Motzkin Numbers

Edgar Valdebenito
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abstract

Motzkin numbers have many combinatorial interpretations. In particular, M_n is the total number of ways in which it is possible to draw non-intersecting chords between n points on a circle.

Keywords: Motzkin numbers, number Pi, hypergeometric functions.

I. Introduction

The Motzkin numbers M_n may be defined by

$$\frac{1-z-\sqrt{1-2z-3z^2}}{2z^2}=\sum_{n=0}^{\infty} M_n z^n \tag{1}$$

They have numerous combinatorial interpretations, see ([1],[2],[3],[4]).

The first Motzkin numbers are:

$$M_n=\{1,\ 1,\ 2,\ 4,\ 9,\ 21,\ 51,\ 127,\ 323,\ 835,\ 2188,\ 5798,\ \ldots\} \tag{2}$$

Some notations:

The Gauss hypergeometric function is defined by

$${}_2F_1(a,\ b;\ c;\ z)=\sum_{n=0}^{\infty}\frac{(a)_n\,(b)_n}{(c)_n\,n!}z^n\ \ ,\ \ |z|<1 \tag{3}$$

where $(a)_n=a\,(a+1)\,(a+2)\ldots(a+n-1)$, $(a)_0=1$, is the Pochhammer symbol.

The Appell hypergeometric function is defined by

$$F_1(a,\ b,\ c,\ d,\ x,\ y)=\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{(a)_{m+n}\,(b)_m\,(c)_n}{m!\,n!\,(d)_{m+n}}x^m y^n \tag{4}$$

The generalized hypergeometric function is defined by

$${}_pF_q(a_1,\ \ldots,\ a_p;\ b_1,\ \ldots,\ b_q\ ;\ z)=\sum_{n=0}^{\infty}\frac{(a_1)_n\ldots(a_p)_n}{(b_1)_n\ldots(b_q)_n}\frac{z^n}{n!} \tag{5}$$

II. Elementary properties of M_n

$$M(z)=\frac{1-z-\sqrt{1-2z-3z^2}}{2z^2}=\frac{2}{1-z+\sqrt{1-2z-3z^2}}=\sum_{n=0}^{\infty} M_n z^n \tag{6}$$

$$M_n = \sum_{k=0}^{\lfloor n/2 \rfloor} \frac{1}{k+1} \binom{2k}{k} \binom{n}{2k} \;, \; n = 0,\, 1,\, 2,\, 3,\, \ldots \tag{7}$$

$$M_n = \frac{2n+1}{n+2} M_{n-1} + \frac{3(n-1)}{n+2} M_{n-2} \quad , \; n = 2,\, 3,\, 4,\, \ldots \tag{8}$$

$$\lim_{n \to \infty} \frac{M_n}{M_{n+1}} = \frac{1}{3} \tag{9}$$

$$M_n = \frac{2}{\pi} \int_0^{\pi} (\sin x)^2 \, (1 + 2\cos x)^n \, dx \;, \; n = 0,\, 1,\, 2,\, 3,\, \ldots \tag{10}$$

$$\frac{1}{M(z)} = \frac{2z^2}{1 - z - \sqrt{1 - 2z - 3z^2}} = 1 - z - \sum_{n=0}^{\infty} M_n \, z^{n+2} \tag{11}$$

$$M(z) = 1 + x\, M(z) + (x\, M(z))^2 \tag{12}$$

$$M_n = {}_2F_1\left(\frac{1-n}{2},\, -\frac{n}{2};\, 2;\, 4\right) ,\; n = 0,\, 1,\, 2,\, 3,\, \ldots \tag{13}$$

III. Pi formulas via Motzkin numbers

Recall that:

