

# A connection between the Darwin term and a non-spherical charge distribution

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## Abstract

A derivation of the Darwin term is given based on assuming a non-spherical charge distribution. The total translational energy of the system is obtained for a static electric field and the corresponding quantum equation is found to contain the Darwin term under certain conditions.

## I. Introduction.

If we make a non-relativistic expansion of the Dirac equation, one of the terms in the expansion is the Darwin term, (for example see Sakurai<sup>1</sup>). It is named for C. G. Darwin<sup>2</sup> who first investigated it. It takes the form  $-\frac{q\hbar^2}{8m^2c^2}\nabla\cdot\mathbf{E}$  where  $\mathbf{E}$  is the electric field,  $m$  is the mass of the particle,  $q$  its charge, and  $c$  is the speed of light. A bold symbol represents a vector.

The most common explanation of the Darwin term is that it is an effect due to Zitterbewegung, (for example see Sakurai<sup>1</sup>). Wilson<sup>3</sup> extends this approach by representing the electron as an oscillator. Yu, Henneberger<sup>4</sup> propose that it is an extension of the spin-orbit

term while Fushchych et al.<sup>5</sup> show that it can be thought of as a non-relativistic effect by expanding the Levi-Leblond equation. Khriplovich and Milstein<sup>6</sup> show that the Darwin term can be considered of the same origin as the spin-orbit term. Faber<sup>7</sup> shows that it can be associated with a random walk. We show that by using a non-spherical charge distribution it is possible to derive it in a way different from the above sources.

## II. Charge Distribution

Consider a rotating cylindrically symmetric charge distribution in a frame where the angular velocity  $\omega$  aligns with the z axis in an x, y, z rectangular coordinate system. Also take the charge density  $\rho$  to be centered at the center of the coordinate system and symmetric with respect to z. Call this the primed frame. We need  $\int \rho dV = q$  where the spatial integral is over the particle. Because of the charge symmetry we also have

$$\int \rho x' dV = \int \rho y' dV = \int \rho z' dV = 0 \quad (1a)$$

$$\int \rho x' y' dV = \int \rho x' z' dV = \int \rho y' z' dV = 0 \quad (1b)$$

In this frame we then define the values  $Q_1$  and  $Q_2$  by the relations

$$Q_1 = \int \rho z'^2 dV \quad (1c)$$

$$Q_2 = \int \rho x'^2 dV = \int \rho y'^2 dV \quad (1d)$$

Note that  $Q_1$  and  $Q_2$  are similar to the moment of inertia for a mass distribution, but are defined in terms of the charge density instead of the mass density. In general for a rotating

object the charge distribution will not be spherically symmetric but will depend upon the angular velocity  $\omega$ . As a result of this  $Q_1$  and  $Q_2$  will also depend upon  $\omega$ .

Now we need  $Q_1 = Q_2$  as  $\omega$  goes to zero, so to second order set  $Q_1 = Q_0(1 + \alpha_1\omega + \alpha_2\omega^2)$  and  $Q_2 = Q_0(1 + \beta_1\omega + \beta_2\omega^2)$  for some constants  $Q_0$ ,  $\alpha_1$ ,  $\alpha_2$ ,  $\beta_1$  and  $\beta_2$ . We want this to be unchanged if we replace  $\omega$  by  $-\omega$ , so we need  $\alpha_1 = \beta_1 = 0$ . To make the units work out correctly we can set  $\alpha_2 = \alpha \frac{I}{mc^2}$  and  $\beta_2 = \beta \frac{I}{mc^2}$  where  $I$  is the moment of inertia of the particle and  $\alpha$  and  $\beta$  are dimensionless constants.  $I$  is the standard moment of inertia for a mass distribution and has units of grams-cm<sup>2</sup> (for example see Goldstein<sup>8</sup>). Thus we have

