Article

Formulas for SU(3) Matrix Generators

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Abstract: The Lie algebra of a Lie group is a set of commutation relations, equations satisfied by the group's generators. For SU(2) and many other Lie groups, the equations have been solved and matrix generators are realized as algebraic expressions. This article derives formulas for a basis of matrix generators for the irreducible representations of the Lie group SU(3). A special sequence of eigenvectors is deduced to assist the derivation. As algebraic functions, the formulas are suited to numerical evaluation, algebraic manipulations and analytic operations.

Keywords: special unitary group; SU(3); matrix representation

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1. Introduction

The special unitary group SU(3) has extensive applications in physics, from particle physics[1–5] and nuclear physics[6,7] to the ubiquitous harmonic oscillator, specifically the 3D isotropic harmonic oscillator.[8] For these investigations, it may be convenient to have quick and easy access to matrix representations of the group. To possibly aid such endeavors, this article derives formulas for basis generators of the group SU(3).

Many Lie groups have formulas for their basis generators. Matrix representations of the group SU(2), one step down from SU(3), can be found for any integer or half-integer spin t by substituting t into well-known formulas.[9–12] The homogeneous Lorentz group has matrix realizations determined by compound SU(2) formulas.[2] Having formulas for these groups is fundamental to many numerical, algebraic and analytic explorations of topics related to SU(2) and the Lorentz group. The same may become true for SU(3).

The formulas derived here from the Lie algebra are more compact and simpler than the formulas of the earlier experimental work,[13] which were inferred, in part, from numerical small-dimensional irreducible representations (irreps) obtained irrep-by-irrep by computer.

Roughly put, the eight generators of the Lie algebra $\mathfrak{su}(3)$ are the nine generators of three $\mathfrak{su}(2)$ algebras with two of the nine 'averaged' together.

To elaborate and introduce the notation, let T, U, V be three sets of generators for $\mathfrak{su}(2)$, a total of nine generators. The eight $\mathfrak{su}(3)$ generators are T^3, Y, $T^{\pm}$, $U^{\pm}$, $V^{\pm}$, where $Y = 2(U^3 + V^3)/3$ is the 'average' that reduces nine generators to eight.

The two generators T^3 and Y are diagonal matrices, by assumption and convention. That makes the components on the diagonals of T^3 and Y the eigenvalues of simultaneous eigenvectors. The simultaneous eigenvalues are pairs of real numbers that, when plotted, make a pattern called a 'multiplet.'[3,4,9,14]

Applying one of the six generators $T^{\pm}$, $U^{\pm}$, $V^{\pm}$ to a simultaneous eigenvector gives another eigenvector with raised or lowered eigenvalues. One says 'the six generators $T^{\pm}$, $U^{\pm}$, $V^{\pm}$ raise and lower eigenvalues.' Starting with one of the points on the multiplet pattern, all the points can be reached by successive application of $T^{\pm}$, $U^{\pm}$, or $V^{\pm}$.

Thus, T^3 and Y, are diagonal matrices and $T^{\pm}$, $U^{\pm}$, $V^{\pm}$ are raising/lowering matrices.

The matrices T^3, Y, and $T^{\pm}$, have formulas from the $\mathfrak{su}(2)$ algebra, the T-matrices, and from the properties of multiplets, the matrix Y. The T,Y-matrices are called 'given' since they have known formulas. The task here is to derive formulas for the four 'unknown' U,V matrices, i.e. $U^{\pm}$ and $V^{\pm}$.

Except that the order of SU(2)-irreps in the direct sum making the T-matrices is arbitrary. Equivalently, there is considerable freedom to choose the sequence of the simultaneous eigenvectors of T^3 and Y. We spend Sections 3, 4, 5, producing a special sequence of eigenvectors that is helpful in the derivation.

The place n of an eigenvector in the special sequence depends on three parameters, $n(a,b,\alpha)$, where a, b, and α are multiplet quantities. The function n is useful because the number of eigenvectors is the same as the number of rows and columns in the matrix generators, both of which are the dimension d of the SU(3)-irrep. See Appendix A for an example.

Make a one-to-one correspondence between row index r and the place n, $r \leftrightarrow n(a_r,b_r,\alpha_r)$, and a one-to-one correspondence for the column index c, $r \leftrightarrow n(a_c,b_c,\alpha_c)$. Then we can use the six quantities $(a_r,b_r,\alpha_r,a_c,b_c,\alpha_c)$, as parameters in the function for the value of the matrix component in the r^{th} row and c^{th} column.

The equations to solve and some initial conditions are found in Section 2. The special sequence of eigenvectors is constructed in Sections 3, 4, 5. In Section 6, formulas are determined that relate the row/column indices r,c of a matrix M^{rc} to the parameters $(a_r,b_r,\alpha_r,a_c,b_c,\alpha_c)$.

Formulas for the four given matrix generators T,Y require some assembly which is presented in Section 7. Sections 8 and 9 have the derivation of formulas for the four remaining U,V matrices. Section 8 considers those CRs that are linear in the U,V matrices. After evaluating several linear CRs, only two unknown functions remain to be determined. In Section 9, the CRs quadratic in the U,V matrices and the Casimir operator equation are solved for the two unknown functions. Inverting the steps of the derivation yields formulas for the matrices $U^{\pm}$ and $V^{\pm}$.

The solutions for the eight generators are shown in Section 10. A brief discussion concludes the work, Section 11.

As Supplemental Material, we provide links to a FORTRAN computer program that calculates the matrix generators for an irreducible representation (irrep) of the user's choice[15,16] The program also appears in Appendix B. A second link provides access to a Wolfram Mathematica program that calculates generators for an irrep.[17,18] The Mathematica program also presents calculations that verify that the formulas satisfy the 28 CRs of the Lie algebra $\mathfrak{su}(3)$ for any finite-dimensional irrep. Based on those verifying calculations, it is expected that the formulas hold for all finite-dimensional SU(3)-irreps.

2. Lie algebra

The equations to solve are the 28 commutation relations (CR) of the Lie algebra $\mathfrak{su}(3)$. For convenience, we include an equation with the quadratic Casimir operator C, which is a well-known consequence of the algebra and therefore does not provide any additional constraints.[3,19,20]

The Lie algebras CRs can be grouped into CRs of given T,Y matrices, CRs that are linear in the unknown U,V matrices, and CRs that are quadratic in unknowns. [3,4,9]

CRs in T,Y matrices only:

$$[T^+,T^-] = 2\,T^3 \quad ; \quad [T^3,T^{\pm}] = \pm\,T^{\pm} \quad ; \tag{1}$$

$$[Y,T^{\pm}] = 0 \quad ; \quad [Y,T^3] = 0 \quad . \tag{2}$$

CRs linear in U,V matrices:

$$[T^3,U^{\pm}] = \mp\frac{1}{2}U^{\pm}\,;\; [T^3,V^{\pm}] = \pm\frac{1}{2}V^{\pm}\,;\; [Y,U^{\pm}] = \pm\,U^{\pm}\,;\; [Y,V^{\pm}] = \pm\,V^{\pm} \tag{3}$$

$$[T^{\pm},U^{\mp}] = [T^{\pm},V^{\pm}] = 0 \quad ; \quad [T^{\pm},U^{\pm}] = \pm\,V^{\pm} \quad ; \quad [T^{\pm},V^{\mp}] = \mp\,U^{\mp} \quad . \tag{4}$$

CRs quadratic in U, V matrices:

$$[U^+, U^-] = \frac{3}{2}Y - T^3 \quad ; \quad [V^+, V^-] = \frac{3}{2}Y + T^3 \quad ; \tag{5}$$

$$[U^\pm, V^\mp] = \pm T^\mp \quad ; \quad [U^\pm, V^\pm] = 0 \quad , \tag{6}$$

The commutator $[A, B]$ of two matrices is the difference of their dot products, $[A, B] \equiv AB - BA$.

The quadratic Casimir equation is [3,19,20]

$$C \equiv \left(\{T^+, T^-\} + \{U^+, U^-\} + \{V^+, V^-\}\right)/2 + {T^3}^2 + 3Y^2/4 = \left(p^2 + pq + q^2 + 3p + 3q\right)/3\,\mathbf{1} \,, \tag{7}$$

where the anti-commutator is $\{A, B\} \equiv AB + BA$, and the identity matrix $\mathbf{1}$ is appropriately dimensioned. Equations (1) - (7) are the equations to solve.

We can reduce the number of cases to consider. Assume that we are given a set of generators T^3, Y, $T^\pm$, $U^\pm$, $V^\pm$ for the (p, q)-irrep. One can show that the negative transpose of the generators make a basis for the (q, p)-irrep, *i.e.* for the reverse order of p and q.[3,4,9] In view of this, the derivation can assume p is greater than or equal to q,

$$p \geq q \quad . \tag{8}$$

Then, when we want formulas for a $(\bar{p}, \bar{q})$-irrep with $\bar{p} < \bar{q}$, we can take the negative transposes of the generators for the (p, q)-irrep with $p = \bar{q} > q = \bar{p}$. In this way we get basis generators for (p, q)-irreps with any nonnegative integers p and q by finding formulas for $p \geq q$.

The eight matrices T^3, Y, $T^\pm$, $U^\pm$, $V^\pm$, are assumed to have real-valued components. They are required to satisfy the 28 CRs of the Lie algebra $\mathfrak{su}(3)$ and the expression for the quadratic Casimir operator for the $p \geq q$, (p, q)-irrep of the algebra.

The four matrices $T^\pm$, T^3, Y can be found with SU(2)-irreps and from the properties of a multiplet. Both the SU(2) formulas and the multiplet properties are well documented, and their derivations are not repeated here.

Thus, we need to derive formulas for the set of four unknown matrices $U^\pm$, $V^\pm$. We start by placing the eigenvectors in a special sequence.

3. Eigenvectors

By the CRs in (1), the T-matrices satisfy the Lie algebra $\mathfrak{su}(2)$. Therefore, we can assume that the T-matrices are completely reduced to direct sums of irreps of SU(2). However, a similarity transformation can rearrange the order of the SU(2) irreps, so we can arrange the SU(2) irreps in a special, useful sequence.

We choose T^3 and Y to be diagonal matrices, which is the standard choice. However, we find that the eigenvalues of Y are not convenient to index the components of the matrix. So we define a matrix Y^0 with eigenvalues that are shifted from those of Y The new matrix Y^0 and its eigenvalues y^0 are

$$Y^0 \equiv Y + 2(p - q)/3\,\mathbf{1} \quad \text{and} \quad y^0 = y + 2(p - q)/3 \quad . \tag{9}$$

The eigenvalues of Y^0 and Y differ by $2(p - q)/3$. The two matrices Y and Y^0 are equivalent in the sense that one may be found given the other. But unlike Y, the matrix Y^0 is neither a generator nor a linear combination of generators because the identity matrix $\mathbf{1}$ is not a combination of generators.

Let α and y^0 be the r^{th} component along the diagonals of T^3 and Y^0, resp, *i.e.* $T^3_{rr} = \alpha$ and $Y^0_{rr} = y^0$. Therefore, α and y^0 are simultaneous eigenvalues for the eigenvector, in 'bra-ket' notation, $|\alpha, y^0\rangle$. We have

$$T^3|\alpha, y^0\rangle = \alpha|\alpha, y^0\rangle \quad , \quad Y^0|\alpha, y^0\rangle = y^0|\alpha, y^0\rangle \quad , \tag{10}$$

with the r^{th} column of the identity matrix $\mathbf{1}$ as $|\alpha, y^0\rangle$,

$$|\alpha, y^0\rangle = (0, ..., 1, ...0)^{\mathrm{T}}, \tag{11}$$

with the 1 in the r^{th} place. For convenience, the transpose T is used to display the column vector. Multiplying the eigenvector by an arbitrary nonzero real number A gives an eigenvector with the same eigenvalue(s). We make note of eigenvector normalization when pertinent. We can assume that the eigenvectors are columns of the identity matrix.

For the (p,q)-irrep, the matrix generators are $d \times d$ square matrices with real-valued components, where the dimension d is known to be[3,4,9]

$$d = \frac{1}{2}(p+1)(q+1)(p+q+2) \quad . \tag{12}$$

Thus, the identity matrix $\mathbf{1}$ has dimension d and the eigenvector $|\alpha, y\rangle$ is a column vector with d components.

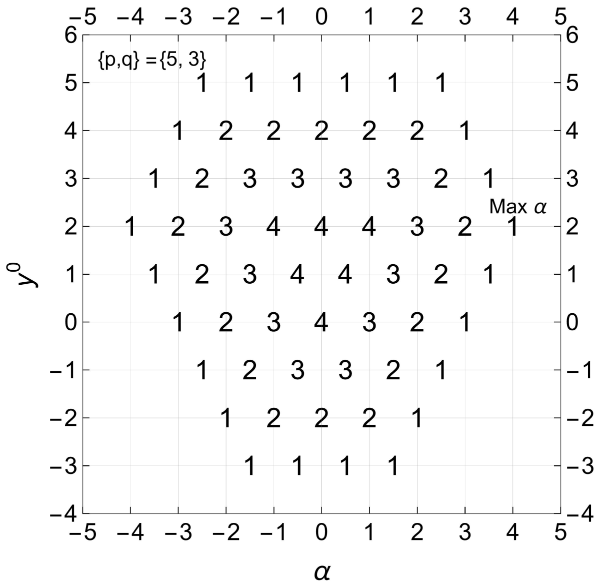

Figure 1. *The* (p,q) = (5,3) *multiplet.* The eigenvalues α and y^0 of the simultaneous eigenvectors $|\alpha, y^0\rangle$ of T^3 and Y^0 make a six-sided figure whose sides have p or q spaces. The point (α, y^0) is marked with the number of eigenvectors, the 'multiplicity,' that share the eigenvalues α and y^0. The multiplicity is one on the rim and increases inward on smaller six-sided figures until reaching the central triangle with multiplicity 4. The point on the rim with the maximum T^3 eigenvalue α = 4 is labeled 'Max α,', often referred to as max weight.

CRs: $[T^3, T^{\pm}], [T^3, U^{\pm}], [T^3, V^{\pm}], [Y, T^{\pm}], [Y, U^{\pm}], [Y, V^{\pm}]$. These equations govern the action of the raising and lowering matrices, $T^{\pm}, U^{\pm}, V^{\pm}$, on the eigenvectors $|\alpha, y^0\rangle$.

Consider the CR with T^3 and one of the raising or lowering matrices R. We have $[T^3, R]$ = rR, for the constant r found in the CR. Applying the CR to an eigenvector gives

$$T^3 R|\alpha, y^0\rangle = RT^3|\alpha, y^0\rangle + rR|\alpha, y^0\rangle = (\alpha + r)R|\alpha, y^0\rangle \Rightarrow R|\alpha, y^0\rangle \propto |\alpha + r, y^0\rangle \quad . \tag{13}$$

Thus, the action of applying the matrix R to an eigenvector $|\alpha, y^0\rangle$ yields an eigenvector $|\alpha + r, y^0\rangle$, which has raised or lowered the eigenvalue α depending on the sign of r. For the Rs and rs in the CRs listed above, we get

$$T^{\pm}|\alpha, y^0\rangle \propto |\alpha \pm 1, y^0\rangle \quad , \quad U^{\pm}|\alpha, y^0\rangle \propto |\alpha \mp 1/2, y^0 \pm 1\rangle \quad , \quad V^{\pm}|\alpha, y^0\rangle \propto |\alpha \pm 1/2, y^0 \pm 1\rangle \quad , \tag{14}$$

which includes the results of raising and lowering both eigenvalues α and y^0. It is known that all eigenvectors of a (p,q)-irrep can be obtained by the action of raising and lowering matrices applied to any eigenvector of the irrep.

