ON THE CLASSIFICATION OF THE OPERATIONAL SPACES

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Abstract. In this paper, we introduce operational spaces that are a special case of what is usually called quotient topological spaces with an equivalence relation of interest. Then we derive some results and conjectures on their classification and an application to define the convergence of a sequence towards a cycle in the general case. We also provide a perspective on the still unaddressed conjectures concerning the operational spaces and we argue that the commonly useless non-uniform almost periodicity is actually useful in this context to understand the classification of such spaces.

1. Operational space

First, let us recall some properties of the quotient spaces that are useful to understand the sequel.

1.1. **Basics on the quotient spaces.** Let E be any set, a pseudo-distance is a function $\mathcal{D} : E \times E \to \mathbb{R}_+$ verifying

$$\forall (u, v) \in E^2, \mathcal{D}(u, v) = \mathcal{D}(v, u)$$
$$\forall (u, v, w) \in E^3, \mathcal{D}(u, w) \leq \mathcal{D}(u, v) + \mathcal{D}(v, w)$$
$$\forall u \in E, \mathcal{D}(u, u) = 0.$$

For a given $\mathcal{D}$ pseudo-distance, we have a natural equivalence relation defined by

$$u \sim v \Longleftrightarrow \mathcal{D}(u, v) = 0.$$

So we also have a quotient space F which is the quotient of E by $\sim$ i.e.

$$F = E/\sim = \{U \in \mathcal{P}(E) \mid \forall (u, v) \in U^2, u \sim v\}.$$

Theorem 1.1.1. *(F, d) is a metric space where d is the natural distance of the topology induced on F defined by*

$$\forall U, V \in F, d(U, V) = \sup_{(u,v) \in U \times V} \mathcal{D}(u, v).$$

Proof. Indeed, d is trivially symmetrical and reflexive. Moreover, for $U, V, W \in F$, we have

$$\forall (u, v, w) \in U \times V \times W, \mathcal{D}(u, w) \leq \mathcal{D}(u, v) + \mathcal{D}(v, w) \leq \sup_{v' \in V} \left(\mathcal{D}(u, v') + \mathcal{D}(v', w)\right).$$

So we have

$$\forall (u, w) \in U \times W, \mathcal{D}(u, w) \leq \sup_{v \in V} \mathcal{D}(u, v) + \sup_{v' \in V} \mathcal{D}(v, w).$$

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Applying the upper bound to the inequality on $U \times W$, we obtain

$$d(U,W) \le d(U,V) + d(V,W).$$

□

The following property is used extensively in the sequel:

Property 1.1.2. *Let $U, V \in F$, for all $(u,v) \in U \times V$, we have*

$$d(U,V) = \mathcal{D}(u,v).$$

Proof. Just note that for all $(u_1,u_2) \in U^2$ and $(v_1,v_2) \in V^2$, we have

$$\mathcal{D}(u_1,v_1) \le \mathcal{D}(u_1,u_2) + \mathcal{D}(u_2,v_2) + \mathcal{D}(v_2,v_1) = \mathcal{D}(u_2,v_2).$$

□

1.2. **Formal definition.** Let (F, d_F) be a complete metric space, and $(\mathcal{A}, \circ)$ be a unital magma of unitary operators $A : F \to F$, i.e. such that

$$(i)\ \forall A \in \mathcal{A}, \forall (u,v) \in F^2, \quad d_F(Au,Av) \le d_F(u,v)$$
$$(ii)\ \forall B \in \mathcal{A}, \quad B \circ \mathcal{A} \subset \mathcal{A}$$
$$(iii)\ (Id_F : u \longmapsto u) \in \mathcal{A}.$$

Theorem 1.2.1. *The function $\mathcal{D}_{\mathcal{A}}$ defined by*

$$\forall (u,v) \in F^2, \quad \mathcal{D}_{\mathcal{A}}(u,v) = \max\left(\inf_{A\in\mathcal{A}} d_F(Au,v), \inf_{A\in\mathcal{A}} d_F(u,Av)\right)$$

is a pseudo-distance on F.