$$Q_1 = Q_0(1 + \alpha \frac{I}{mc^2} \omega^2) \quad (2a)$$

$$Q_2 = Q_0(1 + \beta \frac{I}{mc^2} \omega^2) \quad (2b)$$

In general rectangular coordinates  $x^i$ , where  $i = 1,2,3$ , using eqs. (2a,b), eqs. (1a-d) take the form

$$\int \rho \delta x^i dV = 0 \quad (3a)$$

$$\int \rho \delta x^i \delta x^j dV = Q_0((1 + \beta \frac{I}{mc^2} \omega^2) \delta^{ij} + (\alpha - \beta) \frac{I}{mc^2} \omega^i \omega^j) \quad (3b)$$

where  $\delta x^i$  represents the coordinate distance from the center of the charge.

We will consider a current density  $\mathbf{j} = \rho(\mathbf{v} + \boldsymbol{\omega} \times \delta \mathbf{x})$  where  $\mathbf{v}$  is the velocity of the particle. The magnetic moment  $\boldsymbol{\mu}$  and g factor are defined by (for example see Jackson<sup>9</sup>)

$$\boldsymbol{\mu} = \frac{1}{2c} \int \delta \mathbf{x} \times \mathbf{j} dV \quad (4)$$

$$\boldsymbol{\mu} = \frac{gq}{2mc} \mathbf{s} \quad (5)$$

where  $\mathbf{s}$  is the interior angular momentum of the particle. If we take  $\mathbf{s} = I\boldsymbol{\omega}$ , and use our relation for  $\mathbf{j}$  given above along with eq. (4) and eqs. (3a,b) in eq. (5) we obtain

$$Q_0 = \frac{gq}{2m} I \left( 1 - \beta \frac{I}{mc^2} \omega^2 \right)$$

to order  $1/c^2$ . The  $\alpha$  term cancels out. The moment of inertia can also be a function of  $\omega^2$  so set

$$I = I_0 \left( 1 + \gamma \frac{\omega^2}{c^2} \right) \text{ for some constants } \gamma \text{ and } I_0, \text{ so to order } 1/c^2$$

$$Q_0 = \frac{gq}{2m} I_0 \left( 1 + \gamma \frac{\omega^2}{c^2} - \beta \frac{I_0}{mc^2} \omega^2 \right)$$

In order for  $Q_0$  and  $I_0$  to be independent of  $\omega$  we need  $\gamma = \beta \frac{I_0}{m}$  so that

$$Q_0 = \frac{gq}{2m} I_0 \quad (6)$$

### III. Equations of Motion and Quantization

Now consider the translational equation of motion with only a static electric field  $\mathbf{E}$ .

$$m \frac{d\mathbf{v}}{dt} = \int \rho \mathbf{E} dV \quad (7)$$

Expanding  $\mathbf{E}$  in a Taylor series about the center of the particle we have

$$\mathbf{E} = \mathbf{E}_0 + (\delta\mathbf{x} \cdot \nabla)\mathbf{E}_0 + \frac{1}{2}(\delta\mathbf{x} \cdot \nabla)^2\mathbf{E}_0 \quad (8)$$

where  $\mathbf{E}_0$  is  $\mathbf{E}$  and its derivatives evaluated at the center of the particle, and we have ignored terms higher than quadratic in  $\delta\mathbf{x}$ .

Using eq. (8), eqs. (3a,b), and eq. (6) along with the condition  $\int \rho dV = q$  in eq. (7) we obtain the relation

$$m \frac{dv}{dt} = q \left\{ \mathbf{E}_0 + \frac{I_0}{2m} \left( (1 + \beta \frac{I_0}{mc^2} \omega^2) \nabla^2 \mathbf{E}_0 + (\alpha - \beta) \frac{I_0}{mc^2} (\boldsymbol{\omega} \cdot \nabla)^2 \mathbf{E}_0 \right) \right\} \quad (9)$$

where we have ignored terms higher than quadratic in  $\delta\mathbf{x}$  and dropped terms higher than  $1/c^2$ .

We have also set  $g = 2$ .