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \ldots \tag{14}$$

Pi formulas via Motzkin numbers:

$$\pi = 2\sqrt{3}\, \ln 2 + 24\sqrt{3} \sum_{n=0}^{\infty} M_n \left(\frac{\sqrt{2} - 1}{3}\right)^{n+2} {}_2F_1\left(1,\, n+2;\, n+3;\, 1 - \sqrt{2}\right) \tag{15}$$

$$\pi = 2\sqrt{3}\, \ln 2 + 12\sqrt{6} \sum_{n=0}^{\infty} M_n \left(\frac{\sqrt{2} - 1}{3}\right)^{n+2} {}_2F_1\left(1,\, 1;\, n+3;\, 1 - \frac{1}{\sqrt{2}}\right) \tag{16}$$

$$\pi = 2\sqrt{3}\, \ln 2 + 24\sqrt{3} \sum_{n=0}^{\infty} M_n \left(\frac{2 - \sqrt{2}}{6}\right)^{n+2} {}_2F_1\left(n+2,\, n+2;\, n+3;\, 1 - \frac{1}{\sqrt{2}}\right) \tag{17}$$

$$\pi = \sqrt{3}\, \ln 3 + 12\sqrt{3} \sum_{n=0}^{\infty} M_n \left(\frac{\sqrt{3} - 1}{3}\right)^{n+2} {}_2F_1\left(1,\, n+2;\, n+3;\, 1 - \sqrt{3}\right) \tag{18}$$

$$\pi = 12 - 5\sqrt{3} - 18\sqrt{3} \sum_{n=0}^{\infty} \frac{M_n}{n+3} \left(\frac{\sqrt{3} - 1}{3}\right)^{n+3} \tag{19}$$

$$\pi = 8\sqrt{6} - 6\sqrt{3} - 6 - 36\sqrt{3} \sum_{n=0}^{\infty} \frac{M_n}{n+3} \left(\frac{\sqrt{2} - 1}{3}\right)^{n+3} \tag{20}$$

$$\pi = 6 - \frac{3\sqrt{3}}{2} - 3 \sum_{n=0}^{\infty} \frac{M_n}{n+3}\, 2^{-2n-2}\, {}_2F_1\left(n+1,\, n+3;\, n+4;\, -\frac{1}{4}\right) \tag{21}$$

$$\pi = 6 - \frac{3\sqrt{3}}{2} - 3 \sum_{n=0}^{\infty} \frac{M_n}{n+3}\, 5^{-n-1}\, {}_2F_1\left(n+1,\, 1;\, n+4;\, \frac{1}{5}\right) \tag{22}$$

$$\pi = 6 - \frac{3\sqrt{3}}{2} - 48\sum_{n=0}^{\infty}\frac{M_n}{n+3}\,5^{-n-3}\,{}_2F_1\left(n+3,\,3;\,n+4;\,\frac{1}{5}\right) \tag{23}$$

$$\pi = 6 - \frac{3\sqrt{3}}{2} - \frac{3}{4}\sum_{n=0}^{\infty}\frac{M_n}{n+3}\,5^{-n}\,{}_2F_1\left(1,\,3;\,n+4;\,-\frac{1}{4}\right) \tag{24}$$

$$\pi = \frac{23-6\sqrt{3}}{4} - 3\sum_{n=0}^{\infty}\frac{2^{-4n-6}}{2n+5}\sum_{k=0}^{n}M_k\binom{n}{n-k} \tag{25}$$

$$\pi = \frac{23-6\sqrt{3}}{4} - 3\sum_{n=0}^{\infty}\frac{2^{-4n-6}M_n}{2n+5}\,{}_2F_1\left(1+n,\,\frac{5}{2}+n\,;\,\frac{7}{2}+n\,;\,\frac{1}{16}\right) \tag{26}$$

$$\pi = \frac{23-6\sqrt{3}}{4} - \frac{3}{4}\sum_{n=0}^{\infty}\frac{15^{-n-1}M_n}{2n+5}\,{}_2F_1\left(1+n,\,1\,;\,\frac{7}{2}+n\,;\,-\frac{1}{15}\right) \tag{27}$$

