Chapter 1
On Born Reciprocal Relativity Theory, The Relativistic Oscillator and the Fulling-Davies-Unruh effect

Carlos Castro Perelman[0000−0002−7276−9957]

Abstract A continuation of the Born Reciprocal Relativity Theory (BRRT) program in phase space shows that a natural temperature-dependence of mass occurs after recurring to the Fulling-Davies-Unruh effect. The temperature dependence of the mass $m(T)$ resemblances the energy-scale dependence of mass and other physical parameters in the renormalization (group) program of QFT. It is found in a special case that the effective photon mass is no longer zero, which may have far reaching consequences in the resolution of the dark matter problem. The Fulling-Davies-Unruh effect in a $D = 1 + 1$-dim spacetime is analyzed entirely from the perspective of BRRT, and we explain how it may be interpreted in terms of a linear superposition of an infinite number of states resulting from the action of the group $U(1, 1)$ on the Lorentz non-invariant vacuum $|\tilde{0}\rangle$ of the relativistic oscillator studied by Bars [1]

1.1 Novel findings in Born Reciprocal Relativity Theory

Most of the work devoted to Quantum Gravity has been focused on the geometry of spacetime rather than phase space per se. The first indication that phase space should play a role in Quantum Gravity was raised by [2]. The principle behind the concept of "Born reciprocal relativity theory", or non-inertial relativity to be more precise[1], was advocated by [3], [4], [5] and it was based on the idea proposed long ago by [2] that coordinates and momenta should be unified on the same footing. Consequently, if there is a limiting speed (temporal derivative of the position coordinates) in Nature there should be a maximal force as well, since force is the temporal derivative of the momentum. Hence, a *maximal* speed limit (speed of light) must be accompanied

Bahamas Advanced Science Institute and Conferences
Ocean Heights, Stella Maris, Long Island, the Bahamas
e-mail: perelmanc@hotmail.com

[1] We thank one of the referees for highlighting this fact in order to clarify the point that Born did not propose a reciprocal relativity theory

with a *maximal* proper force (which is also compatible with a *maximal* and *minimal* length duality) [5]. The principle of maximal acceleration was advocated earlier on by [6].

We explored in [5] some novel consequences of Born reciprocal Relativity theory in flat phase-space and generalized the theory to the curved spacetime scenario. We provided, in particular, some specific results resulting from Born reciprocal Relativity and which are *not* present in Special Relativity. These are : momentum-dependent time delay in the emission and detection of photons; relativity of chronology; energy-dependent notion of locality; superluminal behavior; relative rotation of photon trajectories due to the aberration of light; invariance of areas-cells in phase-space and modified dispersion relations.

The generalized velocity and force (acceleration) boosts (rotations) transformations of the *flat* $8D$ Phase space coordinates , where $X^i, t, E, P^i; i = 1,2,3$ are **c** -valued (classical) variables which are *all* boosted (rotated) into each-other, were given by [3] based on the group $U(1,3)$ and which is the Born version of the Lorentz group $SO(1,3)$. The $U(1,3) = SU(1,3) \times U(1)$ group transformations leave invariant the symplectic 2-form $\Omega = - dt \wedge dE + \delta_{ij} dX^i \wedge dP^j; i, j = 1,2,3$ and also the following Born-Green line interval in the *flat* $8D$ phase-space

$$(d\omega)^2 = c^2 (dt)^2 - (dX)^2 - (dY)^2 - (dZ)^2 +$$
$$\frac{1}{b^2} \left((dE)^2 - c^2 (dP_x)^2 - c^2 (dP_y)^2 - c^2 (dP_z)^2 \right) \tag{1.1}$$

The maximal proper force is set to be given by b. The symplectic group is relevant because $U(1,3) = Sp(8,R) \cap O(2,6)$; $U(3,1) = Sp(8,R) \cap O(6,2)$, and $U(2,2) = Sp(8,R) \cap O(4,4)$.

The 16 generators Z_{ab} of the $U(1,3)$ algebra can be decomposed into the 6 Hermitian Lorentz sub-algebra generators $L_{[ab]}$, and the 10 anti-Hermitian "shear"-like generators $iM_{(ab)}$ (note the i factor that converts the Hermitian generators $M_{(ab)}$ into anti-Hermitian ones $iM_{(ab)}$) as follows

$$Z_{ab} \equiv \frac{1}{2} \left(iM_{(ab)} + L_{[ab]} \right) \Rightarrow L_{[ab]} = (Z_{ab} - Z_{ba})$$
$$M_{(ab)} = -i \, (Z_{ab} + Z_{ba}), \; a, b = 0,1,2,3 \tag{1.2}$$

The Weyl unitary trick allows to relate the unitary group $U(p+q)$ and the pseudo-unitary group $U(p,q)$, and explains why one needs to decompose the matrix generators of the non-compact pseudo-unitary group $U(1,3)$ in terms of Hermitian *and* anti-Hermitian matrices. The Weyl unitary trick explains the factor of **i** before the M_{ab} in the definition of the Z_{ab} generators in eq-(1.2).

Given the $U(1,3)$ invariant metric $\eta_{ab} = diag(+1,-1,-1,-1)$, the explicit commutation relations of the M_{ab}, L_{ab} generators are given by

$$[L_{ab}, \; L_{cd}] = i \, (\eta_{bc} L_{ad} - \eta_{ac} L_{bd} - \eta_{bd} L_{ac} + \eta_{ad} L_{bc}). \tag{1.3a}$$
$$[M_{ab}, \; M_{cd}] = -i \, (\eta_{bc} L_{ad} + \eta_{ac} L_{bd} + \eta_{bd} L_{ac} + \eta_{ad} L_{bc}). \tag{1.3b}$$

$$[L_{ab},\ M_{cd}] \ = \ i\ (\eta_{bc} M_{ad} - \eta_{ac} M_{bd} + \eta_{bd} M_{ac} - \eta_{ad} M_{bc}). \qquad (1.3c)$$