4. Multiplets

Plotting the eigenvalues (α, y^0) of the eigenvectors of a (p,q)-irrep makes a three- or six-sided figure called a 'multiplet.' See Figure 1. We list some well-known multiplet properties (MPs).[3,4,9,14]

MP1. The multiplets have a six-sided pattern with $p+1$ points in the row with maximum y^0 and $q+1$ points in the row with minimum y^0. The sides alternately have p spaces and q spaces.

In Figure 1, the top row has six points with $p = 5$ spaces; the bottom row has $q = 3$ spaces.

MP2. The points (α, y^0) on the outer rim each represent one eigenvector, so they have multiplicity one. Each step inward from the rim has points representing an additional eigenvector, until the inner triangle whose points have the maximum multiplicity.

In Figure 1, the outer rim has multiplicity '1,' while the inner triangle is filled with points with multiplicity '4,' meaning each point represents four distinct eigenvectors that happen to have the same eigenvalues.

MP3. The point on the right with the largest eigenvalue $\alpha_{\max}$, which is often called the point with maximum 'weight,' has coordinates $(\alpha_{\max}, y^0) = ((p+q)/2, p-q)$.

Theorem 1 (MP4). *At least one eigenvector has null eigenvalues $\alpha = y^0 = 0$. Thus, a multiplet always has at least one eigenvector $|0,0\rangle$ at the $\alpha = y^0 = 0$ vertex of the inner triangle.*

Proof. Start on the multiplet at the point of maximum weight, $(\alpha_{\max}, y^0) = ((p+q)/2, p-q)$. By (14), after q applications of the raising matrix U^+ and p applications of V^-, we get an eigenvector at the origin $(\alpha, y^0) = (0,0)$, which completes the proof. See Figure 1. □

Theorem 2 (MP5). *At any point of the multiplet, take double the spin component, i.e. 2α, and take y^0 to make the pair $(2\alpha, y^0)$, both integers. The pair never has one odd integer and one even integer. It is always true that the pair are both even integers or they are both odd integers.*

Proof. Since p and q are integers, it follows that their sum and difference $(p+q)$ and $(p-q)$, are even integers or are both odd integers. They are both even when p and q are both even or odd; they are both odd integers when just one of (p,q) is even and the other is odd. Thus, MP5 is true for the eigenvector with maximum weight, the point $(\alpha_{\max}, y^0) = ((p+q)/2, p-q)$, since we take double the spin component $2\alpha_{\max} = (p+q)$. But all eigenvectors of the multiplet can be reached by the action of the raising and lowering matrices, (14), so MP5 is true for all points of the multiplet. □

5. Sequencing Eigenvectors

As stated previously, we determine a special sequence of SU(2) irreps for the reduced T-matrices. The process takes two steps. First the eigenvectors are collected into SU(2) irreps. After collecting SU(2) irreps, the SU(2)-irreps must be sequenced.

Collect the eigenvectors into SU(2)-irreps. An SU(2) irrep with spin t has $2t+1$ eigenvectors $|\alpha, y^0\rangle$ with spin components $\alpha = t, t-1, ..., -t$. We proceed by example.

For example, collect the eigenvectors whose eigenvalues are plotted in the $y^0 = 3$ row of Figure 1. The leftmost point of the row has $\alpha = -7/2$, and the rightmost point has $\alpha = +7/2$, all with $y^0/2 = 3/2$. By the MP2 multiplet property, the multiplicities increase by one from "1" outside to a constant "3" inside of the row.

See Table 1 which records the successive removal of eigenvectors in batches of SU(2)-irreps. Note that, as a consequence of MP2, each SU(2) irrep collected has a unique combination of spin t and half-eigenvalue $y^0/2$.. There is only one $(t, y^0/2)$-SU(2) irrep for a given combination $(t, y^0/2)$.

Table 1. Collecting the eigenvectors in the $y^0 = 3$ row of Figure 1. The first line has the original multiplicities of the eigenvectors with T^3 eigenvalues α. The second and third line have the multiplicities after the removal of eigenvectors collected in the $(t, y^0/2)$-SU(2) irrep in the right column.

α	$-7/2$	$-5/2$	$-3/2$	$-1/2$	$1/2$	$3/2$	$5/2$	$+7/2$	$(t, y^0/2)$
original	1	2	3	3	3	3	2	1	$(7/2, 3/2)$
		1	2	2	2	2	1		$(5/2, 3/2)$
			1	1	1	1			$(3/2, 3/2)$

The row $y^0 = 3$ has $1 + 2 + 3 + 3 + 3 + 3 + 2 + 1 = 18$ eigenvectors that have been collected into three $(t, y^0/2)$-SU(2) irreps $(7/2, 3/2)$, $(5/2, 3/2)$, $(3/2, 3/2)$. These SU(2)-irreps have $2t + 1 = 8, 6, 4,$ eigenvectors each.

The generalization to the other rows of Figure 1 and to the multiplet of any (p, q)-irrep should be apparent.

Theorem 3 (MP6). *Each of the collected irreducible representations $(t, y^0/2)$-SU (2) is unique. At most one SU(2)-irrep has that combination of t and $y^0/2$.*

Proof. Consider the set of all eigenvectors with eigenvalue y^0, a horizontal row of the multiplet. By MP2, only one eigenvector has the maximum eigenvalue α since the eigenvector is on the rim of the multiplet. Again by MP2, there is at least one eigenvector with eigenvalues in unit steps from $-\alpha$ to $+\alpha$. Collecting one eigenvector for each eigenvalue makes exactly one $(t, y^0/2)$ -SU (2) irrep of spin $t = \alpha$. By MP2, the remaining set of eigenvectors has only one eigenvector with the maximum eigenvalue $\alpha - 1$. The collection process produces exactly one $(t, y^0/2)$ -SU (2) irrep of spin $t = \alpha - 1$. By MP2, the process continues until the remaining eigenvectors have a constant multiplicity of one and collecting them makes the final SU(2)-irrep for the y^0 row. Since each collection step makes exactly one SU(2)-irrep and following steps give successively smaller spins t, it follows that none of the $(t, y^0/2)$ -SU (2) irreps have duplicate t. Eigenvectors that have different eigenvalues y^0 are in other constant y^0 sets. Thus, either t or y^0 are different for the $(t, y^0/2)$ -SU (2) irreps and that proves the statement. □

That completes the collection of the eigenvectors into SU(2)-irreps.

Put the SU(2)-irreps in order. The $(t, y^0/2)$-SU(2) irreps can now be placed in order. Spin t and the half-value $y^0/2$ served well in collecting eigenvectors in SU (2) irreps. However, we retire them in favor of their sum and difference, which are more useful quantities for sequencing the SU(2)-irreps.

Let a and b be the difference and sum defined in

$$a \equiv t - y^0/2 \quad ; \quad b \equiv t + y^0/2 \quad . \tag{15}$$

Solving the equations for t and y^0 gives

$$t = (a + b)/2 \quad ; \quad y^0 = b - a \quad . \tag{16}$$

When plotted, as in Figure 2 for the $(p, q) = (5, 3)$-multiplet in Fig 1, the points (a, b) make a rectangle aligned with the a and b-axes.

In view of the new parameters $(t, y^0/2) \to (a, b)$, we give the $(t, y^0/2)$-SU(2) irreps a second name, the '(a, b)-SU(2) irrep.' Since the eigenvectors collected in $(t, y^0/2)$-SU(2) are the same eigenvectors in the (a, b)-SU(2) irrep, we have a corollary to Theorem 3.

Corollary 1 (MP6). *There is exactly one (a, b)-SU(2) irrep for each combination of integers $a = 0, ..., q$ and $b = 0, ..., p$.*

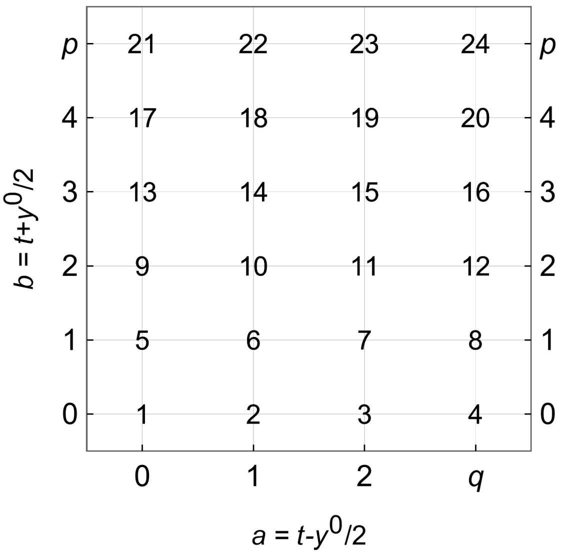

Figure 2. *The sequence of SU(2) irreps in the direct sum of SU(2) irreps making the T-matrices.* For the $(p,q) = (5,3)$ multiplet in Figure 1, simultaneous eigenvectors $|\alpha, y^0\rangle$ are collected in (a,b)-SU(2) irreps. By Corollary 1, each of the $(q+1)(p+1)$ points of the rectangle represents exactly one (a,b)-SU(2) irrep. The sequence of SU(2) irreps k marks the points of the rectangle. The $k = 1$ irrep is the $(a,b) = (0,0)$-SU(2) irrep and the $k = 24$ irrep is the $(a,b) = (p,q) = (5,3)$-SU(2) irrep.

We now show that the points (a,b) are pairs of integers: In a spin t SU(2) irrep, one of the spin components α is $\alpha = t$. From MP5, $2t$ and y^0 are either both even integers or both odd integers. By their definitions as a sum and difference in (15), it follows that a and b are integers.

Plotting the points (a,b) for the (a,b)-SU(2) irreps of the $(p,q) = (5,3)$-multiplet in Figure 1 produces the rectangular array in Figure 2. There are $(p+1)(q+1) = 24$ points in the rectangle with one SU(2) irrep represented at each point.

Since there is exactly one SU(2)-irrep represented at each point, we can assign an order to the SU(2)-irreps by assigning an order to the points of the rectangle. The points are placed in row-by-row order, starting with the bottom row of the rectangle. One can see that the (a,b)-SU(2) irrep is assigned the k^{th} place in the sequence, where

$$k = 1 + a + b\,(q+1) \quad . \tag{17}$$

The range of the SU(2)-irrep index k is $k = 1, \ldots, (q+1)(p+1)$.

For example, in Figure 2, the $k = 15^{\text{th}}$ SU(2) irrep is the $(a,b) = (2,3)$-SU(2) irrep which was earlier called the $(t, y^0/2) = (5/2, 1/2)$-SU(2) irrep.

The reduction of T-matrices into the direct sum of SU(2)-irreps is complete. By (17), the (a,b)-SU(2) irrep is the k^{th} SU(2) irrep in the direct sum, where $a = 0, \ldots, q$, $b = 0, \ldots, p$. We call k the 'single' index for the list of SU(2) irreps. The pair of integers (a,b) is called the 'double' index for the SU(2)-irrep list.

In each SU(2)-irrep, the eigenvectors are placed in descending order. Each (a,b)-SU(2) irrep has spin $t = (a+b)/2$ with $N(a,b) = 2t+1$ eigenvectors $|\alpha, y^0\rangle$. We have

$$N(a,b) = 2t + 1 = a + b + 1 \quad . \tag{18}$$

These eigenvectors are placed in order of the spin component α, largest first, $\alpha = t, \ldots, -t$. That makes $|\alpha, y^0\rangle$ the m^{th} eigenvector in the (a,b)-SU(2) irrep, where

$$m = t - \alpha + 1 = (a+b)/2 - \alpha + 1 \quad . \tag{19}$$

The place number m is a positive integer, $m = 1, \ldots, a+b+1$.

In general, the point (a,b) for the (a,b)-SU(2) irrep is somewhere along the row b in the rectangle. The eigenvectors to the left of (a,b) precede our selected eigenvector $|\alpha, y^0\rangle$. In the b-row, those eigenvectors in the $(\bar{a},b)$-SU(2) irreps with $\bar{a} = 0, \ldots, a-1$ precede the eigenvector $|\alpha, y^0\rangle$ in the sequence of eigenvectors.

The number of eigenvectors in the partial row to the left of (a,b) is

$$\sum_{\bar{a}=0}^{a-1} N(\bar{a},b) = \sum_{\bar{a}=0}^{a-1} (\bar{a}+b+1) = a(a+1)/2 + ab \quad . \tag{20}$$

This does not include the eigenvectors preceding $|\alpha, y^0\rangle$ in the (a,b)-SU(2) irrep itself, which we will count separately below.

For a complete row, $a-1 = q$, so substitute $a = q+1$ in (20). Thus the points $(\bar{a},\bar{b})$ with $0 \leq \bar{a} \leq q$ make up the row with $b = \bar{b}$. The SU(2)-irreps for the row have a number of eigenvectors $n_{\mathrm{row}}(\bar{b})$ given by

$$n_{\mathrm{row}}(\bar{b}) = (q+1)(q+2)/2 + (q+1)\bar{b} \quad . \tag{21}$$

All of the complete rows below the row with the point (a,b) have eigenvectors that precede our eigenvector $|\alpha, y^0\rangle$ in the sequence of eigenvectors.

Our eigenvector $|\alpha, y^0\rangle$ is in the (a,b)-SU(2) irrep which is represented by the point (a,b). Combining the numbers of eigenvectors in the complete rows below (a,b) with the number in the partial row b to the left of (a,b) and placing the eigenvector as the m^{th} eigenvector in the (a,b)-SU(2) irrep, gives us the place n of the eigenvector $|\alpha, y^0\rangle$ in the sequence of d eigenvectors.

Therefore, the place n is given by the formula

$$\begin{aligned} n(a,b,\alpha) &= \left[\sum_{\bar{b}=0}^{b-1} n_{\mathrm{row}}(\bar{b}) + \sum_{\bar{a}=0}^{a-1} N(\bar{a},b)\right] + m \\ &= [(q+1)b(b+1)/2 + b(q+1)(q+2)/2 + a(a+1)/2 + ab] + (t - \alpha + 1) \end{aligned}$$

or

$$n(a,b,\alpha) = \frac{1}{2}[(a+b)(a+b+1) + qb(q+b+2)] + [t - \alpha + 1] \quad , \tag{22}$$

where the first square-bracketed term on the right counts the eigenvectors of the complete rows and the partial row $(\bar{a},\bar{b})$-SU(2) irreps that precede the (a,b)-SU(2) irrep. The second square-bracketed term is the place of the eigenvector $|\alpha, y^0\rangle$ in the (a,b)-SU(2) irrep, which has $t = (a+b)/2$.

For example, in Figure 2, consider the (a,b)= $(3,2)$-SU(2) irrep, which has spin $t = 5/2$. One of its eigenvectors is $|1/2, -1\rangle$, the number $m = 3$ eigenvector in its irrep. The partial row to the left of $(3,2)$, has $3+4+5 = 12$ eigenvectors. And the complete rows below $(3,2)$ have a total of $10+14 = 24$ eigenvectors. Therefore, the eigenvector $|1/2, -1\rangle$ in the $(3,2)$-SU(2) irrep is the $24+12+3 = 39^{\mathrm{th}}$ eigenvector in the sequence of $d = 120$ eigenvectors for the $(p,q) = (5,3)$ irrep of SU(3).

The sequence of eigenvectors is determined. The eigenvector $|\alpha, y^0\rangle$ in the (a,b)-SU(2) irrep is the $n(a,b,\alpha)^{\mathrm{th}}$ in the sequence of d eigenvectors. This is a one-to-one correspondence of the integers n, where $1 \leq n \leq d$, with the d eigenvectors in the collection of simultaneous eigenvectors $|\alpha, y^0\rangle$ of T^3 and Y^0.