$$\pi = \frac{23-6\sqrt{3}}{4} - \frac{48}{15\sqrt{15}}\sum_{n=0}^{\infty}\frac{15^{-n-1}M_n}{2n+5}\,{}_2F_1\left(\frac{5}{2},\,\frac{5}{2}+n\,;\,\frac{7}{2}+n\,;\,-\frac{1}{15}\right) \tag{28}$$

$$\pi = \frac{23-6\sqrt{3}}{4} - \frac{3}{64}\sum_{n=0}^{\infty}\frac{15^{-n}M_n}{2n+5}\,{}_2F_1\left(1,\,\frac{5}{2}\,;\,\frac{7}{2}+n\,;\,\frac{1}{16}\right) \tag{29}$$

$$\pi = 18\sqrt{3}\,\ln\left(\frac{1+\sqrt{13}}{2\sqrt{3}}\right) - \frac{\sqrt{3}\left(3+\sqrt{13}\right)}{2} - \sum_{n=0}^{\infty}\frac{M_n}{2n+5}\,12^{-n-1}\,{}_2F_1\left(\frac{1}{2},\,n+\frac{5}{2};\,n+\frac{7}{2};\,-\frac{1}{12}\right) \tag{30}$$

$$\pi = 4\sqrt{3}\left(\sqrt{\sqrt{2}-1}\right) + 2\sqrt{2} - 4 - 24\sqrt{3}\sum_{n=0}^{\infty}\frac{M_{2n}}{2n+3}\left(\frac{\sqrt{\sqrt{2}-1}}{3}\right)^{2n+3} \tag{31}$$

$$\pi = 6\sqrt{3}\left(\sqrt{3\sqrt{3}-5}\right) + 9 - 6\sqrt{3} - 36\sqrt{3}\sum_{n=0}^{\infty}\frac{M_{2n}}{2n+3}\left(\frac{\sqrt{3\sqrt{3}-5}}{3}\right)^{2n+3} \tag{32}$$

$$\pi = 3\sqrt{6} - 3\sqrt{2} + 18\sqrt{3}\sum_{n=0}^{\infty}\frac{M_n}{n+3}\left(\frac{\sqrt{2}-1}{3}\right)^{n+3} F_1\left(n+3,\,\frac{1}{2},\,\frac{1}{2},\,n+4,\,-\frac{\sqrt{2}-1}{3},\,\sqrt{2}-1\right) \tag{33}$$

$$\pi = 6\sqrt{2+\sqrt{2}} - 6\sqrt{6-3\sqrt{2}} - 36\sqrt{3}\sum_{n=0}^{\infty}\frac{M_n}{n+3}\left(\frac{\sqrt{2-\sqrt{2}}-1}{3}\right)^{n+3} F_1\left(n+3,\,\frac{1}{2},\,\frac{1}{2},\,n+4,\,-\frac{\sqrt{2-\sqrt{2}}-1}{3},\,\sqrt{2-\sqrt{2}}-1\right) \tag{34}$$

$$\pi = 2\sqrt{3}\,\ln\left(\frac{34}{27}+\frac{8\sqrt{2}}{9}\right) - 24\sqrt{3}\sum_{n=0}^{\infty}\frac{M_n}{n+3}\left(\frac{\sqrt{2}-1}{3}\right)^{n+3} F_1\left(n+3,\,1,\,1,\,n+4,\,-\frac{\sqrt{2}-1}{3},\,\sqrt{2}-1\right) \tag{35}$$

$$\pi = 4\sqrt{3}\,\ln\left(\frac{27\left(2-\sqrt{2-\sqrt{2}}\right)}{\left(2+\sqrt{2-\sqrt{2}}\right)^3}\right) + 48\sqrt{3}\sum_{n=0}^{\infty}\frac{M_n}{n+3}\left(\frac{\sqrt{2-\sqrt{2}}-1}{3}\right)^{n+3} F_1\left(n+3,\,1,\,1,\,n+4,\,-\frac{\sqrt{2-\sqrt{2}}-1}{3},\,\sqrt{2-\sqrt{2}}-1\right) \tag{36}$$