Proof. It's easy to check with (iii) that $\mathcal{D}_{\mathcal{A}}(u,v) = \mathcal{D}_{\mathcal{A}}(v,u)$, $\mathcal{D}_{\mathcal{A}}(u,u) = 0$ for all $u, v \in F$. For $u, v, w \in F$, we also have using (i) that

$$\forall (A_1,A_2) \in \mathcal{A}^2, \quad d_F(A_1A_2u,v) \le d_F(A_1A_2u, A_1w) + d_F(A_1w,v)$$

$$\le d_F(A_2u,w) + d_F(A_1w,v).$$

Hence

$$\inf_{A_2\in\mathcal{A}} d_F(A_1A_2u,v) \le d_F(A_1u,w) + \inf_{A_2\in\mathcal{A}} d_F(A_2w,v). \tag{1}$$

We deduce from (ii) that

$$\inf_{A\in\mathcal{A}} d_F(Au,v) \le \inf_{A_1\in\mathcal{A}} \inf_{A_2\in\mathcal{A}} d_F(A_1A_2u,v) \le \inf_{A_1\in\mathcal{A}} d_F(A_1u,w) + \inf_{A_2\in\mathcal{A}} d_F(A_2w,v).$$

Therefore,

$$\begin{aligned}\mathcal{D}_{\mathcal{A}}(u,v) &\le \max\left(\inf_{A\in\mathcal{A}} d_F(Au,w) + \inf_{A\in\mathcal{A}} d_F(Aw,v), \inf_{A\in\mathcal{A}} d_F(Av,w) + \inf_{A\in\mathcal{A}} d_F(Aw,u)\right)\\ &\le \max\left(\inf_{A\in\mathcal{A}} d_F(Au,w), \inf_{A\in\mathcal{A}} d_F(Aw,u)\right) + \max\left(\inf_{A\in\mathcal{A}} d_F(Av,w), \inf_{A\in\mathcal{A}} d_F(Aw,v)\right)\\ &\le \mathcal{D}_{\mathcal{A}}(u,w) + \mathcal{D}_{\mathcal{A}}(w,v).\end{aligned}$$

□

We thus denote $\mathcal{A}_F$ the metric space induced by the pseudo-metric space $(F, \mathcal{D}_{\mathcal{A}})$ and obtain the natural distance $d_{\mathcal{A}}$ on this space (see section 1.1). The resulting metric space $(\mathcal{A}_F, d_{\mathcal{A}})$ is called the operational space associated to $\mathcal{A}$ with respects to F and is convenient for studying properties specific to operators contained in $\mathcal{A}$. The definition of the distance may recall the reader of the Hausdorff's distance which is (see [DieudonnéDieudonné1979], p.61) :

$$d(A, B) = \max\left(\sup_{x\in A}\inf_{y\in B} d(x, y), \sup_{y\in B}\inf_{x\in A} d(x, y)\right)$$

However, since we took a inf instead of sup, we obtain a weaker metric and thus different properties. We will need the following technical definition, which represents a certain form of uniqueness of the limit:

Definition 1.2.2. *We say that $\mathcal{A}_F$ is absorbing if for all $(u, v) \in F^2$, we have :*

$$(\exists (A_n)_n, (B_n)_n \in \mathcal{A}^{\mathbb{N}}, A_n u \underset{n\to+\infty}{\to} v \text{ and } B_n v \underset{n\to+\infty}{\to} u) \Longrightarrow \exists A \in \mathcal{A}, v = Au$$

We also need a fixed-point transmission property, which is reflected in the next definition.

Definition 1.2.3. *$\mathcal{A}_F$ absorbs $B \in \mathcal{A}$ in $u \in F$ if :*

$$\exists (A_n)_{n\in\mathbb{N}} \in \mathcal{A}^{\mathbb{N}}, A_n B u \underset{n\to+\infty}{\to} u$$

In the following, we'll denote $\mathcal{A}_{F,u} = \{B \in \mathcal{A} \mid \mathcal{A}_F \text{ absorbs } B \text{ in } u\}$

Properties 1.2.4. *Trivially:*

$$Id_F \in \mathcal{A}_{F,u} \subset \mathcal{A}$$
$$\forall u \in F, \forall A \in \mathcal{A}, Au = u \Longrightarrow A \in \mathcal{A}_{F,u}$$

These properties justify the following definition.

Definition 1.2.5. *$\mathcal{A}_F$ is absolutely bipolar in $u \in F$ if*

$$(\mathcal{A}_{F,u} \neq \{Id_F\}) \Longrightarrow (\exists A \in \mathcal{A}\backslash\{Id_F\}, Au = u) \Longrightarrow (\mathcal{A}_{F,u} = \mathcal{A})$$

Definition 1.2.6. *$\mathcal{A}_F$ is absolutely bipolar if and only if $\mathcal{A}_F$ is absolutely bipolar in u for all $u \in F$, the same applies to all other definitions.*

In the sequel, we'll also need the following weaker definition.