6. The Eigenvector Sequence and Matrix Indices

Let M^{rc} denote one of the matrix generators. The number of rows in M^{rc} is the same as the number of eigenvectors in the SU(3)-irrep; both being the dimension d of the representation. Hence, the row index r can be put in a one-to-one correspondence with the place number n of the eigenvectors of the SU(3)-irrep. See Appendix A. The same is true for the column index c. We have

$$r = n(a_r, b_r, \alpha) \quad ; \quad c = n(a_c, b_c, \beta) \quad , \tag{23}$$

where $0 \leq a_r, a_c \leq q$, $0 \leq b_r, b_c \leq p$ and the spin indices have ranges $-t_r \leq \alpha \leq t_r$, $-t_c \leq \beta \leq t_c$, with SU(2)-irrep spins $t_r = (a_r + b_r)/2$ and $t_c = (a_c + b_c)/2$.

In this way, the row-column address of a matrix component in the matrix inherently contains parameters, *e.g.* a, b, α, that may be used in the formulas for the component's value. Thus, the indices of the component as well as its value are functions of the parameters (a, b) and α.

It follows that we can identify the row and column with either the pair of integers r, c or with the two sets of three parameters $\{a_r, b_r, \alpha\}$ and $\{a_c, b_c, \beta\}$. In view of this, we introduce a new symbol for the component M^{rc}. We have

$$M^{rc} = M^{(a_r,b_r),(a_c,b_c)}_{\alpha,\beta} \quad . \tag{24}$$

We say that the indices r, c are a pair of 'single' indices, and that the combinations (a_r, b_r), (a_c, b_c) are a pair of 'double' indices.

It is convenient to talk about '(i, j)-blocks.' For fixed double indices (a_i, b_i), (a_j, b_j), the rectangular matrix with rows and columns indexed by α, β is called an '(i, j)-block' defined by

$$M^{(i,j)}_{\alpha,\beta} = M^{(a_i,b_i),(a_j,b_j)}_{\alpha,\beta} \quad , \tag{25}$$

where $-t_i \leq \alpha \leq t_i$, $-t_j \leq \beta \leq t_j$, with $t_i = (a_i + b_i)/2$ and $t_j = (a_j + b_j)/2$. By (17), $i = 1 + a_i + b_i(q + 1)$ and $j = 1 + a_j + b_j(q + 1)$, with $i, j = 1, ..., (p + 1)(q + 1)$.

The block structure of the matrix M is a square array, $(p + 1)(q + 1)$ on one side, of (i, j)-blocks with each block containing a rectangular matrix of components $(2t_i + 1)$ by $(2t_j + 1)$.

When we write the CRs of the algebra, the commutator of two matrices M^1 and M^2 is written in terms of block-block dot products. The dot product of two blocks has the form

$$\left[M^1, M^2\right]^{(a_i,b_i),(a_j,b_j)}_{\alpha,\beta} = {}^1M^{(a_i,b_i),(a_n,b_n)}_{\alpha,\eta}\, {}^2M^{(a_n,b_n),(a_j,b_j)}_{\eta,\beta} - {}^2M^{(a_i,b_i),(a_{\bar{n}},b_{\bar{n}})}_{\alpha,\bar{\eta}}\, {}^1M^{(a_{\bar{n}},b_{\bar{n}}),(a_j,b_j)}_{\bar{\eta},\beta} \quad , \tag{26}$$

with summation understood for repeated block indices and repeated spin components. The labels '1' and '2' are moved to the left to avoid crowding the superscripts on the right.

7. The T-matrices and Y

After a long development based largely on well-known properties of multiplets, it is time to write the formulas for the 'given matrices, the T-matrices and Y.

Since the T-matrices satisfy the Lie algebra $\mathfrak{su}(2)$, they can be, and are here, direct sums of SU(2)-irreps. Invoking standard formulas for the SU(2)-irreps,[2], the components of the T-matrices can be written

$${}^{\pm}T^{(a_i,b_i)(a_j,b_j)}_{\alpha,\beta} = \sqrt{(t_i \pm \alpha)(1 + t_i \mp \alpha)}\, \delta_{\beta,\alpha\mp 1}\, \delta_{i,j} \quad , \tag{27}$$

$${}^{3}T^{(a_i,b_i)(a_j,b_j)}_{\alpha,\beta} = \alpha\, \delta_{\alpha,\beta}\delta_{i,j} \quad , \tag{28}$$

where, by (16), $t_i = (a_i + b_i)/2$, $t_j = (a_j + b_j)/2$, $\alpha = t_i, t_i - 1, ..., -t_i$, $\beta = t_j, t_j - 1, ..., -t_j$. In these equations, the double indices on the right crowd out the identifiers '$\pm$' and '3' which move to the left of 'T.'

By (9) and(16), we can write the matrices Y^0 and Y,

$${}^{0}Y^{(a_i,b_i)(a_j,b_j)}_{\alpha,\beta} = (b_i - a_i)\, \delta_{\alpha,\beta}\delta_{i,j} \quad , \tag{29}$$

$$Y^{(a_i,b_i)(a_j,b_j)}_{\alpha,\beta} = [b_i - a_i - 2(p - q)/3]\, \delta_{\alpha,\beta}\delta_{i,j} \quad . \tag{30}$$

The matrix Y^0 is useful in the preceding work and the work to follow, but it is not a generator. The matrix Y is one of the eight basis generators.

CRs: $[T^+, T^-]$, $[T^3, T^{\pm}]$, $[Y, T^{\pm}]$. The five CRs listed only involve the four generators $T^{\pm}$, T^3, Y. By (27), (28), (30), one can show that CRs (1) and (2) are satisfied.

We have formulas for four of the eight generators, $T^{\pm}$, T^3, Y. That leaves four unknown generators, $U^{\pm}$ and $V^{\pm}$.

8. Commutation relations linear in unknown generators

Let us begin to constrain the components of the four unknown generators, $U^{\pm}$ and $V^{\pm}$, by looking at CRs linear in the unknowns, whose commutators combine a given T, Y generator with an unknown U, V generator, (3) and (4).

CR: $[T^3, U^{\pm}] = \mp U^{\pm}/2$. By T^3 in (28), the CR with T^3 and U^+ gives

$$T^3 U^+ - U^+ T^3 = -U^+/2 \tag{31}$$

$$^3T^{(i,n)}_{(\alpha,\sigma)} \; {}^+U^{(n,j)}_{(\sigma,\beta)} - {}^+U^{(i,\bar{n})}_{(\alpha,\bar{\sigma})} \; {}^3T^{(\bar{n},\beta)}_{(\bar{\sigma},\beta)} = - \; {}^+U^{(i,j)}_{(\alpha,\beta)}/2 \tag{32}$$

$$\alpha \; {}^+U^{(i,j)}_{(\alpha,\beta)} - {}^+U^{(i,j)}_{(\alpha,\beta)}\beta = - \; {}^+U^{(i,j)}_{(\alpha,\beta)}/2 \quad , \tag{33}$$

which implies that either

$$^+U^{(i,j)}_{(\alpha,\beta)} = 0 \quad \text{or} \qquad {}^+U^{(i,j)}_{(\alpha,\beta)} = {}^+U^{(i,j)}_{(\alpha,\alpha+1/2)} \quad . \tag{34}$$

Thus, the only components of the (i, j) block of U^+ that may not vanish have spin components related by $\beta = \alpha + 1/2$. The possibly nonzero components of a U^+ block occupy a diagonal of the block.

CR: $[T^3, V^{\pm}] = \pm V^{\pm}/2$. Calculations for $V^{\pm}$ are similar to those just completed for U^+. The results for all four of these CRs show that only the following components of the unknown matrices can be nonzero,

$$^{\pm}U^{(i,j)}_{(\alpha,\alpha\pm1/2)} \quad ; \quad {}^{\pm}V^{(i,j)}_{(\alpha,\alpha\mp1/2)} \quad . \tag{35}$$

For all four U, V-matrices, the nonzero components can only occur on one diagonal in a block, a diagonal of components with spin indices α and β that differ by plus or minus one half.

The difference of a half in spin indices implies that, of the two spins t_i, t_j, one spin is an integer and the other is a half-integer. Recall that spin t is a half-integer when $2t$ is an odd integer. Thus, $2(t_j - t_i)$ is some odd integer $2m + 1$. It is tempting to simply assume that $m = 0$, and $2(t_j - t_i) = \pm 1$, which is what we do.

Assumption 1. *For the nonzero (i, j) blocks of the unknown matrices $U^{\pm}$ and $V^{\pm}$ the column-row spin difference is one-half,*

$$t_j - t_i = \pm 1/2 \quad . \tag{36}$$

Since a similarity transformation acting on a given basis of generators produces another such basis, it seems reasonable to expect that a general solution would involve unnecessary similarity transformation parameters. Therefore, in the course of the derivation, we make additional assumptions, of which three assumptions are highlighted.

Upper, lower blocks. Since the spins of the block t_i, t_j are not equal, the nonzero blocks of $U^{\pm}$ and $V^{\pm}$ are not diagonal blocks. Those (i, j) blocks that have $j > i$ are above the main diagonal and are called 'upper' blocks. The (i, j) blocks below the diagonal have $i > j$, and are called 'lower' blocks.

Assumption 1 and (16), $t = (a + b)/2$, constrain the (a, b) double indices for non-zero (i, j)-blocks. We have

$$a_j + b_j = a_i + b_i \pm 1 \quad , \tag{37}$$

where the choice $+1$ for the sign applies to nonzero upper blocks, $j > i$, and the choice -1 works with nonzero lower blocks, $i > j$.

CRs: $[Y, U^{\pm}] = \pm U^{\pm}$ and $[Y, V^{\pm}] = \pm V^{\pm}$. These calculations follow the process shown in (31) to (34): the CR is written out, the matrix Y is substituted and the result is simplified. One finds that nonzero components in $U^{\pm}$ and $V^{\pm}$ occur only in (i, j) blocks with

$$y_j - y_i = y^0_j - y^0_i = \mp 1 \quad , \tag{38}$$

where the choice of signs on the right is correlated with the sign in $U^{\pm}$ and $V^{\pm}$. Thus *e.g.* y_j - $y_i = -1$ for U^+ and V^+. The constraint applies to the entire (i, j) block.

The conditions for nonzero (i, j)-blocks have consequences for their (a, b) double indices $(a_i, b_i)(a_j, b_j)$. By (16), $y^0 = b - a$, the difference in y^0s in (38) gives

$$b_j - a_j = b_i - a_i \mp 1 \quad , \tag{39}$$

for $U^{\pm}$ and $V^{\pm}$. Note the $\mp$ sign in (39) is correlated with the the $\pm$ sign in $U^{\pm}$ and $V^{\pm}$ and is the same for both upper and lower blocks of a given U, V-matrix.

There are constraints on the (a, b) double indices, (37) and (39), and constraints on the spin components in (35). These constraints imply that the only potentially nonzero components of the upper blocks of $U^{\pm}$ and $V^{\pm}$ are the following,

$${}^{+}U^{(a-1,b),(a,b)}_{(\beta-1/2,\beta)} \quad ; \quad {}^{+}V^{(a-1,b),(a,b)}_{(\beta+1/2,\beta)} \quad ; \quad {}^{-}U^{(a,b-1),(a,b)}_{(\beta+1/2,\beta)} \quad ; \quad {}^{-}V^{(a,b-1),(a,b)}_{(\beta-1/2,\beta)} \quad , \tag{40}$$

and the potentially nonzero components of their lower blocks are

$${}^{+}U^{(a,b),(a,b-1)}_{(\alpha,\alpha+1/2)} \quad ; \quad {}^{+}V^{(a,b),(a,b-1)}_{(\alpha,\alpha-1/2)} \quad ; \quad {}^{-}U^{(a,b),(a-1,b)}_{(\alpha,\alpha-1/2)} \quad ; \quad {}^{-}V^{(a,b),(a-1,b)}_{(\alpha,\alpha+1/2)} \quad . \tag{41}$$

These eight are the only nonzero component functions.

Each of the eight components in (40) and (41) is determined by three quantities, a, b, and α or β. The values of a and b are those of the rectangle of Figure 2, except when the occurence of $a - 1$ or $b - 1$ removes $a = 0$ or $b = 0$. Therefore, there are $(p + 1)(q + 1) - 1$ nonzero blocks for each of the eight components. Each nonzero block has only one nonzero diagonal with spin indices $(\alpha, \alpha \pm 1/2)$ or $(\beta \pm 1/2, \beta)$.

Thus, the block-block dot product of two matrix generators is the product of two blocks with only one non-zero diagonal each. By (26), the dot product of two blocks is a block with just one non-zero diagonal. This property simplifies the calculations.

By (27), the raising and lowering T -matrices T^- and T^+, are the transpose of each other. This property makes $F_1 = (T^+ + T^-)/2$ and $F_2 = -i(T^+ - T^-)/2$ hermitian matrices. To have similar constructs for the U- and the V-matrices, one requires that the raising and lowering matrices are each other's transpose. The full basis of hermitian generators is discussed in Section 10.

Assumption 2. *The unknown raising and lowering matrices are assumed to be transposes of one another,*

$${}^{+}U = {}^{-}U^{\mathrm{T}} \quad \text{and} \quad {}^{+}V = {}^{-}V^{\mathrm{T}} \quad . \tag{42}$$

We need only deal with one of a pair of CRs that are each other's transpose because the transpose of a CR is satisfied when the CR is satisfied.

The possibly non-zero components are listed in (40) and (41). Assumption 2 requires the pairwise identifications,

$${}^{-}U^{(a,b),(a-1,b)}_{(\beta,\beta-1/2)} = {}^{+}U^{(a-1,b),(a,b)}_{(\beta-1/2,\beta)} \quad , \tag{43}$$

where $-t + 1 \leq \beta \leq t$,

$${}^{-}U^{(a,b-1),(a,b)}_{(\alpha+1/2,\alpha)} = {}^{+}U^{(a,b),(a,b-1)}_{(\alpha,\alpha+1/2)} \quad , \tag{44}$$

where $-t \leq \alpha \leq t-1$,

$$ {}^{-}V^{(a,b),(a-1,b)}_{(\beta,\beta+1/2)} = {}^{+}V^{(a-1,b),(a,b)}_{(\beta+1/2,\beta)} \quad , \tag{45} $$

where $-t \leq \beta \leq t-1$,

$$ {}^{-}V^{(a,b-1),(a,b)}_{(\alpha-1/2,\alpha)} = {}^{+}V^{(a,b),(a,b-1)}_{(\alpha,\alpha-1/2)} \quad , \tag{46} $$

where $-t+1 \leq \alpha \leq t$. In (43) - (46), we have $t = (a+b)/2$.

In view of these transpose properties, all unknown components of ${}^{-}U$ or ${}^{-}V$ can be replaced with their counterparts in ${}^{+}U$ or ${}^{+}V$. Thus the number of unknowns is halved.

CR: $[T^{-}, U^{+}] = 0$. This CR gives recursion relations for components of the matrix generator U^{+}. The T-matrix formulas in (27) give

$$ [(t_i - \alpha + 1)(t_i + \alpha)]^{1/2} \, {}^{+}U^{(a_i,b_i),(a_j,b_j)}_{(\alpha,\beta)} = {}^{+}U^{(a_i,b_i),(a_j,b_j)}_{(\alpha-1,\beta-1)} \left[\left(t_j - \beta + 1\right)\left(t_j + \beta\right) \right]^{1/2} \quad , \tag{47} $$

where $t_i = (a_i + b_i)/2$, $t_j = (a_j + b_j)/2$, and the ranges of the spin indices are $\alpha = t_i, t_i - 1, ..., -t_i + 1$ and $\beta = t_j, t_j - 1, ..., -t_j + 1$. The equation holds for both upper and lower blocks. We first treat the upper blocks.