Therefore, given $Z_{ab} = \frac{1}{2}(iM_{ab} + L_{ab}),\ Z_{cd} = \frac{1}{2}(iM_{cd} + L_{cd})$ after straightforward algebra it leads to the $U(1,3)$ commutators

$$[\ Z_{ab},\ Z_{cd}\ ] \ = \ -i\ (\ \eta_{bc}\ Z_{ad}\ -\ \eta_{ad}\ Z_{cb}\ ). \qquad (1.3d)$$

as expected. The commutators of the Lorentz boosts generators L_{ab} and X_c, P_c are of the form

$$[L_{ab},\ X_c] \ = \ i\ (\ \eta_{bc}\ X_a\ -\ \eta_{ac}\ X_b\ ); \quad [L_{ab},\ P_c] \ = \ i\ (\ \eta_{bc}\ P_a\ -\ \eta_{ac}\ P_b\ ) \quad (1.4)$$

The Hermitian M_{ab} generators are the "reciprocal" boosts/rotation transformations which $exchange$ X for P, in addition to boosting (rotating) those variables, and one ends up with the commutators of M_{ab} and X_c, P_c given by

$$[M_{ab},\ \frac{X_c}{\lambda_l}] \ = \ -\frac{i}{\lambda_p}\ (\ \eta_{bc}\ P_a + \eta_{ac}\ P_b\ ); \quad [M_{ab},\ \frac{P_c}{\lambda_p}] \ = \ -\frac{i}{\lambda_l}\ (\ \eta_{bc}\ X_a + \eta_{ac}\ X_b\ ) \qquad (1.5)$$

where λ_l, λ_p are suitable length and momentum scales which are chosen to be the Planck length and momentum, respectively.

The rotations, velocity and force (acceleration) boosts leaving invariant the symplectic 2-form and the line interval in the $8D$ phase-space are rather elaborate. In four spacetime dimensions the velocity-boosts generators along the x_i spatial directions ($i = 1, 2, 3$) are given by $K_i = L_{0i}$. The force-boots (acceleration boosts) generators along the x_i spatial directions are given by $N_i = M_{0i}$. The rotation generators are $J_i = \epsilon_i^{jk} L_{jk}$. The shear generators are M_{ij}, M_{00}. In general, given the $U(1,3)$ generator $Z = \frac{1}{2}\theta^{AB} Z_{AB}$, the transformations of the four-vectors $\mathbf{X} = (T, X_i)$; $\mathbf{P} = (E, P_i)$ are given by

$$\mathbf{X}' \ = \ e^{\frac{1}{2}\theta^{AB} Z_{AB}}\ \mathbf{X}\ e^{-\frac{1}{2}\theta^{AB} Z_{AB}}, \quad \mathbf{P}' \ = \ e^{\frac{1}{2}\theta^{AB} Z_{AB}}\ \mathbf{P}\ e^{-\frac{1}{2}\theta^{AB} Z_{AB}} \qquad (1.6)$$

leading to

$$\mathbf{X}' \ = \ \mathbf{X} \ + \ [Z,\ \mathbf{X}] \ + \frac{1}{2!}\ [Z,\ [Z,\ \mathbf{X}]] \ + \frac{1}{3!}\ [Z,\ [Z,\ [Z,\ \mathbf{X}]]] \ + \ \ldots \qquad (1.7)$$

and a similar relation for $\mathbf{P}'$ in terms of the nested commutators.

By recurring to the commutation relations (1.5) and the nested commutators in eq-(1.7), one finds that the group transformations of the 8-dim phase space coordinates involving both velocity and force boosts are given by [3] (page 18)

$$t' \ = \ t\ \cosh\xi \ + \ (\frac{\xi_v^i\ X_i}{c} \ + \ \frac{\xi_a^i\ P_i}{b})\ \frac{\sinh\xi}{\xi} \qquad (1.8a)$$

$$E' = E \cosh\xi + (-b\,\xi^i_a\,X_i + c\,\xi^i_v\,P_i)\,\frac{\sinh\xi}{\xi} \qquad (1.8b)$$

$$X'^i = X^i + (\cosh\xi - 1)\,\frac{(\xi^i_v\,\xi^j_v + \xi^i_a\,\xi^j_a)\,X_j}{\xi^2} + (c\,\xi^i_v\,t - \frac{\xi^i_a\,E}{b})\,\frac{\sinh\xi}{\xi} \quad (1.8c)$$

$$P'^i = P^i + (\cosh\xi - 1)\,\frac{(\xi^i_v\,\xi^j_v + \xi^i_a\,\xi^j_a)\,P_j}{\xi^2} + (b\,\xi^i_a\,t + \frac{\xi^i_v\,E}{c})\,\frac{\sinh\xi}{\xi} \quad (1.8d)$$

where ξ^i_v are the velocity-boost rapidity parameters along the e_i directions; ξ^i_a are the force (acceleration) boost rapidity parameters along the e_i directions, $i = 1, 2, 3$, and ξ is the *net* effective rapidity parameter of the primed-reference frame given by

$$\xi = \sqrt{(\xi^i_v)^2 + (\xi^i_a)^2}, \quad i = 1, 2, 3 \qquad (1.9)$$

A straightforward way of understanding how one obtains the above transformations of eqs-(1.8) can be found by simply recalling the most general (Lorentz) velocity boosts transformations of the spacetime coordinates after splitting the three-vectors $\mathbf{X}, \mathbf{P}$ into the parallel $\mathbf{X}_{\parallel}$ and transverse $\mathbf{X}_{\perp}$ components with respect to the velocity boost rapidity parameter
$\boldsymbol{\xi} = (\xi_1, \xi_2, \xi_3); \xi = \sqrt{(\xi_1)^2 + (\xi_2)^2 + (\xi_3)^2}$. Such decomposition is of the form

$$\mathbf{X}_{\parallel} = (\mathbf{X}\cdot\boldsymbol{\xi})\,\frac{\boldsymbol{\xi}}{\xi^2}, \quad \mathbf{X}_{\perp} = \mathbf{X} - \mathbf{X}_{\parallel} = \mathbf{X} - (\mathbf{X}\cdot\boldsymbol{\xi})\,\frac{\boldsymbol{\xi}}{\xi^2} \qquad (1.10)$$

$$\mathbf{P}_{\parallel} = (\mathbf{P}\cdot\boldsymbol{\xi})\,\frac{\boldsymbol{\xi}}{\xi^2}, \quad \mathbf{P}_{\perp} = \mathbf{P} - \mathbf{P}_{\parallel} = \mathbf{P} - (\mathbf{P}\cdot\boldsymbol{\xi})\,\frac{\boldsymbol{\xi}}{\xi^2} \qquad (1.11)$$

so that the Lorentz transformations of $\mathbf{X}, \mathbf{P}$ can be written in vector form as

