (\tan\theta)^2) = 2 + 8(\tan\theta)^2 + 6(\tan\theta)^4. \quad (2.1b)$$

Next we calculate $(D_\theta)^5 \tan\theta$,

$$(D_\theta)^5 \tan\theta = (D_\theta)^2[(D_\theta)^3 \tan\theta] = D_\theta[D_\theta[2 + 8(\tan\theta)^2 + 6(\tan\theta)^4]] =$$
$$D_\theta[[16\tan\theta + 24(\tan\theta)^3](1 + (\tan\theta)^2)] = D_\theta[16\tan\theta + 40(\tan\theta)^3 + 24(\tan\theta)^5] =$$
$$[16 + 120(\tan\theta)^2 + 120(\tan\theta)^4](1 + (\tan\theta)^2) = 16 + 136(\tan\theta)^2 + 240(\tan\theta)^4 + 120(\tan\theta)^6. \quad (2.1c)$$

It is now apparent that for $n = 1, 2, \ldots,$ $(D_\theta)^{2n-1}\tan\theta = \sum_{k=0}^{n} I_k^n (\tan\theta)^{2k}$, *where* $I_0^n, I_1^n, \ldots, I_n^n$ *are positive integers, and* $\left[(D_\theta)^{2n-1}\tan\theta\right]_{\theta=0} = I_0^n$. Therefore Eq. (1.2b) can be more simply restated as,

$$\pi = 2\lim_{n\to\infty}\sqrt{(2n+1)(2n)I_0^n/I_0^{n+1}}, \quad (2.2)$$

and from Eqs. (2.1a), (2.1b) and (2.1c) we have,

$$I_0^1 = 1, I_1^1 = 1; \quad I_0^2 = 2, I_1^2 = 8, I_2^2 = 6; \quad I_0^3 = 16, I_1^3 = 136, I_2^3 = 240, I_3^3 = 120. \quad (2.3)$$

We now work out the recursion from $I_0^n, I_1^n, \ldots, I_n^n$ to $I_0^{n+1}, I_1^{n+1}, \ldots, I_n^{n+1}, I_{n+1}^{n+1}$, where $n = 1, 2, \ldots,$

$$\sum_{k=0}^{n+1} I_k^{n+1}(\tan\theta)^{2k} = (D_\theta)^{2n+1}\tan\theta = (D_\theta)^2[(D_\theta)^{2n-1}\tan\theta] = (D_\theta)^2\left[\sum_{k=0}^{n} I_k^n(\tan\theta)^{2k}\right] =$$
$$\sum_{k=0}^{n} I_k^n (D_\theta)^2[(\tan\theta)^{2k}] = \sum_{k=1}^{n} I_k^n (D_\theta)[(2k)(\tan\theta)^{2k-1}(1 + (\tan\theta)^2)] =$$
$$\sum_{k=1}^{n} I_k^n (D_\theta)[(2k)(\tan\theta)^{2k-1} + (2k)(\tan\theta)^{2k+1}] =$$
$$\sum_{k=1}^{n} I_k^n[(2k)(2k-1)(\tan\theta)^{2k-2} + (2k+1)(2k)(\tan\theta)^{2k}](1 + (\tan\theta)^2) =$$
$$\sum_{k=1}^{n} I_k^n[(2k)(2k-1)(\tan\theta)^{2k-2} + 2(2k)^2(\tan\theta)^{2k} + (2k+1)(2k)(\tan\theta)^{2k+2}] =$$
$$\sum_{j=0}^{n-1}(2j+2)(2j+1)I_{j+1}^n(\tan\theta)^{2j} + \sum_{k=1}^{n} 2(2k)^2 I_k^n(\tan\theta)^{2k} + \sum_{l=1}^{n+1}(2l-1)(2l-2)I_{l-1}^n(\tan\theta)^{2l} =$$
$$2I_1^n + \sum_{k=1}^{n-1}[(2k+2)(2k+1)I_{k+1}^n + 2(2k)^2 I_k^n + (2k-1)(2k-2)I_{k-1}^n](\tan\theta)^{2k} +$$
$$[2(2n)^2 I_n^n + (2n-1)(2n-2)I_{n-1}^n](\tan\theta)^{2n} + (2n+1)(2n)I_n^n(\tan\theta)^{2(n+1)}, \quad (2.4a)$$

so beginning from the initial two positive integers $I_0^1 = 1$ and $I_1^1 = 1$, we have for $n = 1, 2, \ldots$ the recursion,

$$I_0^{n+1} = 2I_1^n,$$
$$\text{if } n > 1, \text{ then } I_k^{n+1} = (2k+2)(2k+1)I_{k+1}^n + 2(2k)^2 I_k^n + (2k-1)(2k-2)I_{k-1}^n \text{ for } k = 1, \ldots, n-1,$$
$$I_n^{n+1} = 2(2n)^2 I_n^n + (2n-1)(2n-2)I_{n-1}^n \text{ and } I_{n+1}^{n+1} = (2n+1)(2n)I_n^n. \quad (2.4b)$$

We now apply the Eq. (2.4b) recursion to further extend the positive-integer numerical results of Eq. (2.3),

$$I_0^4 = 272, I_1^4 = 3968, I_2^4 = 12096, I_3^4 = 13440, I_4^4 = 5040;$$
$$I_0^5 = 7936, I_1^5 = 176896, I_2^5 = 814080, I_3^5 = 1491840, I_4^5 = 1209600, I_5^5 = 362880;$$
$$I_0^6 = 353792, I_1^6 = 11184128, I_2^6 = 71867136, I_3^6 = 191431680, I_4^6 = 250145280, I_5^6 = 159667200,$$
$$I_6^6 = 39916800; \quad I_0^7 = 22368256, I_1^7 = 951878656; \quad I_0^8 = 1903757312. \quad (2.5)$$

3. Characteristics of the pi-convergent sequence

Upon inserting the Eq. (2.3) and (2.5) positive-integer numerical results into the Eq. (2.2) π-convergent sequence, it becomes apparent that that sequence converges to π monotonically from above, and each successive member of the sequence appears to refine the approximation to π by about one significant decimal figure. The values of the first seven members of the Eq. (2.2) π-convergent sequence are,

$$3.46,\ 3.162,\ 3.1436,\ 3.14181,\ 3.141616,\ 3.1415953 \text{ and } 3.141592946. \quad (3.1)$$