For (47) with upper blocks and applying the expression in (40), we have $\alpha = \beta - 1/2$ and double indices $(a_i, b_i) = (a-1, b)$ and $(a_j, b_j) = (a, b)$. By (16), it follows that $t_i = t - 1/2$ and $t_j = t$, where $t = (a+b)/2$. The upper block U^{+} recursion becomes

$$ (t + \beta - 1)^{1/2} \, {}^{+}U^{(a-1,b),(a,b)}_{(\beta-1/2,\beta)} = {}^{+}U^{(a-1,b),(a,b)}_{(\beta-3/2,\beta-1)} (t+\beta)^{1/2} \quad . \tag{48} $$

Starting with $\beta = t$ at the top of the nonzero diagonal in the block, successive application of the recursion in (48), produces an expression for all the nonzero components along an upper U^{+} block's diagonal,

$$ {}^{+}U^{(a-1,b),(a,b)}_{(\beta-1/2,\beta)} = \left(\frac{t+\beta}{2t} \right)^{1/2} {}^{+}U^{(a-1,b),(a,b)}_{(t-1/2,t)} \quad . \tag{49} $$

where $a = 1, ..., q$, $b = 0, ..., p$, $t = (a+b)/2$ and $\beta = t, t-1, ..., -t+1$.

For (47) with the lower blocks of U^{+}, substitute the double indices and spin component requirements from (41), *i.e.* $\alpha + 1/2 = \beta$ and $(a_i, b_i), (a_j, b_j) = (a, b), (a, b-1)$. Then apply the resulting recursion repeatedly to obtain an expression for all components of the block's nonzero diagonal. One finds an expression for the nonzero components along a lower U^{+} block's diagonal

$$ {}^{+}U^{(a,b),(a,b-1)}_{(\alpha,\alpha+1/2)} = (t-\alpha)^{1/2} \, {}^{+}U^{(a,b),(a,b-1)}_{(t-1,t-1/2)} \quad , \tag{50} $$

where $a = 0, , ..., q$, $b = 1, ..., p$, $t = (a+b)/2$ and $\alpha = t-1, t-2, ..., -t$.

Thus, CR $[T^{-}, U^{+}] = 0$ yields recursions that reduce the number of unknowns in an upper or lower non-zero block of U^{+}. A block's nonzero components lie on a diagonal and just one component suffices to get the other components along the diagonal.

CR: $[T^{+}, V^{+}] = 0$. Follow the steps for CR $[T^{-}, U^{+}] = 0$: In the commutator $[T^{+}, V^{+}]$, substitute T^{+} from (27) to find an expression for both upper and lower blocks of V^{+}. For the upper and lower blocks of V^{+}, (40) and (41) give restrictions on the double block indices and spin component indices.

For the nonzero components along an upper V^{+} block's diagonal, one gets

$$ {}^{+}V^{(a-1,b),(a,b)}_{(\beta+1/2,\beta)} = (t-\beta)^{1/2} \, {}^{+}V^{(a-1,b),(a,b)}_{(t-1/2,t-1)} \quad , \tag{51} $$

where $a = 1, ..., q$, $b = 0, ..., p$, $t = (a+b)/2$ and $\beta = t-1, t-2, ..., -t+1$. For the nonzero components along a lower V^{+} block's diagonals, we find

$$ {}^{+}V^{(a,b),(a,b-1)}_{(\alpha,\alpha-1/2)} = \left(\frac{t+\alpha}{2t} \right)^{1/2} {}^{+}V^{(a,b),(a,b-1)}_{(t,t-1/2)} \quad , \tag{52} $$

where $a = 0, , ..., q$, $b = 1, ..., p$, $t = (a+b)/2$ and $\alpha = t, t-2, ..., -t+1$.

Thus, one component determines all components on the diagonal of nonzero components in a block of V^{+}.

CRs: $[T^{\pm}, V^{\mp}] = \mp U^{\mp}$, $[T^{\pm}, U^{\pm}] = \pm V^{\pm}$. These CRs bridge the gap between the matrices U and V.

Start with the CR $[T^{-}, V^{+}] = U^{+}$. Substituting the formula for T^{-} from (27) gives

$$[(t_i-\alpha)(t_i+\alpha+1)]^{1/2+}V^{(i,j)}_{\alpha+1,\beta} - {}^{+}V^{(i,j)}_{\alpha,\beta-1}\left[(t_j+\beta)(t_j-\beta+1)\right]^{1/2} = {}^{+}U^{(i,j)}_{\alpha,\beta} \quad , \tag{53}$$

which holds for both upper and lower blocks. We treat upper and lower blocks separately.

The nonzero components of the upper blocks of U^{+} and V^{+}, $i < j$, are displayed in (40). Therefore, we substitute for (i,j) the double indices $(a-1,b), (a,b)$ and we require $\alpha = \beta - 1/2$. By (16) and with $t = (a+b)/2$, we have $t_i = t - 1/2$ with $t_j = t$. The CR (53) is now

$$[(t-\beta)(t+\beta)]^{1/2+}V^{(a-1,b),(a,b)}_{\beta+1/2,\beta} - {}^{+}V^{(a-1,b),(a,b)}_{\beta-1/2,\beta-1}[(t+\beta)(t-\beta+1)]^{1/2} = {}^{+}U^{(a-1,b),(a,b)}_{\beta-1/2,\beta} \quad , \tag{54}$$

for the upper blocks.

Next, substitute expressions for the components from the recursions in (47),(51). The results is a relationship between the U^{+} and V^{+} upper block unknowns,

$${}^{+}V^{(a-1,b),(a,b)}_{(t-1/2,\,t-1)} = -(2t)^{-1/2+}U^{(a-1,b),(a,b)}_{(t-1/2,\,t)} \quad , \tag{55}$$

where simplification included canceling a common factor of $(t+\beta)$, so $\beta \neq -t$. The equation holds because upper blocks have at least one component with $\beta > -t$, since $a \geq 1$. The equation reduces the number of upper block unknowns from two to one for the matrices U^{+} and V^{+}.

For the lower blocks, substitutions follow from (41), we replace (i,j) by $(a,b), (a,b-1)$ and assume that $\beta = \alpha + 1/2$. With $t = (a+b)/2$, we have $t_i = t$, $t_j = t - 1/2$. The CR for lower blocks $i > j$ is then

$$[(t-\alpha)(t+\alpha+1)]^{1/2+}V^{(a,b),(a,b-1)}_{\alpha+1,\alpha+1/2} - {}^{+}V^{(a,b),(a,b-1)}_{\alpha,\alpha-1/2}[(t-\alpha)(t+\alpha)]^{1/2} = {}^{+}U^{(a,b),(a,b-1)}_{\alpha,\alpha+1/2} \quad , \tag{56}$$

A relationship between the two unknowns in the lower block follows using the recursions (50),(52). One finds that

$${}^{+}V^{(a,b),(a,b-1)}_{(t,\,t-1/2)} = (2t)^{1/2+}U^{(a,b),(a,b-1)}_{(t-1,\,t-1/2)} \quad , \tag{57}$$

where $t = (a+b)/2$. There is just one lower block unknown for U^{+} and V^{+}.

The remaining CRs in the set, $[T^{+}, V^{-}] = -U^{-}$ and $[T^{\pm}, U^{\pm}] = \pm V^{\pm}$, are satisfied by applying constraints that have been derived up to this point.

We can introduce an abbreviated notation. One unknown is the upper U^{+} block diagonal's endpoint

$$U^{(a,b)} \equiv {}^{+}U^{(a-1,b),(a,b)}_{(t-1/2,\,t)} \tag{58}$$

and the other is the lower block diagonal's endpoint

$$U_{(a,b)} \equiv {}^{+}U^{(a,b),(a,b-1)}_{(t-1,\,t-1/2)} \quad , \tag{59}$$

where $U^{(a,b)}$ and $U_{(a,b)}$ depend only on a pair of integers (a,b) whose values are restricted to a $(p+1)$ by $(q+1)$ rectangle, as in Figure 2. The placement of the (a,b) indices as superscript or subscript is meant to be a mnemonic for the upper block and lower block, respectively.

The CRs that are linear in the unknowns $U^{\pm}$ and $V^{\pm}$ have supplied many constraints on the unknown U, V matrices that reduce the number of unknowns to just two, $U^{(a,b)}$ and $U_{(a,b)}$. This prepares us for dealing with equations that are quadratic in the unknown matrices.

9. Equations quadratic in unknown generators

The equations that are quadratic in the unknown matrices are the CRs in (5) and (6) as well as the expression for the Casimir operator (7).

We have an action plan to reduce the number of unknowns in the quadratic equations. The plan has three steps.

Step 1. The matrices U^- and V^- are rewritten using their transposes U^+ and V^+ by (43) - (46).

Step 2. The nonzero components of the blocks of U^+ and V^+ are functions of one unknown per block via recursions (49), (50), (51),(52). The upper and lower block unknowns of U^+ and V^+ are ${}^+U^{(a-1,b),(a,b)}_{(t-1/2,t)}$, ${}^+V^{(a-1,b),(a,b)}_{(t-1/2,t-1)}$, ${}^+U^{(a,b),(a,b-1)}_{(t-1,t-1/2)}$, and ${}^+V^{(a,b),(a,b-1)}_{(t,t-1/2)}$, four unknowns for a given (a,b).

Step 3. By (55) and (57), the V^+ unknowns ${}^+V^{(a,b),(a,b-1)}_{(t-1,t-1/2)}$, and ${}^+V^{(a,b),(a,b-1)}_{(t,t-1/2)}$ can be rewritten in terms of the U^+ unknowns $U^{(a,b)}$ and $U_{(a,b)}$ in (58) and (59).

After applying these three steps to an equation, the only unknowns have two forms $U^{(a,b)}$ and $U_{(a,b)}$, with the possibility that they appear with various parameters (a,b).

The nonzero blocks of the unknown matrices $U^{\pm}$ and $V^{\pm}$ occur off-diagonal, as upper blocks and lower blocks. We make a few remarks about the dot products of two matrices, each with off-diagonal blocks.

The dot product of one upper block with a second upper block gives results in a block that is twice removed from the diagonal blocks. The same is true for lower-lower block dot products, except that the result is twice lowered. Only an upper-lower dot product or a lower-upper dot product can contribute to a diagonal block.

The systematics of the off-diagonal blocks for the unknown matrices makes three equations for each CR that is quadratic in unknown matrices. With one equation for upper-upper block commutators, one for lower-lower block commutators, and one for upper-lower together with lower-upper block commutators, this makes three equations for each CR that is quadratic in the generators U,V.

CR: $[U^+,U^-] = 3Y/2 - T^3$. By selecting just the dot products of the upper-lower blocks of U^+ and U^- and the lower-upper blocks, we get contributions to the diagonal blocks on the left side. On the right side, the matrices Y and T^3, have nonzero diagonal blocks. One has

$$[U^+_{\text{Upper}},U^-_{\text{Lower}}]+[U^+_{\text{Lower}},U^-_{\text{Upper}}] = 3Y/2 - T^3 \tag{60}$$

As just remarked above, the upper-upper and lower-lower dot products do not make nonzero diagonal blocks.

With (40) and (41), writing out the dot products on the left gives the following.

$$\begin{aligned} &{}^+U^{(a,b),(a+1,b)}_{\alpha,\alpha+1/2}\,{}^-U^{(a+1,b),(a,b)}_{\alpha+1/2,\alpha} - {}^-U^{(a,b),(a-1,b)}_{\alpha,\alpha-1/2}\,{}^+U^{(a-1,b),(a,b)}_{\alpha-1/2,\alpha} \\ &\qquad + {}^+U^{(a,b),(a,b-1)}_{\alpha,\alpha+1/2}\,{}^-U^{(a,b-1),(a,b)}_{\alpha+1/2,\alpha} - {}^-U^{(a,b),(a,b+1)}_{\alpha,\alpha-1/2}\,{}^+U^{(a,b+1),(a,b)}_{\alpha-1/2,\alpha} \quad . \end{aligned} \tag{61}$$

Applying Steps 1,2,3, we obtain

$$\left(\frac{t+\alpha+1}{2t+1}\right)U^{(a+1,b)2} - \left(\frac{t+\alpha}{2t}\right)U^{(a,b)^2} + (t-\alpha)U_{(a,b)}{}^2 - (t-\alpha+1)U_{(a,b+1)}{}^2 = 3y^0/2 - p + q - \alpha \quad , \tag{62}$$

where $t = (a+b)/2$, $t \neq 0$, $y^0 = (b-a)$, $a = 0,...,q-1$, $b = 0,...,p-1$.

CR: $[V^+,V^-] = 3Y/2 + T^3$. The upper-lower dot products combined with those of the lower-upper blocks produce nonzero diagonal blocks. One has

$$[V^+_{\text{Upper}},V^-_{\text{Lower}}]+[V^+_{\text{Lower}},V^-_{\text{Upper}}] = 3Y/2 + T^3 \tag{63}$$

By (40) and (41), the two commutators on the left become

$$+V^{(a,b),(a+1,b)}_{\alpha,\alpha-1/2} - V^{(a+1,b),(a,b)}_{\alpha-1/2,\alpha} - -V^{(a,b),(a-1,b)}_{\alpha,\alpha+1/2} + V^{(a-1,b),(a,b)}_{\alpha+1/2,\alpha}$$
$$+ +V^{(a,b),(a,b-1)}_{\alpha,\alpha-1/2} - V^{(a,b-1),(a,b)}_{\alpha-1/2,\alpha} - -V^{(a,b),(a,b+1)}_{\alpha,\alpha+1/2} + V^{(a,b+1),(a,b)}_{\alpha+1/2,\alpha} \quad . \tag{64}$$

Applying Steps 1,2,3, produces

$$\left(\frac{t-\alpha+1}{2t+1}\right)U^{(a+1,b)^2} - \left(\frac{t-\alpha}{2t}\right)U^{(a,b)^2} + (t+\alpha)U_{(a,b)}{}^2 - (t+\alpha+1)U_{(a,b+1)}{}^2 = 3y^0/2 - p + q + \alpha \quad , \tag{65}$$

where $t = (a+b)/2$, $t \neq 0$, $y^0 = (b-a)$, $a = 0, ..., q-1$, $b = 0, ..., p-1$. This is the same equation as (62), aside from some sign differences.

We can eliminate the dependence on α in (62) and (65) by first adding the equations to get

$$\frac{2(t+1)}{2t+1}U^{(a+1,b)^2} - U^{(a,b)^2} + 2tU_{(a,b)}{}^2 - 2(t+1)U_{(a,b+1)}{}^2 = 3y^0 - 2p + 2q \tag{66}$$

Subtracting (65) from (62) gives

$$\frac{1}{2t+1}U^{(a+1,b)^2} - \frac{1}{2t}U^{(a,b)^2} - U_{(a,b)}{}^2 + U_{(a,b+1)}{}^2 = -1 \quad , \tag{67}$$

where we canceled a factor of α which requires at least one spin component $\alpha \neq 0$. In these equations, we have $t = (a+b)/2$, $t \neq 0$, $y^0 = (b-a)$, $a = 0, ..., q-1$, $b = 0, ..., p-1$.

Equations (66) and (67) are two equations in four unknowns, the squares $U^{(a+1,b)^2}$, $U^{(a,b)^2}$, $U_{(a,b)}{}^2$, and $U_{(a,b+1)}{}^2$. Here, the two unknown functions $U^{(a,b)}$ and $U_{(a,b)}$ appear with three sets of parameters: (a,b), $(a+1,b)$, and $(a,b+1)$.