$$\mathbf{X}' = \left(\mathbf{X} - (\mathbf{X}\cdot\boldsymbol{\xi})\,\frac{\boldsymbol{\xi}}{\xi^2}\right) + (\mathbf{X}\cdot\boldsymbol{\xi})\,\frac{\boldsymbol{\xi}}{\xi^2}\,\cosh\xi + \frac{c\,t\,\sinh\xi}{\xi}\,\boldsymbol{\xi} \qquad (1.12)$$

$$\mathbf{P}' = \left(\mathbf{P} - (\mathbf{P}\cdot\boldsymbol{\xi})\,\frac{\boldsymbol{\xi}}{\xi^2}\right) + (\mathbf{P}\cdot\boldsymbol{\xi})\,\frac{\boldsymbol{\xi}}{\xi^2}\,\cosh\xi + \frac{E\,\sinh\xi}{c\,\xi}\,\boldsymbol{\xi} \qquad (1.13)$$

where the modulus $\xi = |\boldsymbol{\xi}|$ of the velocity-boost rapidity parameters, and the modulus $|\mathbf{v}|$ of the velocity $\mathbf{v}$ of the moving frame of reference are related by $tanh(\xi) = \beta = \frac{\sqrt{v_1^2+v_2^2+v_3^2}}{c}$. One then finds that the transverse directions to the velocity remain unaffected by the Lorentz transformations, while the parallel directions are. One can see by simple inspection that by setting the force-boost parameters to zero $\xi^i_a = 0$ in eqs-(1.8), one recovers the standard Lorentz transformations.

These transformations can be *simplified* drastically when the velocity and force (acceleration) boosts are both parallel to the x-direction and leave the transverse directions Y, Z, P_y, P_z intact. There is now a subgroup $U(1,1) = SU(1,1) \times U(1) \subset U(1,3)$ which leaves invariant the following line interval

$$(d\omega)^2 = c^2(dt)^2 - (dX)^2 + \frac{(dE)^2 - c^2(dP)^2}{b^2} =$$

$$(d\tau)^2 \left(1 + \frac{(dE/d\tau)^2 - c^2(dP/d\tau)^2}{b^2}\right) = (d\tau)^2 \left(1 - \frac{F^2}{F^2_{max}}\right), \qquad (1.14)$$

with $F_{max} = b$ and where one has factored out the non-vanishing proper time infinitesimal $(d\tau)^2 = c^2 dt^2 - dX^2 \neq 0$ in (1.14). The numerical quantity F^2 is positive by definition. The proper force on a massive particle is given by $F = ma$, where a is the proper acceleration and m is the rest mass. The case when $(d\tau)^2 = 0$ is discussed below. We refrained from factoring out $(dt)^2$ in (1.14) because it is not Lorentz invariant, whereas $(d\tau)^2$ is Lorentz invariant.

It is very important to emphasize that there are *no* factors of $(1+F^2/b^2)$ appearing in the above factorization process because in the superluminal case $(d\tau)^2 < 0$ (spacelike spacetime interval) one still has $m^2 a^2 < 0$ despite that $a^2 > 0$ (timelike proper acceleration), because $m^2 < 0$ due to the *imaginary* mass of tachyons. Hence we shall always have the factor $(1 - F^2/b^2)$ as expected. This is a consequence of the fact that if $(d\tau)^2 > 0$, then $(dE)^2 - (dP)^2 < 0$, and vice versa, if $(d\tau)^2 < 0$, then $(dE)^2 - (dP)^2 > 0$.

Consequently, the *negative* sign appearing inside the parenthesis in the last term of eq-(1.14) furnishes the analog of the Lorentz relativistic factor in special relativity and it involves the ratio of the square of two *proper* forces. The result (1.14) in the 4-dim phase space can be generalized to the $8D$-dim phase space (and to higher dimensions) whose coordinates are (X_μ, P_μ), $\mu = 0, 1, 2, 3$, where now one has (for a subluminal particle) $c^2(dt/d\tau)^2 - (dX_i/d\tau)^2 > 0$, with $i = 1, 2, 3$, and $(dE/d\tau)^2 - c^2(dP_i/d\tau)^2 = -F^2 < 0$.

The null case $(d\omega)^2 = 0$ in eq-(1.14) occurs naturally when $(d\tau)^2 = 0$, corresponding to a massless particle (like a photon) moving at the speed of light, and which in turn, implies also that $(dE)^2 - c^2(dP)^2 = 0$ because in the massless case one has $E^2 - c^2P^2 = 0 \Rightarrow E = cP \Rightarrow dE = cdP$. Therefore, the first line of eq-(1.14) yields $(d\omega)^2 = 0$ automatically. However, when $m \neq 0 \Rightarrow (d\tau) \neq 0$, the factorization of $(d\tau) \neq 0$ is allowed in eq-(1.14), and one can still have $(d\omega)^2 = 0$ when the massive particle experiences the *maximal* proper force $F = b$. Therefore, one attains $(d\omega)^2 = 0$ when one has a massless particle, or a massive one experiencing the maximal proper force $F_{max} = b$. A thorough study of the spacelike $(d\omega)^2 < 0$, null $(d\omega)^2 = 0$, and timelike $(d\omega)^2 > 0$ intervals in phase space, and their relation to the intervals in space time, can be found in the Appendix.

Caution must be taken in not confusing the proper force associated to a four-vector $F_\mu = \frac{dP_\mu}{d\tau}$, $\mu = 0, 1, 2, 3$ with the *spatial* force associated to a three-vector

$\mathbf{f} = \frac{dP_i}{d\tau}, i = 1,2,3$. The four-force has for components $F_\mu = (\frac{dE}{d\tau}, c\mathbf{f})$ where $\frac{dE}{d\tau}$ is the proper power. By maximal proper force one means that the magnitude-squared $|(dE/d\tau)^2 - c^2(dP_i/d\tau)^2| = |-F^2| = F^2 \leq b^2$ is bounded. However, this does *not* mean that the individual values of $(dE/d\tau)^2$ (square of the proper power) and $c^2(dP_i/d\tau)^2$ (magnitude-squared of the spatial force) are bounded. What is bounded is their $difference$ $|(dE/d\tau)^2 - c^2(dP_i/d\tau)^2| = F^2 \leq b^2$. For example, given the on-shell relation involving the energy-momentum $E^2 - c^2 P_i^2 = m^2 c^4$, this does not mean that each of the values of E^2, P_i^2 are bounded (they $blow$ up when $v = c$). What is bounded is their difference (for a finite mass m).