Definition 1.2.7. *$\mathcal{A}_F$ is bipolar in $u \in F$ if :*

$$(\mathcal{A}_{F,u} \neq \{Id_F\}) \Longrightarrow (\mathcal{A}_{F,u} = \mathcal{A})$$

1.3. **General properties.** We have the following conjecture:

Conjecture 1.3.1. *$(\mathcal{A}_F, d_{\mathcal{A}})$ is complete. It is denoted $\mathcal{O}(\mathcal{A}, F)$ or $\mathcal{A}_F$ for convenience.*

This is well-known in the particular case when $\mathcal{A}_F$ has a group structure since it is basically a quotient distance $\inf_{A\in\mathcal{A}} d(Ax, y)$ in that case. However, it remains unaddressed (as the definition of $\mathcal{A}_F$) in the general case to the best of the author's knowledge.

The following lemma is important for characterizing this space:

Lemma 1.3.2. *If $\mathcal{A}_F$ is absorbing, the element U of $\mathcal{A}_F$ containing u verifies :*

$$U \subset \mathcal{A}u$$

Proof. Let $u, v \in U \in \mathcal{A}_F$. It is assumed that $\mathcal{A}_F$ is absorbing. We have :

$$\inf_{A\in\mathcal{A}} d_F(Au, v) = \inf_{A\in\mathcal{A}} d_F(Av, u) = 0$$

So we have two minimizing sequences $(A_n u)_n$ and $(B_n v)_n$ of elements of F such that $A_n u \underset{n\to+\infty}{\to} v$ and $B_n v \underset{n\to+\infty}{\to} u$. Now $\mathcal{A}_F$ is absorbing, so we have $A \in \mathcal{A}$ such that $v = Au$ which shows the lemma. □

Property 1.3.3. *If $\mathcal{A}_F$ is absorbing, we have the following characterization of the U element of $\mathcal{A}_F$ containing $u \in F$:*

$$U = \mathcal{A}_{F,u} u$$

Proof. Let $v \in U$, we have from the previous lemma $A \in \mathcal{A}$ such that $v = Au$. But we also have $\mathcal{D}_{\mathcal{A}}(v, u) = 0$ and so we have $(A_n)_n \in \mathcal{A}^{\mathbb{N}}$ such that $A_n Au \to u$ when $n \to +\infty$. So $\mathcal{A}$ absorbs A in u, which shows that $v \in \mathcal{A}_{F,u}u$. In the other direction, if we take $B \in \mathcal{A}_{F,u}$ and set $v = Bu \in \mathcal{A}_{F,u}u$, then we have $(A_n)_n \in \mathcal{A}^{\mathbb{N}}$ such that $A_n Bu \to u$ when $n \to +\infty$. So we necessarily have $\mathcal{D}_{\mathcal{A}}(v, u) = 0$ and therefore $v \in U$. □

Corollary 1.3.4. *If $\mathcal{A}_F$ is absorbing and absolutely bipolar, then :*

$$\forall u \in U \in \mathcal{A}_F, U = \begin{cases} \mathcal{A}u \text{ if } \exists A \in \mathcal{A}\backslash\{Id_F\}, Au = u \\ \\ \{u\} \text{ otherwise} \end{cases}$$

Proof. This is obvious from the definition of absolutely bipolar. □

1.4. **The special case of monogenous monoids.** A monogenous operator monoid is defined by $\mathcal{A} = (A^p)_{p\in\mathbb{N}}$ with $A : F \to F$. We then have $A^0 = Id_F$ and

$$\forall n, p \in \mathbb{N},\ A^{n+p} = A^n A^p = A^p A^n. \tag{2}$$

It is assumed that (F, d_F) is complete and that assumption (i) in part (1.2) holds. This allows to define $\mathcal{A}_F$ according to part (1.2). This special case is interesting because we can uniquely identify any $u \in F$ to $(A^p u)_{p\in\mathbb{N}}$ allowing better understanding of the global properties of u (like the mean of a function for example). Furthermore, we have the following obvious result.