$$\pi = 12 \sum_{n=0}^{\infty} \frac{(-1)^n \left(2-\sqrt{3}\right)^{2n+1}}{2n+1} M_n - 12 \sum_{n=0}^{\infty} \frac{(-1)^n \left(2-\sqrt{3}\right)^{2n+5}}{2n+5} \sum_{k=0}^{n} M_k \sum_{m=0}^{n-k} M_m \tag{37}$$

$$\pi = 12 \sum_{n=0}^{\infty} (-1)^n M_n 2^{-n} \int_0^{2-\sqrt{3}} \left(1+x^2-\sqrt{1-2x^2-3x^4}\right)^n dx \tag{38}$$

$$\pi = 12\left(2-\sqrt{3}\right) \sum_{n=0}^{\infty} (-1)^n M_n 2^{-n} \sum_{k=0}^{n} (-1)^k \binom{n}{k} F_1\left(\frac{1}{2}, -\frac{k}{2}, \frac{k-2n}{2}, \frac{3}{2}, 3\left(2-\sqrt{3}\right)^2, -\left(2-\sqrt{3}\right)^2\right) \tag{39}$$

$$\pi = 48 \sum_{n=0}^{\infty} (-1)^n M_n \int_{2+\sqrt{3}}^{\infty} \left(\frac{4+4x^2}{13+28x^2+16x^4}\right)^{n+1} dx \tag{40}$$

IV. Relations

$$\pi M_n = 4\sqrt{2} \sum_{k=0}^{n} \binom{n}{k} (-2)^k \frac{\left((-1)^n + 3^{n-k}\right)}{2k+3} {}_2F_1\left(-\frac{1}{2}, k+\frac{3}{2}; k+\frac{5}{2}; \frac{1}{2}\right) \tag{41}$$

$$M_{2n-2} = 2^{-2n-1} \sum_{k=0}^{n} \binom{2n-k}{k} \binom{4n-2k}{2n-k} \frac{3^k}{4n-2k-1} \quad , \; n = 1, 2, 3, \ldots \tag{42}$$

$$M_{2n-1} = 2^{-2n-2} \sum_{k=0}^{n} \binom{2n-k+1}{k} \binom{4n-2k+2}{2n-k+1} \frac{3^k}{4n-2k+1} \quad , \; n = 1, 2, 3, \ldots \tag{43}$$

$$\pi M_n = 2 \int_0^1 \sqrt{1-x^2} \, ((1+2x)^n + (1-2x)^n) \, dx \quad , \; n = 0, 1, 2, 3, \ldots \tag{44}$$

$$\pi M_n = 2 \int_0^{\infty} ((1+2\operatorname{sech} x)^n + (1-2\operatorname{sech} x)^n) (\tanh x)^2 \operatorname{sech} x \, dx \quad , \; n = 0, 1, 2, 3, \ldots \tag{45}$$

$$\pi M_n = 2 \int_{-\infty}^{\infty} (1+2\tanh x)^n (\operatorname{sech} x)^3 \, dx \quad , \; n = 0, 1, 2, 3, \ldots \tag{46}$$

$$\pi M_n = 4\sqrt{2}\, 3^{n-1} F_1\left(\frac{3}{2}, -\frac{1}{2}, -n, \frac{5}{2}, \frac{1}{2}, \frac{2}{3}\right) + \frac{4\sqrt{2}\,(-1)^n}{3} F_1\left(\frac{3}{2}, -\frac{1}{2}, -n, \frac{5}{2}, \frac{1}{2}, 2\right) \quad , \; n = 0, 1, 2, 3, \ldots \tag{47}$$

$$M_{n+2} = 1 + \sum_{k=0}^{n} \sum_{m=0}^{n-k} M_k M_m \quad , \; n = 0, 1, 2, 3, \ldots; \; M_0 = M_1 = 1 \tag{48}$$