Adopting the units $\hbar = c = k_B = 1$, one may postulate that the maximal proper-force acting on a fundamental particle in four-spacetime dimensions is given by $F_{max} = b \equiv \kappa m_P^2 = \kappa L_P^{-2} = \kappa/G$, where κ is a numerical coefficient. m_P is the Planck mass and L_P is the postulated minimal Planck length. A way to estimate the numerical coefficient κ is by looking at the Hawking temperature T_H associated to a black hole of Planck mass $T_H = \frac{1}{8\pi G m_P} = \frac{m_P}{8\pi}$. Equating T_H with the Unruh temperature $T_U = \frac{a}{2\pi}$ yields a proper acceleration of $a = \frac{m_P}{4}$, so that the corresponding proper force is $F = m_P a = \frac{m_P^2}{4} \Rightarrow \kappa = \frac{1}{4}$, and one recovers precisely the value of the maximum force conjecture proposed by [18].

Another route one may take is by setting the Unruh temperature to be equal to the Planck temperature $T_U = T_P = m_P = \frac{a}{2\pi} \Rightarrow a = 2\pi m_P$, so that the corresponding proper force is now $F = m_P a = 2\pi m_P^2$ leading to a value of $\kappa = 2\pi$. Invoking a minimal/maximal length duality one can also set $b = \kappa M_U/R_H$, where R_H is the Hubble scale and M_U is the observable mass of the universe. Equating both expressions for b leads to $M_U/m_P = R_H/L_P \sim 10^{60}$. The value of $b = \kappa m_P^2$ may also be interpreted as the maximal string tension. Since physics is an experimental science the choice of κ will have to be determined by experiment or observations, if the Born Reciprocal Relativity postulate is obeyed in nature.

The $U(1,1)$ group transformations involving the velocity and force boosts along the X direction of the phase-space coordinates X, t, P, E which leave the interval (1.14) invariant are obtained directly from eqs-(1.8) in this special case as follows

$$t' = t \cosh\xi + (\frac{\xi_v x}{c} + \frac{\xi_a P}{b}) \frac{\sinh\xi}{\xi} \qquad (1.15a)$$

$$E' = E \cosh\xi + (-b\,\xi_a X + c\,\xi_v P) \frac{\sinh\xi}{\xi} \qquad (1.15b)$$

$$X' = X \cosh\xi + (c\,\xi_v t - \frac{\xi_a E}{b}) \frac{\sinh\xi}{\xi} \qquad (1.15c)$$

$$P' = P \cosh\xi + (\frac{\xi_v E}{c} + b\,\xi_a t) \frac{\sinh\xi}{\xi} \qquad (1.15db)$$

ξ_v is the velocity-boost rapidity parameter; ξ_a is the force (acceleration) boost rapidity parameter, and ξ is the net effective rapidity parameter of the primed-reference frame.

The rapidity parameters ξ_a, ξ_v, ξ are defined, respectively, in terms of the spatial velocity $v = dx/dt$, and proper force $F = ma$, as follows

$$tanh(\xi_v) = \frac{v}{c}; \quad tanh(\xi_a) = \frac{F}{F_{max}}, \quad F_{max} = b, \quad \xi = \sqrt{(\xi_v)^2 + (\xi_a)^2} \quad (1.16)$$

When $\xi_v \rightarrow \infty \Rightarrow v \rightarrow c$. And $\xi_a \rightarrow \infty \Rightarrow F \rightarrow F_{max} = b$.

It is straight-forward to verify that the transformations (1.15) leave invariant the phase space interval $c^2(dt)^2 - (dX)^2 + ((dE)^2 - c^2(dP)^2)/b^2$ but *do not* leave separately invariant the proper time interval $(d\tau)^2 = c^2 dt^2 - dX^2$, nor the interval in energy-momentum space $\frac{1}{b^2}[(dE)^2 - (dP)^2]$. Only the *combination*

$$(d\omega)^2 = (d\tau)^2 \left(1 - \frac{F^2}{F_{max}^2}\right) \quad (1.17)$$

is truly left invariant under force (acceleration) boosts. They also leave invariant the symplectic 2-form (phase space areas) $\Omega = -\, dt \wedge dE + dX \wedge dP$.

An interesting feature of the transformations in eqs-(1.15) is that they remain invariant under the following *discrete* transformations $X \rightarrow P; t \rightarrow E; P \rightarrow -X; E \rightarrow -t$, leaving invariant also the Heisenberg canonical commutation relations; combined with $b \rightarrow \frac{1}{b}$ that is reminiscent of the T-duality symmetry (large R versus small R) in string theory toroidal compactifications , and $c \rightarrow \frac{1}{c}$ which resembles the Galilean limit $c \rightarrow \infty$ and Carollian limit $c \rightarrow 0$ of the Lorentz group.

A physical picture of how velocity-boosts and force-boosts transformations operate on reference frames is best captured if one focus on the full phase-space configuration. Since $d\tau \neq d\omega$, one has $F^\mu = \frac{dp^\mu}{d\tau}; F = \sqrt{|F_\mu F^\mu|}$, and $\mathcal{F}^\mu = \frac{dp^\mu}{d\omega}; \mathcal{F} = \sqrt{|\mathcal{F}_\mu \mathcal{F}^\mu|}$, such that

$$\mathcal{F} = \frac{F}{\sqrt{1 - F^2/b^2}} = \frac{m\ a}{\sqrt{1 - F^2/b^2}} = m(F)\ a, \quad m(F) = \frac{m}{\sqrt{1 - F^2/b^2}} \quad (1.18a)$$

which is the BRRT version of the Special Relativistic relation

$$P = \frac{m\ v}{\sqrt{1 - v^2/c^2}} = m(v)\ v, \quad m(v) = \frac{m}{\sqrt{1 - v^2/c^2}} \quad (1.18b)$$

depicting the velocity-dependence of the mass.