Since we are only considering static electric fields, we can set  $\mathbf{E}_0 = -\nabla\phi$  where  $\phi$  is the scalar potential. Expressing  $\mathbf{E}_0$  in this form eq. (9) becomes

$$m \frac{dv}{dt} = -q \nabla \left\{ \phi + \frac{I_0}{2m} \left( (1 + \beta \frac{I_0}{mc^2} \omega^2) \nabla^2 \phi + (\alpha - \beta) \frac{I_0}{mc^2} (\boldsymbol{\omega} \cdot \nabla)^2 \phi \right) \right\} \quad (10)$$

The right hand side can be viewed as the gradient of a potential so the total energy  $E$  of the system can be written as

$$\begin{aligned} E &= \frac{1}{2} m v^2 + q \left\{ \phi + \frac{I_0}{2m} \left( (1 + \beta \frac{I_0}{mc^2} \omega^2) \nabla^2 \phi + (\alpha - \beta) \frac{I_0}{mc^2} (\boldsymbol{\omega} \cdot \nabla)^2 \phi \right) \right\} \\ &= \frac{1}{2m} p^2 + q \left\{ \phi + \frac{I_0}{2m} \nabla^2 \phi + \frac{1}{2m^2 c^2} (\beta s^2 \nabla^2 + (\alpha - \beta) (\mathbf{s} \cdot \nabla)^2) \phi \right\} \end{aligned} \quad (11)$$

where  $\mathbf{p}$  is the momentum, and again ignoring terms higher than  $1/c^2$ .

We will quantize the system by expressing the energy in eq. (11) as an operator by replacing  $\mathbf{p}$  by  $-i\hbar\nabla$  and  $\mathbf{s}$  by  $\frac{1}{2}\hbar\boldsymbol{\sigma}$  where  $\boldsymbol{\sigma}$  are the Pauli spin matrices in vector form (for example see Saxon<sup>10</sup>) so that our Schrodinger type equation takes the form

$$\begin{aligned} i\hbar \frac{\partial \psi}{\partial t} &= \left[ -\frac{\hbar^2}{2m} \nabla^2 + q\left\{ \phi + \frac{I_0}{2m} \nabla^2 \phi + \frac{\hbar^2}{8m^2 c^2} (\beta \sigma^2 \nabla^2 + (\alpha - \beta)(\boldsymbol{\sigma} \cdot \nabla)^2) \phi \right\} \right] \psi \\ &= \left[ -\frac{\hbar^2}{2m} \nabla^2 + q\left\{ \phi + \frac{\hbar^2}{8m^2 c^2} (2\beta + \alpha) \nabla^2 \phi \right\} \right] \psi \end{aligned} \quad (12)$$

We have used the properties of the  $\boldsymbol{\sigma}$  matrices so that  $\sigma^2 = 3$  and  $(\boldsymbol{\sigma} \cdot \nabla)^2 = \nabla^2$  and have taken the limit of  $I_0$  going to zero. If we think of a rotating object then as  $\omega$  increases the equator moment should expand and the moment parallel to  $\boldsymbol{\omega}$  should reduce in size. Therefore  $\beta$  should be positive and  $\alpha$  negative. If we set  $-\alpha = \beta = 1$  we obtain the Darwin term.

### Conclusion.

One interesting thing about this derivation is that the  $1/8$  in front of the Darwin term comes out naturally, although there is no apparent reason why  $2\beta + \alpha$  should be  $1$ . The other  $c^{-2}$  corrections in the non-relativistic expansion of the Dirac eq., the spin-orbit and relativistic mass correction terms, are due to a relativistic correction to the velocity while in our case the Darwin term appears to be a relativistic correction to the spin.

Instead of including the rotational equations and using a finding a Lagrangian for the whole system, the translational energy has just been used. It turns out that if we try to find a Lagrangian for the translational and rotational equations we run into problems.

In spite of these assumptions it is interesting that the Darwin term can be obtained by using a non-spherical charge distribution, and perhaps a more sophisticated derivation will lead to a better understanding.

## References

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