The Casimir equation, (7), is quadratic in the matrices $U^{\pm}$ and $V^{\pm}$ and otherwise depends on block diagonal matrices, the T-matrices, Y, and the unit matrix $\mathbf{1}$. As noted previously, the dot products of upper-lower combined with lower-upper blocks produce nonzero diagonal blocks. Putting the unknowns on the left and the knowns on the right, we have

$$\{U^+_{\text{Upper}}, U^-_{\text{Lower}}\} + \{U^+_{\text{Lower}}, U^-_{\text{Upper}}\} + \{V^+_{\text{Upper}}, V^-_{\text{Lower}}\} + \{V^+_{\text{Lower}}, V^-_{\text{Upper}}\} =$$
$$-\{T^+, T^-\} - 2T^{3^2} - 3Y^2/2 + \left[2\left(p^2 + pq + q^2 + 3p + 3q\right)/3\right]\mathbf{1} \quad . \tag{68}$$

Rather than displaying the anti-commutators in detail, we refer to the commutators displayed in (61) and (64). By carefully replacing the minus signs with pluses, we obtain the appropriate expression for the anti-commutators on the left.

On the right side of (68), the matrices are known. The T matrices can be evaluated with (27) and (28). By (30), the Y-matrix is known. Substituting these known matrices in (68) gives

$$\frac{2(t+1)}{2t+1}U^{(a+1,b)^2} + U^{(a,b)^2} + 2tU_{(a,b)}{}^2 + 2(t+1)U_{(a,b+1)}{}^2 =$$
$$-2t(t+1) - \frac{3}{2}\left[y^0 - 2(p-q)/3\right]^2 + \frac{2}{3}\left(p^2 + pq + q^2 + 3p + 3q\right) \quad , \tag{69}$$

where $t = (a+b)/2$, $y^0 = (b-a)$, $a = 0, ..., q-1$, $b = 0, ..., p-1$. This equation has the same unknowns as (66) and (67).

The three equations (66),(67),(69) involve the four squares, the two upper block unknowns $U^{(a+1,b)^2}$, $U^{(a,b)^2}$, and the two lower block unknowns $U_{(a,b)}{}^2$, $U_{(a,b+1)}{}^2$. We shall show that linear combinations of these three equations produce two recursions.

First, let us separate the two upper block unknowns from the two lower block unknowns. Multiply (66),(67),(69) by $(t+1/2), 2t(t+1), 1/2$, resp. Adding, rearranging and replacing t by $(a+b)/2$ yields

$$[(a+1)+b+1]{U^{(a+1,b)}}^2 = (a+b+1){U^{(a,b)}}^2 - 3a^2 + a(-2p+2q-3) + 2q + pq \ , \quad (70)$$

which gives us a recursion in the double index parameter a for the quantity $(a+b+1){U^{(a,b)}}^2$. Since the parameter a in the equation has the integer values $a = 0, ..., q-1$, it follows that

$$(a+b+1){U^{(a,b)}}^2 = (b+1){U^{(0,b)}}^2 + \sum_{\bar{a}=0}^{\bar{a}=a}\left[-3\bar{a}^2 + \bar{a}(-2p+2q-3) + 2q + pq\right] \quad . \quad (71)$$

We find that

$$(a+b+1){U^{(a,b)}}^2 = (b+1){U^{(0,b)}}^2 + a(p+a+1)(q-a+1) \quad . \quad (72)$$

This equation shows how the combination $(a+b+1)U^{(a,b)}$ depends on the double index parameter a.

Next, multiply (66),(67),(69) by $-(t+1/2), 2t(t+1), 1/2$, resp. Adding, rearranging and replacing t by $(a+b)/2$ yields the lower block equation

$$[a+(b+1)][a+(b+1)+1]{U_{(a,b+1)}}^2 = (a+b)(a+b+1){U_{(a,b)}}^2 - 3b^2 + b(2p-2q-3) + 2p + pq \ . \quad (73)$$

This is a recursion in the double index parameter b for the quantity $(a+b)(a+b+1){U_{(a,b)}}^2$. Since the parameter b takes the integer values $b = 0, ..., p-1$ in the equation, it follows that

$$(a+b)(a+b+1){U_{(a,b)}}^2 = a(a+1){U_{(a,0)}}^2 + \sum_{\bar{b}=0}^{\bar{b}=b}\left[-3\bar{b}^2 + \bar{b}(2p-2q-3) + 2p + pq\right] \quad . \quad (74)$$

We find that

$$(a+b)(a+b+1){U_{(a,b)}}^2 = a(a+1){U_{(a,0)}}^2 + b(p-b+1)(q+b+1) \quad . \quad (75)$$

This equation shows how the combination $(a+b)(a+b+1)U_{(a,b)}$ depends on the double index parameter b.

For the third linear combination of (66),(67),(69), we ignore (66) and multiply (67) by $2(t+1)$ and subtract (69). One finds

$$(a+b+1){U^{(a,b)}}^2 + (a+b)(a+b+1){U_{(a,b)}}^2 = \\ -a^3 - b^3 - (p-q)\left(a^2-b^2\right) + (a+b)(pq+p+q+1) \quad . \quad (76)$$

Substitute (72) and (75) in (76) and simplify. One gets

$$(b+1){U^{(0,b)}}^2 + a(a+1){U_{(a,0)}}^2 = 0 \quad , \quad (77)$$

which holds for $a = 0, ..., q-1$ and $b = 0, ..., p-1$.

For $a = 0$, one has $(b+1){U^{(0,b)}}^2 = 0$, where $b = 0, ..., p-1$. Putting that back in (77) gives $a(a+1){U_{(a,0)}}^2 = 0$ for $a = 0, ..., q-1$. Though not all cases of a and b are covered, we drop the constants in (72) and (75), *i.e.*

$$(b+1){U^{(0,b)}}^2 = 0 \quad ; \quad a(a+1){U_{(a,0)}}^2 = 0 \quad . \quad (78)$$

Dropping these constants simplifies the formulas.

By (72), the upper block unknown $U^{(a,b)}$ satisfies

$$(a+b+1){U^{(a,b)}}^2 = a(p+a+1)(q-a+1) \tag{79}$$

and, by (75), the lower block unknown $U_{(a,b)}$ obeys

$$(a+b)(a+b+1){U_{(a,b)}}^2 = b(p-b+1)(q+b+1) \quad . \tag{80}$$

Taking positive square roots to get $U^{(a,b)}$ and $U_{(a,b)}$ from their squares is an assumption. However, one can show that any of the four choices for the two sign factors would produce generators that satisfy the CRs. To avoid the clutter of introducing two sign variables ϵ_1 and ϵ_2, we simply take the positive roots.

Assumption 3. *In (79) and (80, take the positive square roots to get the two unknowns $U^{(a,b)}$ and $U_{(a,b)}$.*

By Assumption 3, one has

$$U^{(a,b)} = {}^{+}U^{(a-1,b),(a,b)}_{(t-1/2,t)} = [a(p+a+1)(q-a+1)/(a+b+1)]^{1/2} \tag{81}$$

and

$$U_{(a,b)} = {}^{+}U^{(a,b),(a,b-1)}_{(t-1,t-1/2)} = [b(p-b+1)(q+b+1)/((a+b)(a+b+1))]^{1/2} \quad . \tag{82}$$

Thus we have found the one remaining unknown quantity for an upper block and the one remaining unknown for a lower block. With all the unknowns determined, we can construct a set of basis generator formulas.

10. Matrix generator formulas

The formulas for the basis generators of the (p,q)-irrep of SU(3) are presented in two parts. First, the formulas for the four matrix generators T^3, Y, T^+, T^-, then, second, the formulas for the four generators $U^{\pm}$ and $V^{\pm}$.

It may be appropriate to repeat here some notation details. For the double index block notation, a component of a matrix M is denoted

$$M^{(a_i,b_i)(a_j,b_j)}_{\alpha,\beta} \quad , \tag{83}$$

where we have parameters $(a_i,b_i)(a_j,b_j)$ for the row and column of the (i,j)-block in the array of blocks.

The block itself is a matrix with its own rows and columns. The rows and columns of block $(a_i,b_i)(a_j,b_j)$ are indicated by the spin components α and β, with $\alpha = t_i, t_i-1, ..., -t_i$ and $\beta = t_j, t_j-1, ..., -t_j$. The t_i and t_j are the spins of the block given by $t_i = (a_i+b_i)/2$ and $t_j = (a_j+b_j)/2$.

The formulas below are in double index notation. They can be transformed into single index notation. The single index notation M^{rc} has row r and column c indices ranging from 1 to d, the dimension of the matrices. The formulas for r and c are given parametrically in terms of $\{a_i,b_i,\alpha\}$, $\{a_j,b_j,\beta\}$, resp. We have

$$M^{rc} = M^{(a_i,b_i)(a_j,b_j)}_{\alpha,\beta} \quad ; \quad r = n(a_i,b_i,\alpha) \quad ; \quad c = n(a_j,b_j,\beta) \quad , \tag{84}$$

from (23). The function $n(a,b,\alpha)$ is shown in (22).

T^3. The matrix T^3 is one of the two diagonal matrix generators. By (28), we have

$${}^{3}T^{(a,b)(a,b)}_{\alpha,\alpha} = \alpha \quad , \tag{85}$$

where $\alpha = t, ..., -t$, $t = (a+b)/2$, $a = 0, ..., q$, $b = 0, 1, ..., p$. The index parameters of the row r and column c in (84) take the values $a_i = a_j = a$, $b_i = b_j = b$, $\alpha = \beta$.

Y. For the other diagonal matrix, by (30), we have

$$Y^{(a,b)(a,b)}_{\alpha,\alpha} = b - a - 2(p-q)/3 \quad , \tag{86}$$

where $\alpha = t, ..., -t$, $t = (a+b)/2$, $a = 0, ..., q$, $b = 0, 1, ..., p$. The index parameters of the row r and column c in (84) take the values $a_i = a_j = a$, $b_i = b_j = b$, $\alpha = \beta$. Since the value depends on a and b, which determine the block indices (i, j), but not on the spin component α, each diagonal block of Y is proportional to an identity matrix of dimension $2t + 1 = a + b + 1$.

T^+. By (27), the nonzero components of T^+ are

$$^+T^{(a,b)(a,b)}_{\alpha,\alpha-1} = [(t+\alpha)(1+t-\alpha)]^{1/2} \quad , \tag{87}$$

where $\alpha = t, ..., -t+1$, $t = (a+b)/2$, $a = 0, ..., q$, $b = 0, 1, ..., p$. The index parameters of the row r and column c in (84) take the values $a_i = a_j = a$, $b_i = b_j = b$, $\beta = \alpha - 1$. Since the spin components are in decreasing order, the condition $\beta = \alpha - 1$ means places the non-zero components of T^+ just above the diagonal.

T^-. By (27), a nonzero block of T^- has components

$$^-T^{(a,b)(a,b)}_{\alpha,\alpha+1} = [(t-\alpha)(1+t+\alpha)]^{1/2} \quad , \tag{88}$$

where $\alpha = t-1, ..., -t$, $t = (a+b)/2$, $a = 0, ..., q$, $b = 0, 1, ..., p$. The index parameters of the row r and column c in (84) take the values $a_i = a_j = a$, $b_i = b_j = b$, $\beta = \alpha + 1$. The nonzero components of T^- are just below its diagonal. T^- is the last of the four matrices that are non-zero in diagonal blocks.

We now outline the procedure for determining the four generator matrices $U^\pm$ and $V^\pm$. One substitutes the formulas for $U^{(a,b)}$ and $U_{(a,b)}$ in (81) and (82) back through the derivation. There are the U-V bridge equations, (54), (56). Then there are the recursions (49), (50), (51), (52), which fill up the non-zero diagonal in each U^+ and V^+ block. That completes U^+ and V^+. The final two matrices U^- and V^- are determined because they are the transposes of the U^+ and V^+ matrices, respectively, by (43) - (46).

By construction, each U, V matrix formula is dependent on either $U^{(a,b)}$ or $U_{(a,b)}$. We prefer to write the U, V matrix formulas in terms of two other, related functions g and h, which are defined as

$$g(p,q,a,b) = [a(p+a+1)(q-a+1)]/[(a+b)(a+b+1)] \tag{89}$$

$$h(p,q,a,b) = [b(p-b+1)(q+b+1)/((a+b)(a+b+1))] \quad .$$

Note that g is the function $U^{(a,b)}$ with an additional factor in the denominator and that h is just $U_{(a,b)}$. The presence of U appearing in the formulas would possibly be confusing, hence the change in notation.

The four matrix generators U^+, U^-, V^+, V^-, each have non-zero components in their upper and lower blocks, as listed in (40) and (41). So, there are two formulas, upper and lower, for the nonzero components of each of these generators.

U^+, upper blocks. By (49) and (81), we get

$$^+U^{(a-1,b),(a,b)}_{(\beta-1/2,\beta)} = [g(p,q,a,b)(t+\beta)]^{1/2} \, , \tag{90}$$

where $a = 1, ..., q$, $b = 0, 1, ..., p$, $t = (a+b)/2$. The row r and column c index parameters take the values $a_i = a - 1$, $a_j = a$, and $b_i = b_j = b$, $\alpha = \beta - 1/2$.

U^+, lower blocks. By (50) and (82), we get

$$^+U^{(a,b),(a,b-1)}_{(\alpha,\alpha+1/2)} = [h(p,q,a,b)(t-\alpha)]^{1/2} \, , \tag{91}$$

where $a = 0, \ldots, q$, $b = 1, \ldots, p$, $t = (a+b)/2$. The row r and column c index parameters are $a_i = a_j = a$, and $b_i = b$, $b_j = b-1$, $\beta = \alpha + 1/2$.

U^-, upper blocks. By (44), (50) and (82), we get

$$ {}^{-}U^{(a,b-1),(a,b)}_{(\beta+1/2,\beta)} = [h(p,q,a,b)(t-\beta)]^{1/2}\,, \tag{92} $$

where $a = 0, \ldots, q$, $b = 1, \ldots, p$, $t = (a+b)/2$. The row r and column c index parameters have the values $a_i = a_j = a$, and $b_i = b-1$, $b_j = b$, $\alpha = \beta + 1/2$.

U^-, lower blocks. By (43), (49) and (81), we get

$$ {}^{-}U^{(a,b),(a-1,b)}_{(\alpha,\alpha-1/2)} = [g(p,q,a,b)(t+\alpha)]^{1/2}\,, \tag{93} $$

where $a = 1, \ldots, q$, $b = 0, 1, \ldots, p$, $t = (a+b)/2$. The row r and column c index parameters are given by $a_i = a$, $a_j = a-1$, and $b_i = b_j = b$, $\beta = \alpha - 1/2$.

V^+, upper blocks. By (51), (55) and (81), we get

$$ {}^{+}V^{(a-1,b),(a,b)}_{(\beta+1/2,\beta)} = -[g(p,q,a,b)(t-\beta)]^{1/2}\,, \tag{94} $$

where $a = 1, \ldots, q$, $b = 0, 1, \ldots, p$, $t = (a+b)/2$. The row r and column c index parameters take the values $a_i = a-1$, $a_j = a$, and $b_i = b_j = b$, $\alpha = \beta + 1/2$.

V^+, lower blocks. By (52), (57) and (82), we get

$$ {}^{+}V^{(a,b),(a,b-1)}_{(\alpha,\alpha-1/2)} = [h(p,q,a,b)(t+\alpha)]^{1/2}\,, \tag{95} $$

where $a = 0, \ldots, q$, $b = 1, \ldots, p$, $t = (a+b)/2$. The row r and column c index parameters have values $a_i = a_j = a$, and $b_i = b$, $b_j = b-1$, $\beta = \alpha - 1/2$.