From eq-(1.18) one learns that when $F = b \Rightarrow m(F = b) = \infty$, when the proper (rest) mass $m \neq 0$. Hence, a natural UV cutoff at $F = b$ appears. In case of a photon whose proper mass is zero, one may recall the (right) Rindler wedge corresponding to a family of accelerated observers describing hyperbolic trajectories in $D = 1 + 1$ spacetime given by $t = \frac{1}{a}\sinh(a\tau); x = \frac{1}{a}\cosh(a\tau)$ in $c = 1$ units. a is the proper acceleration and τ is the proper time. In the limit $a \rightarrow \infty$, the hyperbolas degenerate into the light-cone lines (Rindler horizon) corresponding to the null trajectories of a massless particle (like a photon). Consequently, one may choose to have the following double scaling limit

$$m \rightarrow 0, \quad a \rightarrow \infty, \quad m\, a \rightarrow F_0 \leq b \tag{1.19}$$

and which corresponds to reaching the null-lines of the Rindler horizon. One then learns that $m(F_0 < b) = 0$ and the effective mass remains zero. However, if $F_0 = b$, then $m(F_0) = \frac{m}{\sqrt{1-F_0^2/b^2}} = \frac{0}{0}$ is undetermined when $m = 0$ and $F_0 = b$. In this limiting case, when $ma \rightarrow F_0 = b$, the effective (photon) mass is no longer zero and resembles the introduction of an infrared cutoff. This may have far reaching consequences in the resolution of the dark matter problem. See [5] for further applications of BRRT in Cosmology. Conversely, one may look at the reciprocal case when $m \rightarrow \infty$, $a \rightarrow 0$ such that $ma \rightarrow F_0 \leq b$. Hence for very small accelerations and very large masses one can still recapture Born relativistic effects.

The Fulling-Davies-Unruh effect [7] states that a uniformly accelerating observer experiences the vacuum state of a quantum field in Minkowski spacetime as a mixed state in thermodynamic equilibrium. Such mixed state is comprised of a thermal bath (warm gas) of Rindler particles whose temperature is proportional to the acceleration. By invoking the expression of Unruh's temperature in terms of the acceleration $T = \frac{a}{2\pi}$ one finds that the Planck temperature $T_P = m_P$ corresponds to an acceleration $a_P = 2\pi m_P$ [2] so that the ratio $\frac{T}{T_P} = \frac{a}{a_P} = \frac{a}{2\pi m_P}$. Hence, if one sets the maximal proper force to be given by $F_{max} = b = 2\pi m_P^2$ (instead of m_P^2), then one arrives at

$$\frac{F}{b} = \frac{m\, a}{2\pi m_P^2} = \frac{m}{m_P}\, \frac{a}{2\pi m_P} = \frac{m}{m_P}\, \frac{a}{a_P} = \frac{m}{m_P}\, \frac{T}{T_P} \tag{1.20}$$

and in doing so, one may rewrite $m(F)$ in terms of the temperature $m(T)$ as follows

$$m(F) = \frac{m}{\sqrt{1 - F^2/b^2}} \Rightarrow m(T) = \frac{m}{\sqrt{1 - \frac{m^2}{m_P^2}\, T^2/T_P^2}} \tag{1.21}$$

and one arrives at an interesting relationship between mass and temperature. The reader might object invoking the Fulling-Davies-Unruh effect (which requires the use of Quantum Field Theory) in obtaining eq-(1.21) and which involves a classical theory. However one may notice that the $\hbar$ factors cancel out (decouple) in the ratio a/a_P. This can be verified by simply reinstating the dimensionful constants $\hbar = c = k_B = 1$ that were set to unity in the ratio

$$\frac{a}{a_P} = \frac{(\hbar a/2\pi k_B c)}{(\hbar a_P/2\pi k_B c)} = \frac{T}{T_P} \tag{1.22}$$

The Hawking radiation of black holes was derived based on treating the gravitational field as classical field while the matter was treated quantum mechanically. It

[2] Reinstating the units one has $2\pi m_P c^3/\hbar = a_P = 2\pi c^2/L_P$ furnishing a *huge* acceleration associated with the Planck temperature

is important to emphasize that one is *not* trying to derive the Fulling-Davies-Unruh effect from the BRRT. The right-hand side of eq-(1.21) is obtained solely after invoking the Unruh relation between proper acceleration and temperature and which is based on Quantum Field Theory (QFT). This is our only heuristic assumption. The temperature-dependence of the mass $m(T)$ bears some resemblance (analogy) to the energy-scale dependence of the mass and other physical parameters in the Renormalization program of QFT, like the difference between the bare and renormalized mass. The Weyl unitary trick allows to relate the $U(1,3)$ group with the $U(2,2) = SU(2,2) \times U(1)$ group, where $SU(2,2)$ is the conformal group in $D = 4$ involving dilations/scalings. In this algebraic aspect, one finds a common feature behind the above analogy. It would be desirable to explore this analogy further.

1.2 The Relativistic Oscillator and the Fulling-Davies-Unruh effect

We begin this section with the study of the Relativistic Oscillator in an arbitrary number of spatial dimensions d ($D = d+1$-dim spacetime), and then we focus in the very special case $d = 1$ ($D = 1 + 1$ dim spacetime) to analyze the Fulling-Davies-Unruh effect entirely from the perspective of BRRT, and explain how it may be interpreted in terms of a linear superposition of an infinite number of states resulting from the action of the group $U(1,1)$ on the Lorentz non-invariant vacuum $|\tilde{0}\rangle$ of the relativistic oscillator studied by Bars [1].

The author [1] has rigorously shown that the familiar Fock space commonly used to describe the relativistic harmonic oscillator, for example as part of string theory, is insufficient to describe all the states of the relativistic oscillator. He found that there are three different vacua leading to three disconnected Fock sectors, all constructed with the same creation-annihilation operators. These have different spacetime geometric properties as well as different algebraic symmetry properties or different quantum numbers. Two of these Fock spaces include negative norm ghosts (as in string theory) while the third one is completely free of ghosts. He discussed a gauge symmetry in a worldline theory approach that supplies appropriate constraints to remove all the ghosts from all Fock sectors of the single oscillator. The resulting ghost free quantum spectrum in $D = d + 1$ dimensions is then classified in unitary representations of the Lorentz group SO(d,1). Moreover all states of the single oscillator put together make up a single infinite dimensional unitary representation of a hidden global symmetry $SU(d,1)$, whose Casimir eigenvalues are computed. One of the purpose of this section is to exploit the $U(d,1)$ symmetry.