$$\int_0^1 (1+2x)^n \sqrt{1-x^2} \, dx = \frac{\pi}{4} M_n + \frac{2n}{3} {}_3F_2\left(1, \frac{1-n}{2}, 1-\frac{n}{2}; \frac{3}{2}, \frac{5}{2}; 4\right) \quad , \; n = 0, 1, 2, 3, \ldots \tag{49}$$

$$\int_0^1 (1-2x)^n \sqrt{1-x^2} \, dx = \frac{\pi}{4} M_n - \frac{2n}{3} {}_3F_2\left(1, \frac{1-n}{2}, 1-\frac{n}{2}; \frac{3}{2}, \frac{5}{2}; 4\right) \quad , \; n = 0, 1, 2, 3, \ldots \tag{50}$$

$$\sqrt{1+2z-3z^2} - \sqrt{1-2z-3z^2} = 2z + 4 \sum_{n=0}^{\infty} M_{2n+1} z^{2n+3} \tag{51}$$

$$\sqrt{1+2z-3z^2} + \sqrt{1-2z-3z^2} = 2 - 4 \sum_{n=0}^{\infty} M_{2n} z^{2n+2} \tag{52}$$

$$M_{2n+4} - M_{2n+3} = M_{n+1}^2 + 2 \sum_{k=0}^{n} M_k M_{2n-k+2} \quad , \; n = 0, 1, 2, 3, \ldots \tag{53}$$

$$M_{2n+3} - M_{2n+2} = 2 \sum_{k=0}^{n} M_k M_{2n-k+1} \quad , \; n = 0, 1, 2, 3, \ldots \tag{54}$$

$$M_{2n} = \frac{(-4)^n}{(n+1)!}\left(\frac{1}{2}-n\right)_n {}_2F_1\left(-n, -1-n\,; \frac{1}{2}\,; \frac{1}{4}\right) \;,\; n = 0, 1, 2, 3, \ldots \tag{55}$$

$$M_{2n+1} = \frac{(-4)^n}{(n+1)!}\left(-\frac{1}{2}-n\right)_n {}_2F_1\left(-n, -1-n\,; \frac{3}{2}\,; \frac{1}{4}\right) \;,\; n = 0, 1, 2, 3, \ldots \tag{56}$$

$$M(z) = \frac{1}{1-3z} - \frac{1}{1-3z}\sum_{n=0}^{\infty}(3M_n - M_{n+1})\, z^{n+1} \tag{57}$$

V. Differential equations

$$\left(z - 2z^2 - 3z^3\right)\frac{dM}{dz} + \left(2 - 3z - 3z^2\right)M = 2 \quad,\; M(0) = 1 \tag{58}$$

$$\left(1 - 2z - 3z^2\right)\frac{dU}{dz} = 1 - U^2 \;,\; U(0) = 0 \;,\; U(z) = z\,M(z) \tag{59}$$

$$(1 - z - U)\frac{dU}{dz} = 4z + U \;,\; U(0) = 0 \;,\; U(z) = 2z^2 M(z) \tag{60}$$

$$U\frac{dU}{dz} + 3z + 1 = 0 \;,\; U(0) = 1 \;,\; U(z) = 1 - z - 2z^2 M(z) \tag{61}$$

VI. Endnote

$$\frac{\pi}{8\sqrt{3}} = \sum_{n=0}^{\infty}\frac{(-1)^n}{2n+1} M_n \left(\frac{\sqrt{2}-1}{\sqrt{3}}\right)^{2n+1} - \sum_{n=0}^{\infty}\frac{(-1)^n}{2n+3}(3M_n - M_{n+1})\left(\frac{\sqrt{2}-1}{\sqrt{3}}\right)^{2n+3} {}_2F_1\left(1, n+\frac{3}{2}\,; n+\frac{5}{2}\,; -3\left(\frac{\sqrt{2}-1}{\sqrt{3}}\right)^2\right) \tag{62}$$

VII. References

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