Theorem 1.4.1. *The injection $F \hookrightarrow \mathcal{A}_F$ is continuous.*

In the sequel, we write that $u \in F$ converges to $u^* \in F$ in $\mathcal{A}_F$ which means that the

The following result will prove to be quite confusing later on.

Theorem 1.4.2. *The following two assertions are true.*

$$\forall u \in F, (\exists B \in \mathcal{A}\backslash\{Id_F\}, Bu = u) \Longrightarrow (\mathcal{A}_{F,u} = \mathcal{A}) \tag{3}$$

$$\mathcal{A}_F \text{ is absorbing} \Longleftrightarrow \mathcal{A}_F \text{ is absolutely bipolar} \tag{4}$$

Proof. We start by proving (3). If we have $k \in \mathbb{N}^*$ such that $A^k u = u$, then for any $i \geq 1$, we have $j = i(k-1) \in \mathbb{N}$ such that $A^j A^i u = A^{ik} u = u$ from (2), which shows that $\mathcal{A}_{F,u} = \mathcal{A}$.

Now let's prove (4). Suppose $\mathcal{A}_F$ is absorbent. Let $u \in F$ be such that $\mathcal{A}_{F,u} \neq \{Id_F\}$, we have $p \in \mathbb{N}^*$ and $(k_n)_{n\in\mathbb{N}}$ such that $A^{k_n} A^p u \underset{n\to+\infty}{\to} u$. But we also have

$A^p u \underset{n\to+\infty}{\to} A^p u$ which implies (since $\mathcal{A}_F$ is assumed to be absorbing) that there exists $q \in \mathbb{N}$ such that $u = A^q A^p u = A^{p+q} u$. Since $p + q \geqslant 1$, we deduce with (3) that $\mathcal{A}_F$ is absolutely bipolar in u. Since this holds for all $u \in F$, $\mathcal{A}_F$ is absolutely bipolar.

In the other direction, assume that $\mathcal{A}_F$ is absolutely bipolar. Let's show that $\mathcal{A}_F$ is then necessarily absorbing. Given $u, v \in F$ and $(k_n)_n, (j_n)_n$ such that $A^{k_n} u \to v$ and $A^{j_n} v \to u$, if $\liminf\limits_{n\to+\infty} k_n = 0$, we have $u = v$ passing to the limit. From now on, we'll assume that $\exists N \in \mathbb{N}, \forall n \geq N, k_n \geq 1$ (which is equivalent to contradicting $\liminf\limits_{n\to+\infty} k_n = 0$). We then have,

$$A^{k_n+j_n-1} A u = A^{k_n}(A^{j_n} u) \to u,$$

so $A \in \mathcal{A}_{F,u}$. Since $\mathcal{A}_F$ is absolutely bipolar, there exists $k \in \mathbb{N}^*$ such that $u = A^k u$. Let $h_n = \lfloor \frac{k_n}{k} \rfloor k$ and $i_n = k_n - h_n$. Then for all $n \in \mathbb{N}$, $i_n \in [0, k-1] \cap \mathbb{N}$. Applying the Bolzano-Weierstrass theorem, we have an extracted subsequence $(i_{\varphi(n)})_n$ converging to $i \in \mathbb{N}$ such that

$$A^{k_{\varphi(n)}} u = A^{h_{\varphi(n)}+i_{\varphi(n)}} u = A^{i_{\varphi(n)}} u \to A^i u$$

But we also have $A^{k_{\varphi(n)}} u \to v$. By uniqueness of the limit (since F is a metric space), we obtain $v = A^i u$. □

In the general case, we can only give the following result on the classification of $\mathcal{A}_F$.

Theorem 1.4.3. *$\mathcal{A}_F$ is bipolar.*

Proof. If we have $i \geq 1$ and $(j_n)_n \in \mathbb{N}^{\mathbb{N}}$ such that $A^{j_n} A^i u \to u$ when $n \to +\infty$, then there are only two possibilities :

∘ If $(j_n)_n$ is bounded: The Bolzano-Weierstrass theorem provides an extracted subsequence $(j_{\varphi(n)})_n$ which converges to $j^* \in \mathbb{N}$ and is therefore almost constant from a certain rank, such that $A^{j^*+i} u = A^{j^*} A^i u = u$ according to (2). Thus, for $k = j^* + i$, for all $l \geq 1$, we have $m = l(k-1) \in \mathbb{N}$ such that $A^m A^l u = A^{lk} u = u$ according to (2), which shows that $\mathcal{A}_{F,u} = \mathcal{A}$.