V^-, upper blocks. By (46), (52), (57) and (82), we get

$$ {}^{-}V^{(a,b-1),(a,b)}_{(\beta-1/2,\beta)} = [h(p,q,a,b)(t+\beta)]^{1/2}\,, \tag{96} $$

where $a = 0, \ldots, q$, $b = 1, \ldots, p$, $t = (a+b)/2$. The row r and column c index parameters are $a_i = a_j = a$, and $b_i = b-1$, $b_j = b$, $\alpha = \beta - 1/2$.

V^-, lower blocks. By (45), (51), (55) and (81), we get

$$ {}^{-}V^{(a,b),(a-1,b)}_{(\alpha,\alpha+1/2)} = -[g(p,q,a,b)(t-\alpha)]^{1/2}\,, \tag{97} $$

where $a = 1, \ldots, q$, $b = 0, 1, \ldots, p$, $t = (a+b)/2$. The row r and column c index parameters here take values $a_i = a$, $a_j = a-1$, and $b_i = b_j = b$, $\beta = \alpha + 1/2$.

The eight formulas for the U, V matrices in Section 10 have some common characteristics. Each formula has a factor of $g^{1/2}$ or $h^{1/2}$. The upper blocks have $(t \pm \beta)^{1/2}$, while the lower blocks have $(t \pm \alpha)^{1/2}$.

By inspection, one sees that U^- is the transpose of U^+ and V^- is the transpose of V^+. For example, consider the upper block of V^-, (96) and the lower block of V^+, (95). They are related by ${}^{-}V \leftrightarrow {}^{+}V$, $(a, b-1) \leftrightarrow (a, b)$, and $(\beta - 1/2,\ \beta) \leftrightarrow (\alpha,\ \alpha - 1/2)$.

Thus, one can calculate the nonzero components of a generator M from the equations for $M^{(a_i,b_i),(a_j,b_j)}_{(\alpha,\beta)}$ and then place the value in the matrix M^{rc} by calculating r, c from the parameters $a_i, b_i, a_j, b_j, \alpha, \beta$.

For example, the matrices T^+ and U^+ for the $(p,q) = (2,1)$-irrep are shown in Appendix A. The FORTRAN program in Appendix B implements the formulas in the program's Sections 6-13.

The basis derived in this article includes nonsymmetrical matrices with real-valued components. Such matrices are not hermitian.

Following conventional treatments, such as those in Refs. [1,3,4,9], an hermitian basis F_j, $j = 1, 2, ..., 8$, can be constructed. Define the hermitian basis F_j by

$$F_1 = (T^+ + T^-)/2\,;\; F_2 = -i(T^+ - T^-)/2\,;\; F_3 = T^3\,;\; F_4 = (V^+ + V^-)/2\,;$$
$$F_5 = -i(V^+ - V^-)/2\,;\; F_6 = (U^+ + U^-)/2\,;\; F_7 = -i(U^+ - U^-)/2\,;\; F_8 = \sqrt{3}Y/2\,. \quad (98)$$

Note that, in terms of the matrices F_j, the quadratic Casimir operator, (7), is given by

$$C = \sum_{j=1}^{8} F_j F_j\,. \quad (99)$$

Also note that a similarity transformation can be applied to the F_j to obtain an equivalent hermitian basis.

11. Discussion

Without further validation, the formulas in Section 10 should be considered tentative. Various simplifying assumptions were made, such as the spin difference Assumption 1 and the positive root Assumption 3. Such additional constraints may cause the formulas to fail.

Partial validation, case by case for any given SU(3)-irrep, is provided by numerical calculation of the matrix generator basis and subsequent confirmation that the CRs of the $\mathfrak{su}(3)$ Lie algebra are obeyed.

Two computer programs are linked as Supplementary Materials that can perform the necessary calculations.[15,17] The FORTRAN program is included here in the Appendix B. The author has tested and verified that each matrix generator basis satisfies the CRs of $\mathfrak{su}(3)$ for all (p, q) SU(3)-irreps with $p + q \leq 20$.

Supporting and convincing algebraic evidence that the formulas work is provided in the Supplementary Materials; see Section 3 in the Wolfram Mathematica notebook [17]. There, each CR is tested using symbolic algebraic computer techniques, the essence of a proof that the formulas give a valid SU (3) irrep for all finite nonnegative integer pairs (p, q). A detailed proof may be the topic of a future article.

Like the T, U, V-matrices with SU(3), all Lie algebras $\mathfrak{su}(n)$ can be set with embedded copies of SU(2). A strategy for obtaining formulas for matrix generators of $\mathfrak{su}(n)$ could be based on the derivation presented in this article for $\mathfrak{su}(3)$. Or, perhaps, such formulas could be obtained in some other way. Either way, formulas for matrix generator bases of general $\mathfrak{su}(n)$ Lie algebras could be useful tools to have in the toolbox.

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Appendix A. Aspects of the (2,1)-irrep

Why $(2, 1)$? The dimensions of irreducible representations of SU(3) (irreps) grow rapidly with p and q. The matrix generators for the $(p, q) = (2, 1)$ SU(3)-irrep are 15×15 matrices, since the dimension d is $d = (p + 1)(q + 1)(p + q + 2)/2 = 15$. The dimension is $d = 120$ for the $(p, q) = (5, 3)$-irrep of Figures 1 and 2 in the text is too large for this Appendix. This Appendix has examples of quantities for the $d = 15$ dimensional $(2, 1)$ SU(3)-irrep.

Recall that the sequence of eigenvectors in Section 5 provides the parameters a, b, and α for the matrix indices in (84) and the non-zero components in (85)-(97).

Parameters				Indices		
a	b	α	y^0	k	m	n
0	0	0	0	1	1	1
1	0	$\frac{1}{2}$	-1	2	1	2
1	0	$-\frac{1}{2}$	-1	2	2	3
0	1	$\frac{1}{2}$	1	3	1	4
0	1	$-\frac{1}{2}$	1	3	2	5
1	1	1	0	4	1	6
1	1	0	0	4	2	7
1	1	-1	0	4	3	8
0	2	1	2	5	1	9
0	2	0	2	5	2	10
0	2	-1	2	5	3	11
1	2	$\frac{3}{2}$	1	6	1	12
1	2	$\frac{1}{2}$	1	6	2	13
1	2	$-\frac{1}{2}$	1	6	3	14
1	2	$-\frac{3}{2}$	1	6	4	15

Table A1. *The Sequence n of SU(3) Eigenvectors.* The eigenvectors $|\alpha, y^0\rangle$ of T^3 and Y^0 were sequenced by their eigenvalues α and y^0. The eigenvalues α, y^0 are used to sort the eigenvectors into SU(2)-irreps, each SU(2)-irrep identified by the index k with $k = 1, ..., (p+1)(q+1) = 6$. The m^{th} eigenvector in an SU(2)-irrep has a distinct spin component α. There are $a+b+1$ spin components α in each SU(2)-irrep, so $m = 1, ..., a+b+1$. The $d = 15$ eigenvectors are indexed by $n = 1, ..., 15$.

For the allowed values of the integers a and b, the (a, b) SU(2)-irrep is the k^{th} irrep in the direct sums of SU(2)-irreps that make the matrices T^+, T^-, and T^3. In each SU(2)-irrep, the T^3 eigenvalue α is the m^{th} eigenvalue. Combining the irreps, the parameters a, b, and α identify the m^{th} eigenvector in the k^{th} SU(2)-irrep, which is the n^{th} eigenvector in the sequence of eigenvectors. A list of these quantities is shown in Table A1 for the $(2, 1)$-irrep.

The formulas for the generators, (85)-(97), are formulas for the components of blocks. One must combine blocks to make a generator. Thus, for the $(2, 1)$ SU(3)-irrep, the matrix T^+ is

$$T^+ = \left(\begin{array}{|c|cc|cc|ccc|ccc|cccc|}
\hline
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\hline
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\hline
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\hline
0 & 0 & 0 & 0 & 0 & 0 & \sqrt{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \sqrt{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\hline
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \sqrt{2} & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \sqrt{2} & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\hline
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \sqrt{3} & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \sqrt{3} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\hline
\end{array}\right) , \tag{A1}$$

with its blocks outlined. Note that non-zero components occur only in diagonal blocks.

For example, consider the component ${}^{+}T^{14,15}$, the component in the right column 15 of row 14, which is the row just above the bottom row. The double indices $(a_i, b_i)(a_j, b_j)$ and the spin components α, β can be read from Table A1. We have $(a_i, b_i, \alpha) = (1, 2, -1/2)$ since $i = 14$ and $(a_j, b_j, \beta) = (1, 2, -3/2)$

since $j = 15$. Now, the value of the component can be calculated from (87). With $t_i = (a_i + b_i)/2 = 3/2$, one finds that ${}^{+}T^{14,15} = [(t_i + \alpha)(1 + t_i - \alpha)]^{1/2}\,\delta_{\beta,\alpha-1} = \sqrt{3}$, which is consistent with the value in (A1).

The U,V-matrices are nonzero in the upper and lower off-diagonal blocks. The matrix U^{+} is

$$U^{+} = \left(\begin{array}{|c|cc|cc|ccc|ccc|cccc|}
\hline
0 & \sqrt{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\hline
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\hline
0 & 0 & 0 & 0 & 0 & \frac{2}{\sqrt{3}} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\sqrt{3} & 0 & 0 & 0 & 0 & 0 & \sqrt{\frac{2}{3}} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\hline
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & \sqrt{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\hline
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & \frac{2}{\sqrt{3}} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \sqrt{\frac{2}{3}} & 0 & 0 \\
0 & 0 & 0 & 0 & 2\sqrt{\frac{2}{3}} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{\sqrt{3}} & 0 \\
\hline
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & \sqrt{\frac{2}{3}} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & \frac{2}{\sqrt{3}} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \sqrt{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\hline
\end{array}\right), \qquad \text{(A2)}$$

where all of the nonzero components occur in the upper and lower blocks just above and below the diagonal blocks.

For the U^{+} matrix, let us look at the component ${}^{+}U^{14,7}$, a component in a lower block of U^{+}. By Table A1, we have $(a_i, b_i, \alpha) = (1, 2, -1/2)$ since $i = 14$ and $(a_j, b_j, \beta) = (1, 1, 0)$ since $j = 7$. By (91), with $a = a_i = 1$, $b = b_i = 2$, $\alpha = -1/2$, $t = (a_i + b_i)/2 = 3/2$, we have

$${}^{+}U^{14,7} = \left[\frac{b(p - b + 1)(q + b + 1)}{(a + b)(a + b + 1)}\right]^{1/2} (t - \alpha)^{1/2} = \frac{2}{\sqrt{3}}\,,$$

which confirms the value of the component ${}^{+}U^{14,7}$ in the matrix, (A2). The other non-zero components of U^{+} can be verified in the same way.

Appendix B. Computer Program

The following Fortran-90 program calculates a basis of matrix generators for a given (p, q)-irrep.[15] As of this writing, access to Fortran compilers is available to the public.[16]