As it is customary, Bars [1] began his results by *absorbing* all the dimensionful parameters, as well as the frequency of the oscillator, by rescaling the phase space coordinates x^μ, p^μ, such that the relativistic oscillator eigenvalue equation (in units $\hbar = c = 1$) turns out to be

$$\frac{1}{2}\,(-\partial^\mu\partial_\mu \,+\, x_\mu x^\mu)\Psi_\lambda(x^\mu) \;=\; \lambda\,\Psi_\lambda(x^\mu) \tag{2.1}$$

One of the key findings in [1] was in understanding that the symmetry properties of the solutions of eq-(2.1) were based precisely on the $U(1,3)$ group ($U(3,1)$ group depending on the signature). As usual, eq-(2.1) can be recast as an operator equation $Q\Psi_\lambda = \lambda\Psi_\lambda$ in terms of Lorentz covariant oscillators

$$a_\mu \;=\; \frac{1}{\sqrt{2}}\,(x_\mu + ip_\mu),\;\; \bar{a}_\mu \;=\; \frac{1}{\sqrt{2}}\,(x_\mu - ip_\mu) \tag{2.2}$$

where the operator Q is

$$Q \;=\; \frac{1}{2}\,(p_\mu p^\mu \,+\, x_\mu x^\mu) \;=\; \eta^{\mu\nu}\,\bar{a}_\mu\,a_\nu \,+\, \frac{d+1}{2} \tag{2.3}$$

The hidden symmetry is $U(d,1)$ with generators : $\bar{a}_\mu a_\nu$, and all of these $(d+1)^2$ generators commute with Q.

The author [1] remarked that in a unitary Hilbert space the operators x_μ, p_μ are Hermitian; in that case $\bar{a}_\mu$ is the Hermitian conjugate of a_μ i.e. $\bar{a}_\mu = a^\dagger_\mu$. A unitary Hilbert space without ghosts (negative norm states) is possible only if and only if x_μ, p_μ are hermitian or equivalently if $\bar{a}_\mu = a^\dagger_\mu$. The canonical commutation relations are

$$[x_\mu,\; p_\nu] \;=\; i\eta_{\mu\nu},\;\; [a_\mu,\; \bar{a}_\nu] \;=\; \eta_{\mu\nu} \;=\; diag\;(-1,1,1,1) \tag{2.4}$$

Lorentz covariant solutions based on a vacuum state $\Psi_{vac} \sim e^{-\frac{1}{2}(x^\mu x_\mu)}$ (that is a Lorentz invariant Gaussian) have a number of problems, including issues of infinite norm and negative norm states. Bars [1] explored the spacelike $x_\mu x^\mu > 0$, and the timelike $x_\mu x^\mu < 0$ cases, and assigned the vacuum states $|0\rangle \leftrightarrow e^{-\frac{1}{2}(x^\mu x_\mu)}$; $|0'\rangle \leftrightarrow e^{\frac{1}{2}(x^\mu x_\mu)}$, respectively, to the spacelike and timelike cases.

The relevant vacuum state we shall explore in this work is the unitary Fock space based on the non-symmetric vacuum (Lorentz non-invariant vacuum) in $D = d+1$ spacetime, $\Psi_{vac} \sim e^{-\frac{1}{2}(x^i x_i + t^2)}, i = 1,2,\ldots,d$. Note that $x^\mu x_\mu \neq x^i x_i + t^2$. This vacuum state was labeled as $|\tilde{0}\rangle$ by [1].

The solutions we are interested in are found by separating the spatial variables from the temporal one as follows

$$\frac{1}{2}\,[\;(-\partial^i\partial_i \,+\, x_i x^i) \,-\, (-\partial_t^2 \,+\, t^2)\;]\;\Psi_\lambda(x^i,t) \;=\; \lambda\,\Psi_\lambda(x^i,t),\;\; i = 1,2,\ldots,d \tag{2.5}$$

A factorization

$$\Psi_\lambda(x^i,t) \;=\; \Psi_{\lambda_x}(x^i)\;\Psi_{\lambda_t}(t),\;\; \lambda \;=\; \lambda_x \,-\, \lambda_t \tag{2.6}$$

yields the equations for the Euclidean harmonic oscillator in d dimensions and 1 dimension, respectively

$$\frac{1}{2}\,(-\partial^i\partial_i + x_i x^i)\,\Psi_{\lambda_x}(x^i) \;=\; \lambda_x\,\Psi_{\lambda_x}(x^i),\;\; \frac{1}{2}\,(-\partial_t^2 + t^2)\,\Psi_{\lambda_t}(t) \;=\; \lambda_t\,\Psi_{\lambda_t}(t) \quad (2.7)$$

and whose solutions are well known. The possible eigenvalues for Euclidean harmonic oscillator in d spatial dimensions and 1 temporal dimension are respectively

$$\lambda_x \;=\; n_1 + n_2 + n_3 \;+\; \ldots \;+ n_d \;+\; \frac{d}{2},\;\; \lambda_t \;=\; n_0 \;+\; \frac{1}{2},\;\; \Rightarrow$$

$$\lambda \;=\; \lambda_x \;-\; \lambda_t \;=\; n_1 + n_2 + n_3 \;+ \ldots \;+ n_d \;-\; n_0 \;+\; \frac{d-1}{2} \;=\; N \;+\; \frac{d-1}{2} \quad (2.8)$$

For any given eigenvalue $\lambda = N + \frac{d-1}{2}$, with $N = 0, \pm 1, \pm 2, \pm 3, \ldots$ there is an infinite degeneracy of values $n_0, n_1, n_2, n_3, \ldots, n_d$

All solutions have the form $\Psi_\lambda(x^\mu) \sim e^{-\frac{1}{2}(t^2 + x_i x^i)} \times$ Hermite polynomials in the variables x_i, t. The wavefunction of an arbitrary excited state of the d-dimensional Euclidean (isotropic) harmonic oscillator with eigenvalue $n + \frac{d}{2}$ and $SO(d)$ orbital angular momentum quantum number l, has the form [1]

$$\Psi^{nl}_{i_1 i_2 \ldots i_l}(\mathbf{x}) \;=\; e^{-\mathbf{x}^2/2}\;|\mathbf{x}|^l\; L_n^{l-1+d/2}\;(\mathbf{x}^2)\; T_{i_1 i_2 \ldots i_l}(\frac{x_i}{|\mathbf{x}|}) \quad (2.9)$$

where $T_{i_1 i_2 \ldots i_l}(\frac{x_i}{|\mathbf{x}|})$ is the symmetric traceless tensor of rank l constructed from the unit vector $\hat{x}_i = \frac{x_i}{|\mathbf{x}|}$ and which can also be recast in terms of the hyper-spherical harmonic functions based on the $d-1$ angles associated with the hyper-sphere S^{d-1}. The function $L_n^{l-1+d/2}\;(\mathbf{x}^2)$ is a generalized Laguerre polynomial. The excitation level n is any positive integer $n = 0, 1, 2, 3, \ldots$ while at fixed n the allowed values of l are $l = n, (n-2), (n-4), \ldots\ldots$ (1 or 0) .