∘ If $(j_n)_n$ is not bounded : Then we have an extracted subsequence $(j_{\varphi(n)})_n$ such that $j_{\varphi(n)} \to +\infty$ when $n \to +\infty$. Let $k \in \mathbb{N}$, so we have a certain rank $N \in \mathbb{N}$ such that :

$$\forall n \geqslant N, j_{\varphi(n)} \geqslant k - i$$

This allows to write :

$$A^{j_{\varphi(n)}+i-k} A^k u \underset{n\to+\infty}{\to} u$$

□

In the isolated case, a better classification is obtained by the following theorem.

Theorem 1.4.4. *If E is isolated (i.e. $\bar{E}$ is discrete), then $\mathcal{A}_F$ is absolutely bipolar.*

Proof. Let $u \in F$, we have two implications to show.

$(\exists A \in \mathcal{A} \backslash \{Id_F\}, Au = u) \Longrightarrow (\mathcal{A}_{F,u} = \mathcal{A})$: If we have $k \in \mathbb{N}^*$ such that $A^k u = u$, then for any $i \geq 1$, we have $j = i(k-1) \in \mathbb{N}$ such that $A^j A^i u = A^{ik} u = u$ from (2), which shows that $\mathcal{A}_{F,u} = \mathcal{A}$.

$\underline{(\mathcal{A}_{F,u} \neq \{Id_F\}) \Longrightarrow (\exists A \in \mathcal{A}\backslash\{Id_F\}, Au = u)}$: If we have $i \geq 1$ and $(j_n)_n \in \mathbb{N}^{\mathbb{N}}$ such that $A^{j_n}A^i u \to u$ when $n \to +\infty$. Since E is isolated, $(A^{i+j_n}u)_n$ is eventually equal to u, say for $n \geq N \in \mathbb{N}$, by a classical argument. We then have $u = A^{i+j_N}u$. Since this reasoning is true for all $u \in F$, we deduce that $\mathcal{A}_F$ is absolutely bipolar. ☐

We also have funny results such as the following.

Property 1.4.5. *Let $P \in \mathbb{Z}[X]$ be a polynomial of degree $d \in \mathbb{N}$ with non-negative coefficients and $u \in F$, $(k_n)_n \in (\mathbb{N}^*)^{\mathbb{N}}$ such that $A^{k_n}u \underset{n\to+\infty}{\overset{F}{\to}} u$. We have :*

$$(\forall n \in \mathbb{N}, k_n = P(n)) \Longrightarrow A^{P^{(d)}(0)}u = u$$

Proof. $(*)$: We define the endomorphism on $\mathbb{Z}[X]$

$$\Delta : Q \longmapsto Q(X+1) - Q(X)$$

and the assertions (for $k \in \mathbb{N}$)

$$G(k) : \text{the coefficients of } \Delta^k P \text{ are non-negative and } \tau^{(\Delta^k P)(n)}u \underset{n\to+\infty}{\overset{F}{\to}} u.$$

$$H(k) : \forall Q \in \mathbb{Z}_k[X], \quad \Delta^k Q = Q^{(k)}(0).$$

$G(0)$ and $H(0)$ are clearly verified with the assumptions.

Let $k \in \mathbb{N}$, and assume that $G(k)$ and $H(k)$ are true. Denoting $e_i = X^i$ the vectors of the canonical basis of $\mathbb{Z}_d[X]$, we have

$$\forall i \geqslant 0, \Delta e_i = (X+1)^i - X^i = \sum_{j=0}^{i-1} \binom{j}{i} X^j \in \mathbb{Z}[X],$$

which shows that Δe_i has non-negative coefficients. By hypothesis, the coefficients of $\Delta^k P$ in this base are non-negative. We deduce that $\Delta^{k+1}P = \Delta(\Delta^k P)$ also has non-negative coefficients. But then $(\Delta^{k+1}P)(n) \geqslant 0$ for all $n \in \mathbb{N}$ which allows to consider $A^{(\Delta^{k+1}P)(n)}$. Thus, the triangular inequality gives

$$d_F(A^{(\Delta^{k+1}P)(n)}u, u) \leq d_F(A^{(\Delta^{k+1}P)(n)}u, A^{(\Delta^k P)(n+1)}u) + d_F(A^{(\Delta^k P)(n+1)}u, u).$$