See Section 1 of the program for some guidance.

```
!RULER789112345678921234567893123456789412345678951234567896123456789712345678981
!
!Formulas for SU(3) Matrix Generators, the FORTRAN Program,
! by Richard Shurtleff, Wentworth Institute of Technology, Boston,
! MA, USA - retired
! email: shurtleffr(at)wit.edu, momentummatrix(at)yahoo.com
!
!------- CONTENTS -----------------------
!---0. Preamble
!---1.  Readme
!---2.  Program Start, Interface Blocks
!---3.  Type declarations
!---4.  Set (p,q), MaxErrLimit
!---5.  Preliminaries
!---6.  Make T3
```

```
!---7.  Make Y
!---8.  Make Tp
!---9.  Make Tm
!--10.  Make Up
!--11.  Make Um
!--12.  Make Vp
!--13.  Make Vm
!--14.  Extra steps needed when p < q
!--15.  Check 28 commutation relations and the Casimir equation
!--16.  Save the results to a file, end program
!--17.  External functions, end-of-file
!
!----------------------------0. Preamble---------------------------------
!This FORTRAN program implements the calculations in the Main article [1],
! 'Formulas for SU(3) Matrix Generators' by Richard Shurtleff,
!   Main article's Abstract:
! The Lie algebra of a Lie group is a set of commutation relations, equations
! satisfied by the group's generators. For SU(2) and many other Lie groups,
! the equations have been solved and matrix generators are realized as
! algebraic expressions. This article derives formulas for a basis of matrix
! generators for the irreducible representations of the Lie group SU(3).
! A special sequence of eigenvectors is deduced to assist the derivation. As
! algebraic functions, the formulas are suited to numerical evaluation,
! algebraic manipulations and analytic operations.
!
!   Acknowledgment
! I would like to express my appreciation for Patrick Koh's debugging of a
! previous program which guided the writing of this program.
!
!   Copyright: CC BY-SA. Public use and modification of this code are allowed
!   provided that the preprint[1] or any subsequent published version is cited.
!
!   References
!
!   [1] R. Shurtleff, "Formulas for SU(3) Matrix Generators"
!       To be submitted to MDPI Modern Mathematical Physics
!   [2] This program can be downloaded by following the link:
! https://www.dropbox.com/scl/fi/fq1jck3s4wbeyc35kngdi/SU3GENpq.MDPIb.f90?rl
! key=brxlx9254rwfmjat14hhxycrk&st=2fwnu1c5&dl=0
!
!
!--------------------------1. Readme-------------------------------------
!
! This program calculates 8 basis generator matrices for the (p,q) irrep of
! the su(3) Lie algebra.  Each generator is constructed in a separate section.
! The equations are implicit in the code.
! See the article Ref. [1] for details.
!
! This FORTRAN 90 program ran successfully on a Windows 11 computer with a
!  GNU fortran compiler Code::Blocks 20.03, Created: 2010/25/05 11:52
```

```
!  Updated: 2010/25/05 11:52, Author: HighTec EDV-Systeme GmbH,
!  Copyright 2010 HighTec EDV-Systeme GmbH, available at
!  https://gcc.gnu.org/fortran/ and http://www.codeblocks.org
!
! USER GUIDE:
! INPUT your value of p and q in Sec. 4 Begin as p0 and q0.
! SET the tolerance for error, MaxErrLimit, if you wish.
! I set MaxErrLimit to 1.D-10. Check the code; it is easily changed.
!
! OUTPUT to standard screen
! Some data is sent to the standard output. This includes the integers
! p0 and q0 that identify the irrep, the dimension of the irrep's matrices,
! the error tolerance, the maximum error found in 29 equations that the
! generators are required to solve.
!
! OUTPUT to a DATA File named "p#q#SU3MatrGen.dat" where # stands for the
! values of p,q. The DATA File's records are
! Record 1: the integers p, q, and the max error in 29 su(3) algebra
! equations: the 28 commutation relations plus the Casimir expression.
! The Format of Record 1: FORMAT(2I3,1ES16.7).
!
! Records 2,3,...: the components of the eight BASIS matrices, a total
!  of 8n**2 real numbers, where n is the matrix dimension dimREP, n = dimREP.
!  Each of the 8 TYUV matrices X has n**2 components arranged by rows, i.e.
!   X(1,1), X(1,2),X(1,3),...,X(1,n),X(2,1),X(2,2),X(2,3),...,X(n,n).
!  The sequence of TYUV matrices is X = T3, Y, Tp, Tm, Up, Um, Vp, Vm.
! The Format of Record 2,3,...: FORMAT(5F16.12),
!Thus the matrix components appear 5 real numbers per line until the last line.
!
! Sample records from (p,q) = (3,1) irrep file "p3q1SU3MatrGen.dat"
! Record #1:
! 3  1   3.5527137E-15
! Record #12:
! -0.500000000000  0.000000000000  0.000000000000 0.000000000000 0.000000000000
! Record #541:
! 0.000000000000  1.118033988750  0.000000000000  0.000000000000 0.000000000000
! Record #923
! 0.000000000000  0.000000000000  0.000000000000
!
!  Given n = dimREP = 24 for (p,q) = (3,1), the four Records shown tell us that
!   Record 1: p = 3, q = 1, MaxErr = 3.5527137E-15
!   Record 12: 5*(11-1)+1 = 51 = 2*n + 3 implies T3(3,3) = -0.5
!   Record 541: 5*(540-1)+2 = 4*24**2 + 16*24 + 9 implies Up(17,9) = 1.11803399
!   Record 923: 5*(922-1) + 3 = 4608 = 8*24**2 = 8*n**2, Total # of components
!
! Best wishes, may you find the program of use.
!
PROGRAM SU3Generators
!
    IMPLICIT NONE
```

```
!------------------------2. Program Start, Interface Blocks -------------------
!
INTERFACE
    FUNCTION nSEQUENCE(p,q,a,b,alpha) RESULT  (w)    !
    IMPLICIT NONE        ! The nth place in the sequence belongs to
        INTEGER :: p,q     ! the T3 eigenvector with eigenvalue alpha
        INTEGER :: a,b,m    ! in the (a,b)-SU2 irrep
        REAL(8) :: alpha,w  !the eigenvector is in the mth place in the irrep,
!                           ! m = (a+b)/2 -alpha + 1
    END FUNCTION
END INTERFACE
INTERFACE
    FUNCTION g(p,q,a,b) RESULT  (w)    !
    IMPLICIT NONE        ! Four of the U,V formulas have
        INTEGER :: p,q     ! the factor SQRT(w), where
        INTEGER :: a,b    !w = a*(p+a+1)*(q-a+1)/[(a+b)(a+b+1)]
        REAL(8) :: w
    END FUNCTION
END INTERFACE
INTERFACE
    FUNCTION h(p,q,a,b) RESULT  (w)    !
    IMPLICIT NONE        ! Four of the U,V formulas have
        INTEGER :: p,q     ! the factor SQRT(w), where
        INTEGER :: a,b    !w = b*(p-b+1)*(q+b+1)/[(a+b)(a+b+1)]
        REAL(8) :: w
    END FUNCTION
END INTERFACE
!
!------------------------3. Type declarations -----------------------
!
INTEGER ::  p0,q0,p,q           ! integers p,q identify the (p,q)SU(3) irrep
INTEGER ::  a,b,c     ! (a,b) identifies an SU2-irrep in the reduced T-matrices
INTEGER ::  ai,bi,aj,bj ! i for row and j for column
REAL(8), ALLOCATABLE :: unitMatrix(:,:)
REAL(8),ALLOCATABLE :: T3(:,:),Y(:,:),Tp(:,:),Tm(:,:),Up(:,:),Um(:,:)
REAL(8),ALLOCATABLE :: Vp(:,:),Vm(:,:)
!
REAL(8) :: MaxErr,MaxErrLimit !Max error found in required equations, tolerance
INTEGER ::  dimREP    !  TYUV matrix dimension
REAL(8) ::  alpha,beta ! spin components as block indices
REAL(8) ::  t ! spin
INTEGER  :: i,j,k,m,n,r,w   ! dummy indices; r row, w column, n dimREP
!
REAL(8),ALLOCATABLE:: T3Tpcomm(:,:),T3Tmcomm(:,:),T3Upcomm(:,:),T3Umcomm(:,:)
REAL(8),ALLOCATABLE:: T3Ycomm(:,:),T3Vpcomm(:,:),T3Vmcomm(:,:),TpTmcomm(:,:)
REAL(8),ALLOCATABLE:: TpUpcomm(:,:),TpUmcomm(:,:),TpYcomm(:,:),TpVpcomm(:,:)
REAL(8),ALLOCATABLE:: TpVmcomm(:,:),TmUpcomm(:,:),TmUmcomm(:,:),TmYcomm(:,:)
REAL(8),ALLOCATABLE:: TmVpcomm(:,:),TmVmcomm(:,:),YUpcomm(:,:),YUmcomm(:,:)
REAL(8),ALLOCATABLE:: YVpcomm(:,:),YVmcomm(:,:),UpUmcomm(:,:),UpVpcomm(:,:)
REAL(8),ALLOCATABLE:: UpVmcomm(:,:),UmVpcomm(:,:),UmVmcomm(:,:),VpVmcomm(:,:)
```

```
REAL(8),ALLOCATABLE:: Casimir(:,:)
!
LOGICAL:: p0GEq0,MaxErrNotTooBig  ! p0 >= q0?, matrices all numerical?
CHARACTER (len=20) :: file_name         ! file name for output data file
!
!
!---------------------4. Set (p,q), MaxErrLimit ----------------------
!
! The (p,q)-irrep is calculated.
p0 = 5        ! Initially, the irrep identifiers (p,q)
q0 = 3       !  are called (p0,q0).
WRITE(*,*) 'p,q = ', p0,q0
WRITE(*,*)    'The program calculates basis generators &
                for the (p,q) irrep of SU(3)'
WRITE(*,*) 'The program produces an output file with the &
            values of p and q in the file-name.'
!
   MaxErrLimit = 1.D-10 ! Set the tolerance, the largest allowed error in
   !                        any component of the 29 eqns
!                                                ! Main article, Eq.(12)
dimREP = (p0+1)*(q0+1)*(p0+q0+2)/2 !Matrices are nxn square, n = dimREP
!WRITE(*,*) 'p0,q0,dimREP = ', p0,q0,dimREP
!
!------------------------5. Preliminaries ----------------------
!
IF (p0 .GE. q0) THEN  ! The formulas are set up for p >= q.
    p0GEq0 = .TRUE.     ! For p < q, we swap p and q for the calculation
    p = p0              ! and take negative transposes to get the generators.
    q = q0
    ELSE IF (p0 < q0) THEN  !When p < q,
        p0GEq0 = .FALSE.
        p = q0
        q = p0
  END IF
 ALLOCATE( unitMatrix(dimREP,dimREP)) ! The unit matrix is the identity matrix
    DO  i = 1,dimREP
        DO  j = 1,dimREP      ! The unit matrix has
    unitMatrix(i,j) = 0.0_8  !  zeros everywhere,except
        END DO
            unitMatrix(i,i) = 1.0_8    ! ones along the diagonal.
    END DO
!
!
!-----------------6. Make T3 ----------------------
!
ALLOCATE( T3(dimREP,dimREP))
!
DO  i = 1,dimREP
    DO  j = 1,dimREP
        T3(i,j) = 0._8
```

```
    END DO
END DO
!
DO b = 0,p
    DO a = 0,q
        DO m = 1,a+b+1
            alpha = DBLE((a+b)/2._8 - m + 1._8) ! Main article, Eq.(19)
            r = nSEQUENCE(p,q,a,b,alpha)
            T3(r,r) = alpha                     ! Main article, Eq.(85)
        END DO
    END DO
END DO
!
!-----------------7. Make Y -----------------------
!
ALLOCATE( Y(dimREP,dimREP))
!
DO  i = 1,dimREP
    DO  j = 1,dimREP
        Y(i,j) = 0._8
    END DO
END DO
!
DO b = 0,p
    DO a = 0,q
        DO m = 1,a+b+1
            alpha = DBLE((a+b)/2._8 - m + 1._8)
            r = nSEQUENCE(p,q,a,b,alpha)
            Y(r,r) = DBLE(b-a-2._8*(p-q)/3._8)
            ! (bi - ai - 2 (pp - qq)/3)
        END DO
    END DO
END DO
!
!-----------------8. Make Tp -----------------------
!
ALLOCATE( Tp(dimREP,dimREP))
!
DO  i = 1,dimREP
    DO  j = 1,dimREP
        Tp(i,j) = 0._8
    END DO
END DO
!
DO b = 0,p
    DO a = 0,q
        DO m = 1,a+b
            alpha = DBLE((a+b)/2._8 - m + 1._8)
            beta = alpha - 1._8
             r = nSEQUENCE(p,q,a,b,alpha)
```

```
             c = nSEQUENCE(p,q,a,b,beta)
              t = (a+b)/2._8
        Tp(r,c) = DSQRT((t+alpha)*(1._8+t-alpha))
!(((ai + bi)/2 + \[Alpha]) (1 + (ai + bi)/2 - \[Alpha]))^(1/2)
        END DO
    END DO
END DO
!
!-----------------9. Make Tm ----------------------
!
ALLOCATE( Tm(dimREP,dimREP))
!
DO  i = 1,dimREP
    DO  j = 1,dimREP
        Tm(i,j) = 0._8
    END DO
END DO
!
DO b = 0,p
    DO a = 0,q
        DO m = 2,a+b+1
            alpha = DBLE((a+b)/2._8 - m + 1._8)
            beta = alpha + 1._8
             r = nSEQUENCE(p,q,a,b,alpha)
             c = nSEQUENCE(p,q,a,b,beta)
              t = (a+b)/2._8
        Tm(r,c) = DSQRT((t-alpha)*(1._8+t+alpha))
!(((ai + bi)/2 + \[Alpha]) (1 + (ai + bi)/2 - \[Alpha]))^(1/2)
        END DO
    END DO
END DO
!
!-----------------10. Make Up ----------------------
!
ALLOCATE( Up(dimREP,dimREP))
!
!WRITE(*,*) "Up"
! Make the upper blocks of Up
DO  i = 1,dimREP
    DO  j = 1,dimREP
        Up(i,j) = 0._8
    END DO
END DO
!
DO b = 0,p                              !upper block
    DO a = 1,q
        DO m = 1,a+b+1
            ai=a-1
            beta = DBLE((a+b)/2._8 - m + 1._8)
            alpha = beta - 0.5_8
```

```
    IF ((alpha .LT. -(ai+b)/2._8).OR.(alpha .GT. (ai+b)/2._8)) THEN
    !WRITE(*,*)  "ai,b,ti,r,alpha = ", ai,b,(ai+b)/2._8,r,alpha
    CYCLE
    END IF
             r = nSEQUENCE(p,q,ai,b,alpha)
             c = nSEQUENCE(p,q,a,b,beta)
            t = (a+b)/2._8
         Up(r,c) = DSQRT(g(p,q,a,b)*(t+beta))
!((+aj) (1 + pp + aj) (1 + qq - aj)/((aj + bj) (aj + bj + 1)))**(1/2)
!g = a*(p+a+1._8)*(q-a+1._8)/((a+b)*(a+b+1._8))
        END DO
    END DO
END DO
! Make the lower blocks of Up
DO b = 1,p                                !lower block
    DO a = 0,q
        DO m = 1,a+b+1  ! replace 1 by 2 and drop IF
            bj = b - 1
            alpha = DBLE((a+b)/2._8 - m + 1._8)
            beta = alpha + 0.5_8
    IF ((beta .LT. -(a+bj)/2._8).OR.(beta .GT. (a+bj)/2._8)) THEN
    !WRITE(*,*)  "a,bj,(a+bj)/2._8,c,beta = ", a,bj,(a+bj)/2._8,c,beta
    CYCLE
    END IF
             r = nSEQUENCE(p,q,a,b,alpha)
             c = nSEQUENCE(p,q,a,bj,beta)
            t = (a+b)/2._8
        Up(r,c) = DSQRT(h(p,q,a,b)*(t-alpha))
!(+b(1+p-b)(1+q+b)/((a+b)(a+b+1)))**(1/2)
! h = b*(p-b+1)*(q+b+1)/((a+b)*(a+b+1._8))
        END DO
    END DO
END DO
!
!-----------------11. Make Um -----------------------
!
ALLOCATE( Um(dimREP,dimREP))
!
!WRITE(*,*) "Um"
! Make the upper blocks of Um
DO  i = 1,dimREP
    DO  j = 1,dimREP
        Um(i,j) = 0._8
    END DO
END DO
!
! Make the upper blocks of Um