The solutions associated with the Lorentz symmetric invariant vacuum $|0\rangle \leftrightarrow e^{-\frac{1}{2}x_\mu x^\mu}$ are of the form

$$\Psi_k \;\sim\; e^{-\frac{1}{2}x^2}\; L_k^{\frac{d-1}{2}}(x^2),\;\; x^2 = x_\mu x^\mu > 0 \quad (2.10)$$

and whose eigenvalue λ is

$$\lambda \;=\; n_1 + n_2 + n_3 + \ldots + n_d \;-\; n_0 \;+\; \frac{d+1}{2} \;=\; 2k \;+\; \frac{d+1}{2} \quad (2.11)$$

The solutions for the other Lorentz invariant vacuum $|0'\rangle$ are obtained by replacing $x^2 = x_\mu x^\mu \rightarrow -x^2 = -x_\mu x^\mu$ and $\lambda \rightarrow -\lambda$ [1].

The $U(1,1)$ algebra generators associated to a 4-dim phase corresponding to a 1 + 1-dim spacetime, can be realized in terms of the creation and annihilation operators as follows

$$J_{\mu\nu} \;=\; \bar{a}_\mu\, a_\nu,\;\; \mu, \nu = 0, 1 \quad (2.12)$$

However there is a *subtlety* in assigning the creation and annihilation operator for the temporal components : $a_0 = x_0 - ip_0 = -x^0 - ip_0 = -x^0 + \frac{\partial}{\partial x^0}$ is an annihilation operator for the Lorentz invariant symmetric vacuum $e^{-\frac{1}{2}[-(x^0)^2 + (x^i)^2]}$, but it is a creation operator for the Lorentz non-invariant vacuum $e^{-\frac{1}{2}[(x^0)^2 + (x^i)^2]}$.

Consequently, the double creation and double annihilation operators for the Lorentz non-invariant vacuum in $D = 1+1$ are respectively given by [1]

$$J_{10} = \bar{a}_1 a_0, \; J_{01} = \bar{a}_0 a_1 \tag{2.13}$$

and such that $J_{01} \neq J_{10}$.

In a $D = d+1 = 1+1$ spacetime one has $\lambda = n_1 - n_0 + \frac{d-1}{2} = n_1 - n_0$. In the particular case when $n_1 = n_0$ one has $\lambda = 0$ and the infinite tower of states originating from $|\tilde{0}\rangle_{\lambda=0}$ is obtained by successive applications of the double creation operator $J_{10} = \bar{a}_1 a_0$ as follows

$$(Tower)_{\lambda=0} = \bigoplus_{k=0}^{\infty} (\bar{a}_1^k \; a_0^k) \; |\tilde{0}\rangle_{\lambda=0} \tag{2.14}$$

There are an infinite number of towers, parametrized by the eigenvalue of $\lambda = n_1 - n_0$, and with an infinite amount of degeneracy (an infinite number of states within each tower). This result is consistent with the fact that unitary irreducible representations of non-compact groups are infinite-dimensional.

Defining the state in the Fock space with $n_0 = n_1 = k$ as

$$|k, \; k\rangle \equiv |n_1 = k, \; n_0 = k\rangle = \frac{\bar{a}_1^k}{\sqrt{k!}} \frac{a_0^k}{\sqrt{k!}} \; |\tilde{0}\rangle_{\lambda=0} \tag{2.15}$$

one has the following infinite superposition of states belonging to the infinite tower

$$|\Psi\rangle_{\lambda=0} = e^{\theta_{10} J_{10}} \; |\tilde{0}\rangle_{\lambda=0} = e^{\theta_{10}\bar{a}_1 a_0} \; |\tilde{0}\rangle_{\lambda=0} =$$
$$\sum_{k=0}^{\infty} \theta_{10}^k \; \frac{\bar{a}_1^k}{\sqrt{k!}} \; \frac{a_0^k}{\sqrt{k!}} \; |\tilde{0}\rangle_{\lambda=0} = \sum_{k=0}^{\infty} \theta_{10}^k \; |k, k\rangle \tag{2.16}$$

and which bears some analogy to the construction of coherent states by applying the displacement operator to the ground state

$$|z\rangle = D(z)|0\rangle = e^{za^\dagger - \bar{z}a}|0\rangle = e^{-|z|^2/2} \sum_{n=0}^{\infty} \frac{z^n \; (a^\dagger)^n}{n!}|0\rangle \tag{2.17}$$

with $z = x + ip$ complex; $|n\rangle = \frac{(a^\dagger)^n}{\sqrt{n!}}|0\rangle$, and $\hat{a}|z\rangle = z|z\rangle$. Consequently, one could have rewritten in the above eq-(2.16) : $|\Psi\rangle_{\lambda=0} = |\theta_{10}\rangle$. Multimode minimal uncertainty squeezed states based on the Quaplectic group, given by the semi-direct product of $U(1,3)$ with the Weyl-Heisenberg group $H(1,3)$, were constructed by [8] from the action of the exponential of the Quaplectic algebra generators on the ground state, in the same vein as displayed in eq-(2.16), and involving the judicious group parameters multiplying the generators.