Or

$$\Delta^{k+1}P + (\Delta^k P)(X) = (\Delta^k P)(X+1),$$

so

$$d_F(A^{(\Delta^{k+1}P)(n)}u, A^{(\Delta^k P)(n+1)}u) = d_F(A^{(\Delta^{k+1}P)(n)}u, A^{(\Delta^{k+1}P)(n)}A^{(\Delta^k P)(n)}u) \quad \leqslant d_F(u, A^{(\Delta^k P)(n)}u)$$

and thus :

$$d_F(A^{(\Delta^{k+1}P)(n)}u, u) \leq d_F(u, A^{(\Delta^k P)(n)}u) + d_F(A^{(\Delta^k P)(n+1)}u, u) \underset{n\to+\infty}{\to} 0.$$

This gives $G(k+1)$. We also have for all $Q \in \mathbb{Z}_{k+1}[X]$ a certain $R \in \mathbb{Z}_k[X]$ such that

$$Q = \frac{Q^{(k+1)}(0)}{(k+1)!}X^{k+1} + R.$$

So we get

$$\Delta^{k+1}Q = Q^{(k+1)}(0) + \Delta\Delta^k R = Q^{(k+1)}(0).$$

So $H(k+1)$ is true. Hence, we have shown by induction that

$$\forall k \in \mathbb{N}, A^{(\Delta^k P)(n)} u \underset{n \to +\infty}{\to} u \text{ and } \forall Q \in \mathbb{Z}_k[X], \Delta^k Q = Q^{(k)}(0).$$

. In particular, we have $A^{(\Delta^d P)(n)} u \underset{n \to +\infty}{\to} u$ and $\Delta^d P = P^{(d)}(0)$, i.e.

$$A^{P^{(d)}(0)} u = u.$$

□

1.4.6. *An important monogenous example: periodic convergence.* We take $F \subset E^{\mathbb{N}}$ (with E a normed vector space, for example), $\mathcal{A} = \mathcal{T} = \{tau^p \mid p \in \mathbb{N}\}$ defining the operator *shift* $\tau : F \to F$ by:

$$\forall u \in F, \quad \tau u = (u_{n+1})_{n \in \mathbb{N}}$$

If E is isolated ($\mathbb{N}$ for example), we obtain with the theorem 1.4.4 and (4) that $\mathcal{T}_F$ is absorbing and absolutely bipolar. Using the general characterization of operational spaces, we can deduce that,

$$\forall u \in U \in \mathcal{T}_F, \quad U = \begin{cases} \{\tau^k u \mid k \in \mathbb{N}\} \text{ if } \exists p \in \mathbb{N}^*, \tau^p u = u \in \mathbb{N}^* \\ \{u\} \text{ otherwise} \end{cases}. \tag{5}$$

Informally, $\mathcal{T}_F$ are spaces where periodic sequences are merged with all their translations. This makes it possible to define convergence to a cycle of a sequence with a separable topology with the following definition :

Definition 1.4.7. *We say that $u \in F$ converges to $v \in F$ if the sequence $(\tau^p u)_{p \in \mathbb{N}}$ converges to $v \in F$ in $\mathcal{T}_F$.*

In the non-discrete case, this same characterization property (5) turns out to be false. To illustrate, assume that E is an $\mathbb{K}$-e.v.n. with $\mathbb{K}$ an arbitrary field. Recall that we can then define $\|u\|_{\ell^\infty(\mathbb{N},E)} = \sup\limits_{i \in \mathbb{N}} \|u_i\|_E$ and :

$$\ell^\infty(\mathbb{N}, E) = u \in E^{\mathbb{N}} \mid \|u\|_{\ell^\infty(\mathbb{N},E)} < +\infty\}.$$

We then obtain that $\ell^\infty(\mathbb{N}, E)$ is complete, which gives meaning to the following result, illustrating the problem.