DO b = 1,p                                !upper block
    DO a = 0,q
        DO m = 2,a+b+1
```

```
            bi = b-1
            beta = DBLE((a+b)/2._8 - m + 1._8)
            alpha = beta + 0.5_8
    IF ((alpha .LT. -(a+bi)/2._8).OR.(alpha .GT. (a+bi)/2._8)) THEN
    !WRITE(*,*)  "a,bi,ti,r,alpha = ", a,bi,(a+bi)/2._8,r,alpha
    CYCLE
    END IF
             r = nSEQUENCE(p,q,a,bi,alpha)
             c = nSEQUENCE(p,q,a,b,beta)
            t = (a+b)/2._8
        Um(r,c) = DSQRT(h(p,q,a,b)*(t-beta))
!(+b(1+p-b)(1+q+b)/((a+b)(a+b+1)))**(1/2)
! h = b*(p-b+1)*(q+b+1)/((a+b)*(a+b+1._8))
        END DO
    END DO
END DO
!
DO b = 0,p                                !lower block
    DO a = 1,q
        DO m = 1,a+b+1  !for m = 1, alpha = t = (a+b)/2 MOVE TO Up
            aj = a - 1  ! tj = (a+b-1)/2
            alpha = DBLE((a+b)/2._8 - m + 1._8)
            beta = alpha - 0.5_8 !For m = 1, beta = (a+b-1)/2
    IF ((beta .LT. -(aj+b)/2._8).OR.(beta .GT. (aj+b)/2._8)) THEN
    !WRITE(*,*)  "aj,b,(aj+b)/2._8,c,beta = ", aj,b,(aj+b)/2._8,c,beta
    CYCLE
    END IF
             r = nSEQUENCE(p,q,a,b,alpha)
             c = nSEQUENCE(p,q,aj,b,beta)
            t = (a+b)/2._8
        Um(r,c) = DSQRT(g(p,q,a,b)*(t+alpha))
!(+a(1+p+a)(1+q-a)/((a+b)(a+b+1)))**(1/2)
!g = a*(p+a+1._8)*(q-a+1._8)/((a+b)*(a+b+1._8))
        END DO
    END DO
END DO
!
!-----------------12. Make Vp -----------------------
!
ALLOCATE( Vp(dimREP,dimREP))
!
!WRITE(*,*) "Vp"
! Make the upper blocks of Vp
DO  i = 1,dimREP
    DO  j = 1,dimREP
        Vp(i,j) = 0._8
    END DO
END DO
!
DO b = 0,p                                !upper block
```

```
    DO a = 1,q
        DO m = 1,a+b+1
            ai = a-1
            beta = DBLE((a+b)/2._8 - m + 1._8)
            alpha = beta + 0.5_8
    IF ((alpha .LT. -(ai+b)/2._8).OR.(alpha .GT. (ai+b)/2._8)) THEN
    !WRITE(*,*)  "ai,b,ti,r,alpha = ", ai,b,(ai+b)/2._8,r,alpha
    CYCLE
    END IF
             r = nSEQUENCE(p,q,ai,b,alpha)
             c = nSEQUENCE(p,q,a,b,beta)
            t = (a+b)/2._8
        Vp(r,c) = -DSQRT(g(p,q,a,b)*(t-beta))
!((+aj) (1 + pp + aj) (1 + qq - aj)/((aj + bj) (aj + bj + 1)))**(1/2)
!g = a*(p+a+1._8)*(q-a+1._8)/((a+b)*(a+b+1._8))
        END DO
    END DO
END DO
! Make the lower blocks of Vp
DO b = 1,p                              !lower block
    DO a = 0,q
        t = DBLE((a+b)/2._8)
        DO m = 1,a+b+1  ! 1 replaced by 2
            bj = b - 1
            alpha = DBLE(t - m + 1._8)
            beta = alpha - 0.5_8
    IF ((beta .LT. -(a+bj)/2._8).OR.(beta .GT. (a+bj)/2._8)) THEN
    !WRITE(*,*)  "a,bj,(a+bj)/2._8,c,beta = ", a,bj,(a+bj)/2._8,c,beta
    CYCLE
    END IF
             r = nSEQUENCE(p,q,a,b,alpha)
             c = nSEQUENCE(p,q,a,bj,beta)
            t = (a+b)/2._8
        Vp(r,c) = DSQRT(h(p,q,a,b)*(t+alpha))
!(+b(1+p-b)(1+q+b)/((a+b)(a+b+1)))**(1/2)
! h = b*(p-b+1)*(q+b+1)/((a+b)*(a+b+1._8))
        END DO
    END DO
END DO
!
!-----------------13. Make Vm ----------------------
!
ALLOCATE( Vm(dimREP,dimREP))
!
!WRITE(*,*) "Vm"
! Make the upper blocks of Vm
DO  i = 1,dimREP
    DO  j = 1,dimREP
        Vm(i,j) = 0._8
    END DO
```

```
END DO
!
! Make the upper blocks of Vm
DO b = 1,p                              !upper block
    DO a = 0,q
        t = (a+b)/2._8
        DO m = 1,a+b+1
            bi = b-1
            beta = DBLE(t - m + 1._8)
            alpha = beta - 0.5_8
    IF ((alpha .LT. -(a+bi)/2._8).OR.(alpha .GT. (a+bi)/2._8)) THEN
    !WRITE(*,*)  "a,bi,ti,r,alpha = ", a,bi,(a+bi)/2._8,r,alpha
    CYCLE
    END IF
             r = nSEQUENCE(p,q,a,bi,alpha)
             c = nSEQUENCE(p,q,a,b,beta)
        Vm(r,c) = DSQRT(h(p,q,a,b)*(t+beta))
!(+b(1+p-b)(1+q+b)/((a+b)(a+b+1)))**(1/2)
! h = b*(p-b+1)*(q+b+1)/((a+b)*(a+b+1._8))
        END DO
    END DO
END DO
!
DO b = 0,p                              !lower block
    DO a = 1,q
        t = (a+b)/2._8
        DO m = 1,a+b+1
            aj = a - 1
            alpha = DBLE(t - m + 1._8)
            beta = alpha + 0.5_8
    IF ((beta .LT. -(aj+b)/2._8).OR.(beta .GT. (aj+b)/2._8)) THEN
    !WRITE(*,*)  "aj,b,(aj+b)/2._8,c,beta = ", aj,b,(aj+b)/2._8,c,beta
    CYCLE
    END IF
               r = nSEQUENCE(p,q,a,b,alpha)
             c = nSEQUENCE(p,q,aj,b,beta)
        Vm(r,c) = -DSQRT(g(p,q,a,b)*(t-alpha))      ! Main article, Eq.(97)
!(+a(1+p+a)(1+q-a)/((a+b)(a+b+1)))**(1/2)
!g = a*(p+a+1._8)*(q-a+1._8)/((a+b)*(a+b+1._8))
        END DO
    END DO
END DO
!
!-----------------14. Extra steps needed when p < q ---------------------
!
! If the given irrep integers p0 and q0 satisfy  p0 >= q0, then the
! 8 matrices found above make the (p0,q0) irrep.
! For  p0 < q0, we calculated with p = q0 and q = p0, so p > q. The negative
! transpose of the 8 matrices of the (p,q)-irrep make the (p0,q0) irrep.
! For p0 < q0, we need one more step to get the matrices.
```

```
  IF (.NOT.p0GEq0) THEN !  needed for p0 < q0
    Tp = -TRANSPOSE(Tp)
    Tm = -TRANSPOSE(Tm)
    T3 = -TRANSPOSE(T3)
    Up = -TRANSPOSE(Up)
    Um = -TRANSPOSE(Um)
    Y = -TRANSPOSE(Y)
    Vp = -TRANSPOSE(Vp)
    Vm = -TRANSPOSE(Vm)
  END IF
!
! ------15. Check 28 commutator relations and the Casimir equation ----------
!
! The 8 matrices T3,Y,Tp,Tm,Up,Um,Vp,Vm have been calculated. Check them.
!
n = dimREP  !n - the dimension of the matrices. 'n' is shorter than dimREP
!
ALLOCATE(T3Tpcomm(n,n),T3Tmcomm(n,n),T3Upcomm(n,n),T3Umcomm(n,n))
ALLOCATE(T3Ycomm(n,n),T3Vpcomm(n,n),T3Vmcomm(n,n),TpTmcomm(n,n))
ALLOCATE(TpUpcomm(n,n),TpUmcomm(n,n),TpYcomm(n,n),TpVpcomm(n,n))
ALLOCATE(TpVmcomm(n,n),TmUpcomm(n,n),TmUmcomm(n,n),TmYcomm(n,n))
ALLOCATE(TmVpcomm(n,n),TmVmcomm(n,n),YUpcomm(n,n),YUmcomm(n,n))
ALLOCATE(YVpcomm(n,n),YVmcomm(n,n),UpUmcomm(n,n),UpVpcomm(n,n))
ALLOCATE(UpVmcomm(n,n),UmVpcomm(n,n),UmVmcomm(n,n),VpVmcomm(n,n))
ALLOCATE(Casimir(n,n))
!                                                 ! Main article, Eqs.(1-6)
        T3Tpcomm = MATMUL(T3,Tp) - MATMUL(Tp,T3) - Tp                   ! 1
        MaxErr = MAXVAL(ABS( T3Tpcomm ))
            T3Tmcomm = MATMUL(T3,Tm) - MATMUL(Tm,T3) + Tm
            MaxErr = MAXVAL(ABS( (/MaxErr, T3Tmcomm/) ))
                T3Upcomm = MATMUL(T3,Up) - MATMUL(Up,T3) + Up/2._8
                MaxErr = MAXVAL(ABS( (/MaxErr, T3Upcomm/) ))
                    T3Umcomm = MATMUL(T3,Um) - MATMUL(Um,T3) - Um/2._8
                    MaxErr = MAXVAL(ABS( (/MaxErr, T3Umcomm/) ))
                        T3Ycomm = MATMUL(T3,Y) - MATMUL(Y,T3)        ! 5
                        MaxErr = MAXVAL(ABS( (/MaxErr, T3Ycomm/) ))
                            T3Vpcomm = MATMUL(T3,Vp) - MATMUL(Vp,T3) - Vp/2.
                            MaxErr = MAXVAL(ABS( (/MaxErr, T3Vpcomm/) ))
                              T3Vmcomm = MATMUL(T3,Vm) - MATMUL(Vm,T3) + Vm/2.
                                MaxErr = MAXVAL(ABS( (/MaxErr, T3Vmcomm/) ))
 !
        TpTmcomm = MATMUL(Tp,Tm) - MATMUL(Tm,Tp) - 2._8*T3
        MaxErr = MAXVAL(ABS( (/MaxErr, TpTmcomm/) ))
            TpUpcomm = MATMUL(Tp,Up) - MATMUL(Up,Tp) - Vp
            MaxErr = MAXVAL(ABS( (/MaxErr, TpUpcomm/) ))
                TpUmcomm = MATMUL(Tp,Um) - MATMUL(Um,Tp)                ! 10
                MaxErr = MAXVAL(ABS( (/MaxErr, TpUmcomm/) ))
                    TpYcomm = MATMUL(Tp,Y) - MATMUL(Y,Tp)
                    MaxErr = MAXVAL(ABS( (/MaxErr, TpYcomm/) ))
                        TpVpcomm = MATMUL(Tp,Vp) - MATMUL(Vp,Tp)
```

```
                    MaxErr = MAXVAL(ABS( (/MaxErr, TpVpcomm/) ))
                        TpVmcomm = MATMUL(Tp,Vm) - MATMUL(Vm,Tp) + Um
                        MaxErr = MAXVAL(ABS( (/MaxErr, TpVmcomm/) ))
        TmUpcomm = MATMUL(Tm,Up) - MATMUL(Up,Tm)
        MaxErr = MAXVAL(ABS( (/MaxErr, TmUpcomm/) ))
            TmUmcomm = MATMUL(Tm,Um) - MATMUL(Um,Tm) + Vm               ! 15
            MaxErr = MAXVAL(ABS( (/MaxErr, TmUmcomm/) ))
                TmYcomm = MATMUL(Tm,Y) - MATMUL(Y,Tm)
                MaxErr = MAXVAL(ABS( (/MaxErr, TmYcomm/) ))
                    TmVpcomm = MATMUL(Tm,Vp) - MATMUL(Vp,Tm) - Up
                    MaxErr = MAXVAL(ABS( (/MaxErr, TmVpcomm/) ))
                        TmVmcomm = MATMUL(Tm,Vm) - MATMUL(Vm,Tm)
                        MaxErr = MAXVAL(ABS( (/MaxErr, TmVmcomm/) ))
!
        YUpcomm = MATMUL(Y,Up) - MATMUL(Up,Y) - Up
        MaxErr = MAXVAL(ABS( (/MaxErr, YUpcomm/) ))
            YUmcomm = MATMUL(Y,Um) - MATMUL(Um,Y) + Um               ! 20
            MaxErr = MAXVAL(ABS( (/MaxErr, YUmcomm/) ))
                YVpcomm = MATMUL(Y,Vp) - MATMUL(Vp,Y) - Vp
                MaxErr = MAXVAL(ABS( (/MaxErr, YVpcomm/) ))
                    YVmcomm = MATMUL(Y,Vm) - MATMUL(Vm,Y) + Vm
                    MaxErr = MAXVAL(ABS( (/MaxErr, YVmcomm/) ))
        UpUmcomm = MATMUL(Up,Um) - MATMUL(Um,Up) - (3._8*Y/2._8 - T3)
        MaxErr = MAXVAL(ABS( (/MaxErr, UpUmcomm/) ))
            UpVpcomm = MATMUL(Up,Vp) - MATMUL(Vp,Up)
            MaxErr = MAXVAL(ABS( (/MaxErr, UpVpcomm/) ))
                UpVmcomm = MATMUL(Up,Vm) - MATMUL(Vm,Up) - Tm           ! 25
                MaxErr = MAXVAL(ABS( (/MaxErr, UpVmcomm/) ))
        UmVpcomm = MATMUL(Um,Vp) - MATMUL(Vp,Um) + Tp
        MaxErr = MAXVAL(ABS( (/MaxErr, UmVpcomm/) ))
            UmVmcomm = MATMUL(Um,Vm) - MATMUL(Vm,Um)
            MaxErr = MAXVAL(ABS( (/MaxErr, UmVmcomm/) ))
        VpVmcomm = MATMUL(Vp,Vm) - MATMUL(Vm,Vp) - (3._8*Y/2._8 + T3)       ! 28
        MaxErr = MAXVAL(ABS( (/MaxErr, VpVmcomm/) ))
!                                                ! Main article, Eq.(7)
        Casimir = (MATMUL(Tp,Tm)+MATMUL(Tm,Tp)+MATMUL(Up,Um)+ &
            MATMUL(Um,Up)+MATMUL(Vp,Vm)+MATMUL(Vm,Vp))/2.+MATMUL(T3,T3) &
        +3._8*MATMUL(Y,Y)/4._8 - (dble(p**2+p*q+q**2)/3._8+ &
         dble(p + q))*unitMatrix                                        ! 29
        MaxErr = MAXVAL(ABS( (/MaxErr, Casimir/) ))
!
    WRITE(*,*)  'The 28 commutation relations plus the quadratic Casimir &
                expression are obeyed: ',(MaxErr.LE.MaxErrLimit)
WRITE(*,*) 'The largest error in the 29 equations is ',MaxErr,','
WRITE(*,*)  'which should be smaller than MaxErrLimit = ',MaxErrLimit,' which &
                is the allowed tolerance.'
    WRITE(*,*)      ! a blank line
!
    IF(MaxErr.LE.MaxErrLimit) THEN    ! Is the largest error within tolerance?
        MaxErrNotTooBig = .TRUE.        ! TRUE means Yes.
```

```
    ELSE IF (MaxErr >MaxErrLimit)    THEN
        MaxErrNotTooBig = .FALSE.       ! FALSE means No, not within tolerance.
    END IF
!
!----------16. Save the results to a file, end program -----------------------
!
 !MaxErrNotTooBig = .FALSE.    ! Use this statement to test the failure option
 ! MaxErrNotTooBig = .TRUE.    ! Use this statement to make an output
!
!WRITE(*,*) 'p,q,MaxErrNotTooBig = ', p0,q0,MaxErrNotTooBig
    IF (MaxErrNotTooBig) THEN              ! No Failure(s) found
        WRITE (file_name,"('p',i0,'q',i0,'SU3MatrGen.dat')")p0,q0
        OPEN(Unit=5,file=file_name)
        WRITE(5,3000) p0,q0, MaxErr   ! Start the data file with p0,q0,MaxErr.
        CLOSE(5)    ! Next, the 8*dimREP**2 components of the 8 basis matrices
    OPEN(Unit=5,file=file_name,STATUS='OLD', POSITION='APPEND') !Append T,Y,U,V
    WRITE(5,4000) TRANSPOSE(T3),TRANSPOSE(Y),TRANSPOSE(Tp), TRANSPOSE(Tm),&
                    TRANSPOSE(Up), TRANSPOSE(Um), TRANSPOSE(Vp), TRANSPOSE(Vm)
      CLOSE(5)
    END IF
!
  3000 FORMAT(2I3,1ES16.7) !FORMAT statement for two integers and a real number
  4000 FORMAT(5F16.12)!The FORMAT statement for the matrices,5 numbers per line
!
                        END PROGRAM Su3Generators
!
!-----------------17. External functions, end-of-file-----------------------
!
FUNCTION nSEQUENCE(p,q,a,b,alpha) RESULT  (w)       ! Main article, Eq.(22)
    IMPLICIT NONE        ! The nth place in the sequence belongs to
        INTEGER :: p,q     ! the T3 eigenvector with eigenvalue alpha
        INTEGER :: a,b,m    ! in the (a,b)-SU2 irrep
        REAL(8) :: alpha,w  !the eigenvector is in the mth place in the irrep,
!                           ! m = (a+b)/2 -alpha + 1
    w = ((a+b)*(a+b+1._8)+q*b*(q+b+2._8))/2._8 + ((a+b)/2._8-alpha+1._8)
    END FUNCTION
FUNCTION g(p,q,a,b) RESULT  (w)                     ! Main article, Eq.(89)
    IMPLICIT NONE        ! Four of the U,V formulas have
    INTEGER :: p,q     ! the factor SQRT(g), where
        INTEGER :: a,b    !g = a*(p+a+1)*(q-a+1)/[(a+b)(a+b+1)]
        REAL(8) :: w
    w = a*(p+a+1._8)*(q-a+1._8)/((a+b)*(a+b+1._8))
    END FUNCTION
FUNCTION h(p,q,a,b) RESULT  (w)                     ! Main article, Eq.(89)
    IMPLICIT NONE        ! Four of the U,V formulas have
INTEGER :: p,q     ! the factor SQRT(h), where
        INTEGER :: a,b    !h = b*(p-b+1)*(q+b+1)/[(a+b)(a+b+1)]
        REAL(8) :: w
    w = b*(p-b+1)*(q+b+1)/((a+b)*(a+b+1._8))
    END FUNCTION
```

```
! end-of-file
```

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