Theorem 1.4.8. *Let $\tilde{E} \subset E$, consider $F = \ell^\infty(\mathbb{N}, E) \cap \tilde{E}^{\mathbb{N}}$ fitted with its natural distance $d_F(u, v) := \|u - v\|_{\ell^\infty(\mathbb{N},E)}$. If $\mathcal{T}_F$ is absolutely bipolar, then $\tilde{E}$ is discrete (for the induced topology).*

Proof. We reason by contrapositive, assuming that $\tilde{E}$ is not discrete. We then have an injective sequence $(a_k)_{k \in \mathbb{N}} \in \tilde{E}^{\mathbb{N}}$ converging to $a \in \tilde{E}$. We can define the sequence $u = (u_n)_{n \in \mathbb{N}} \in \tilde{E}^{\mathbb{N}}$by strong induction (with $u_0 = a$),

$$\forall n \in \mathbb{N}, \forall k \in [2^n, 2^{n+1} - 1] \cap \mathbb{N}, u_k = \begin{cases} u_{k-2^n} \text{ si } k > 2^n \\ a_n \text{ si } k = 2^n \end{cases}$$

It is clear that $u \in F$ because $(a_n)_{n \in \mathbb{N}} \in F$. Let $k \geqslant 2$, we pose for all for $n \in \mathbb{N}$,

$$C_k = \{2^j \mid j \in \mathbb{N}\}, \quad D_k = \{2^j - 2^k \mid j \geqslant k+1\}, \quad B_k^{(n)} = ([0, 2^n - 1] \cap \mathbb{N}) \setminus (C_k \cup D_k).$$

We also define $B_k = \bigcup_{n\in\mathbb{N}} B_k^{(n)}$. In this way, we obtain the partition $\mathbb{N} = B_k \cup C_k \cup D_k$. For all $n \in \mathbb{N}$,

$$b = \sup_{i\in B_k} \|u_{i+2^k} - u_i\|_E$$
$$b_n = \sup_{i\in B_k^{(n)}} \|u_{i+2^k} - u_i\|_E$$
$$c = \sup_{i\in C_k} \|u_{i+2^k} - u_i\|_E$$
$$d = \sup_{i\in D_k} \|u_{i+2^k} - u_i\|_E.$$

By definition, we have,

$$\forall n \in \mathbb{N}^*, \forall i \in B_k^{(n)} \backslash B_k^{(n-1)}, u_{i+2^k} - u_i = u_{i-2^n+2^k} - u_{i-2^n} \text{ and } i - 2^n \in B_k^{(n-1)}.$$

This gives,

$$\forall n \in \mathbb{N}^*, \sup_{i\in B_k^{(n)}} \|u_{i+2^k} - u_i\|_E = \sup_{i\in B_k^{(n-1)}} \|u_{i+2^k} - u_i\|_E.$$

So we have,

$$\forall n \in \mathbb{N}^*, b_n = b_{n-1} = b_0,$$

with $b_0 = \|u_{2^k} - u_0\|_E = \|a_k - u_0\|_E$. We deduce that $b = b_0$. We also have,

$$c = \max\left(\sup_{j\geqslant k}\|u_{2^j+2^k} - u_{2^j}\|_E, \sup_{j<k}\|u_{2^j+2^k} - u_{2^j}\|_E\right) = \max(c^{(1)}, c^{(2)}),$$

where clearly,

$$c^{(2)} = \sup_{j<k}\|u_{2^j+2^k} - u_{2^j}\|_E = \sup_{j<k}\|u_{2^j} - u_{2^j}\|_E = 0,$$

. by definition of u. We deduce that $c = c^{(1)}$. Furthermore, we have,

$$\forall j \geqslant k+1, u_{2^j-2^k} = u_{2^{j-1}+(2^{j-1}-2^k)} = u_{2^{j-1}+(2^{j-1}-2^k)} = u_{2^{j-1}-2^k},$$

where $\forall j \geqslant k, u_{2^j-2^k} = u_0$. From this we deduce that $d = \sup_{j>k}\|u_0 - u_{2^j}\|_E = \sup_{j>k}\|a_j - u_0\|_E$. We obtain,

$$\|tau^{2^k}u - u\| = \sup_{i\in\mathbb{N}}\|u_{i+2^k} - u_i\|_E = \max(b, c, d) = \max(b_0, c^{(1)}, d).$$

But clearly $\max(b_0, d) = \sup_{j\geqslant k}\|a_j - u_0\|_E$ and,

$$c^{(1)} = \sup_{j\geqslant k}\|u_{2^j+2^k} - u_{2^j}\|_E = \sup_{j\geqslant k}\|u_{2^k} - u_{2^j}\|_E = \sup_{j\geqslant k}\|a_k - a_j\|_E$$

. Finally (remembering that $u_0 = a$),

$$\|tau^{2^k}u - u\|_{\ell^\infty} = \max(\sup_{j\geqslant k}\|a_k - a_j\|_E, \sup_{j>k}\|a_j - a\|_E) \underset{k\to+\infty}{\to} 0$$

On the other hand, u is clearly not periodic. To see this, simply note that,

$$\forall k \geqslant 2, \forall i \in B_k^{(n)}, \exists j \leqslant n, u_i = a_j$$

Thus, since $(a_n)_{n\in\mathbb{N}}$ is injective, we have,

$$\forall k \in \mathbb{N}^*, u_{(2^k-k)+k} = a_k \neq u_{2^k-k}.$$

So $\mathcal{T}_F$ is not absolutely bipolar. □

2. A non-uniform almost periodicity

2.1. **Motivations.** The definitions we've introduced can be used to classify operational spaces, but we may wonder what they really mean. In the end, if $u, v \in U \in \mathcal{A}_F$, then :

$$\inf_{A \in \mathcal{A}} d_F(Au, v) = \inf_{A \in \mathcal{A}} d_F(Av, u) = 0$$

So there exists $(A_n)_n, (B_n)_n$ such that :

$$A_n u \underset{n \to +\infty}{\to} v \text{ and } B_n v \underset{n \to +\infty}{\to} u$$

Therefore :

$$A_n B_n u \underset{n \to +\infty}{\to} u$$

Thus, either $v = u$, or there exists $N \in \mathbb{N}$ such that

$$\forall n \geq N, A_n, B_n \neq Id_F.$$

For a monogenic monoid, for example, we get $C_n = A_n B_n \in (\mathcal{A} \setminus \{Id_F\})$ such that $C_n u \to u$. If $\mathcal{A} = \mathcal{T}$, this is exactly the characterization of a non-uniform almost-periodic sequence presented below.

2.2. **Definition and first approach.** We say that a sequence $u = (u_n)_{n \in \mathbb{N}} \in F$ is almost-periodic of period $k = (k_n)_{n \in \mathbb{N}}$ if :

$$\tau^{k_n} u \overset{F}{\underset{n \to +\infty}{\to}} u \text{ and } \forall n \in \mathbb{N}, k_n \geq 1.$$

We assume that we can always define the operational space $\mathcal{T}_F$. We then have the following properties.

Properties 2.2.1. *Let $P \in \mathbb{Z}[X]$ be a polynomial of degree $d \in \mathbb{N}$ with positive or zero coefficients and $u \in F$, $(k_n)_n \in (\mathbb{N}^*)^{\mathbb{N}}$ such that $\tau^{k_n} u \overset{F}{\underset{n \to +\infty}{\to}} u$. We have the following results:*

- $(*)$ $k_n = P(n) \Longrightarrow u$ *is periodic of period* $P^{(d)}(0)$.
- $(**)$ $k_n = 2^n \Longrightarrow \exists v \in F, \tau^{k_n} v \overset{F}{\underset{n \to +\infty}{\to}} v$ *and* v *non-periodic.*

Proof. $(*)$: This is a direct consequence of the property 1.4.5.

$(**)$: the proof is the same as for the theorem 1.4.8 except that we need to consider a distance instead of the ℓ^∞ norm. □

The reader may wonder why such a consideration is interesting. Indeed, we already know very well uniform almost-periodicity (that differs a bit from non-uniform almost-periodicity) [Muchnik, Semenov, and UshakovMuchnik et al.2003] [BesicovitchBesicovitch1926]. However, we showed that non-uniform almost-periodic functions (or sequences) are related to the classification problem of operational spaces. That is why these non-uniform definitions almost inexistent in the literature are of interest here.

2.3. **Conjectures.** There are still many conjectures to prove concerning those spaces. For instance, does the following result holds in the monogeneous monoid case ?

Conjecture 2.3.1. *Let $P \in \mathbb{Z}[X]$ be a polynomial of degree $d \in \mathbb{N}$ with positive or zero coefficients. There exists $u \in F$ such that $A^{2^n} u \underset{n \to +\infty}{\overset{F}{\to}} u$ and $u \notin \mathcal{A}u$.*

Or can we find a general classification depending on $\mathcal{A}$? Or is it possible to exploit the properties of those spaces to prove other conjectures, such as the Collatz conjecture ?

3. Conclusion

We established the classification of the operational spaces in some obvious cases and paved the way towards a better understanding of such spaces. However, some questions remain unaddressed such as : How to generalize those spaces as topological spaces ? Do the results still hold in a certain form for those topological spaces ?

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