The symmetric part $M_{01} = \frac{1}{2}(J_{01} + J_{10})$ is the generator of force-boosts transformations along the x_1 direction, and the antisymmetric part $L_{10} = \frac{1}{2}(J_{01} - J_{10})$ is the generator of Lorentz boost along x_1. When θ_{10} is chosen to be complex-valued (as it occurs in the construction of coherent states $|z\rangle$) then $\theta_{10}J_{10} = \theta_{(10)}M_{10} + i\theta_{[10]}L_{10}$, where $\theta_{(10)} = \xi_a$ is the force-boost rapidity parameter, and $\theta_{[10]} = \xi_v$ is the velocity-boost rapidity parameter. This decomposition results from breaking the complex-valued $\theta_{10} \equiv \theta_{(10)} + i\theta_{[10]}$ into a real-symmetric and imaginary-antisymmetric piece in the indices. [3]

The relativistic oscillator can be studied from two different frames of reference (two different observers) given by a fixed frame of reference, and another frame with a linear uniform acceleration with respect to the first one. The result found in eq-(1.20) given by

$$tanh(\xi_a = \theta_{(10)}) \ = \ \frac{F}{b} \ = \ \frac{m}{m_p} \frac{T}{T_P} \ \Rightarrow \ T \ = \ T_P \ \frac{m_P}{m} \ tanh(\xi_a = \theta_{(10)}) \quad (2.18)$$

and taking into account that one has *absorbed* all the dimensionful parameters, as well as the frequency of the oscillator, by rescaling the phase space coordinates x^μ, p^μ, as follows

$$p_\mu \ p^\mu \leftrightarrow \frac{p_\mu \ p^\mu}{m}, \ \ x_\mu \ x^\mu \leftrightarrow (m\omega^2) x_\mu \ x^\mu, \ \ \lambda \leftrightarrow \lambda \ \omega \quad (2.19)$$

it allows us to identify the mass m appearing in eq-(2.18) with the relativistic oscillator proper mass appearing in eq-(2.19)[4].

One should note once more that the action of $e^{\theta_{10}J_{10}}$ (when θ_{10} are complex-valued parameters) on the vacuum $|\tilde{0}\rangle$ is tantamount of a *combined* velocity and force-boost transformation on the vacuum $|\tilde{0}\rangle_{\lambda=0}$ and generating a superposition of an *infinite* number of states $|n_1 = k, n_0 = k\rangle$ in a Fock space, with complex-valued coefficients. This picture can be contrasted with a warm gas (thermal bath) of Rindler scalar particles of mass m at an equilibrium temperature of $T = T_P(\frac{m_P}{m}) tanh(\xi_a = \xi)$ and experienced by a uniformly accelerated observer in Minkowski space whose acceleration is $a = 2\pi T$. A Planck relativistic oscillator of mass $m = m_P$ yields a temperature of $T = T_P \ tanh(\xi_a = \xi) \leq T_P$. When the force-boost rapidity parameter is $\xi_a = \xi = \infty$, one ends up with $T = T_P$ making contact with the Thermal Relativity proposal in [9] where the Planck temperature is postulated as the maximal temperature.

More recently, Popov [10] has studied the relativistic oscillator from a Geometric Quantization point of view. He showed that turning on the interaction of relativistic spinless particles with the vacuum of relativistic quantum mechanics leads to the replacement of the Klein-Gordon equation with the Klein-Gordon oscillator equation. In this case, coordinate time becomes an operator and free relativistic particles go into a virtual state. He also discussed the geometry associated with classical and quantum Klein-Gordon oscillators, and its relation to the geometry underlying the description of free particles.

---

[3] Velocity boosts with imaginary parameters are equivalent to ordinary rotations, and vice versa, rotations with imaginary angles are equivalent to velocity boosts

[4] One may note that $m \neq \lambda\omega$

1.3 Concluding Remarks

A continuation of the Born Reciprocal Relativity Theory (BRRT) program corresponding to a 4-dim phase space, and associated with a $D = 1 + 1$ dim spacetime, showed that a natural temperature-dependence of mass occurs after recurring to the Fulling-Davies-Unruh effect [7]. The temperature dependence of the mass $m(T)$ resemblances the energy-scale dependence of mass and other physical parameters in the renormalization (group) program of QFT. It was found in a special case that the effective photon mass is no longer zero, which may have far reaching consequences in the resolution of the dark matter problem.

The Fulling-Davies-Unruh effect in a $D = 1+1$-dim spacetime was then analyzed entirely from the perspective of BRRT, and we explained how it may be interpreted in terms of a linear superposition of an infinite number of states resulting from the action of the group $U(1,1)$ on the Lorentz non-invariant vacuum $|\tilde{0}\rangle$ of the relativistic oscillator studied by [1].

It is worth mentioning that a granularity of the $4D$ spacetime within the context of Born reciprocity and the Schrödinger-Robertson inequality for relativistic position and momentum operators $X^\mu, P^\nu, \mu, \nu = 0, 1, 2, 3$ has been proposed by [8] and leading to the more generalized uncertainty relation

$$\Delta X_0 \; \Delta X_1 \; \Delta X_2 \; \Delta X_3 \; \Delta P_0 \; \Delta P_1 \; \Delta P_2 \; \Delta P_3 \;\geq\; (\frac{\hbar}{2})^4 \tag{3.1}$$

The authors [8], [11] studied the constraint quantization of a worldline system invariant under Born Reciprocal Relativity. The reciprocal transformations are *not* spin-conserving in general. The physical state space is vastly enriched as compared with the covariant approach, and contains towers of integer spin massive states (like the string theory spectrum), as well as unconventional massless representations, with continuous Euclidean momentum and arbitrary integer helicity.

The Dirac-Born oscillator as the "square-root" of the relativistic oscillator in $D = 3 + 1$ for a spinless particle was studied by [12]. For some other aspects based on maximal proper acceleration rather than maximal proper force see the classic work by [6], and more recently in [13], where the effect of a hypothetical maximal proper acceleration on the inertial mass of charged particles is investigated in the context of particle accelerators, and also on the effect of a maximal Unruh temperature in hyperbolic paths and of the maximal Hawking temperature in a black hole evaporation process. A similar equation to (1.21) has been derived in a different context. The role of quantum groups, non-commutative Lorentzian spacetimes and curved momentum spaces see [14].

The $8D$ curved phase-space within the context of Finsler Geometry was studied in [15] and allowed us to find novel avenues to tackle the cosmological constant problem [15]. A study of Born deformed reciprocal complex gravitational theory and noncommutative gravity can be found in [16]. The role of maximal acceleration in strings with dynamical Tension and Rindler worldsheets was analyzed in [17].

To finalize, the study of observers moving in $uniform$ circular motion (besides linear uniform acceleration) deserves careful consideration within the realm of this work. The problem is more subtle since it requires instantaneous reference frames where the velocity-boost and force-boosts *directions* change at every instant.

Acknowledgements We are indebted to M. Bowers for assistance and to Prof. Eckhard Hitzer for his very kind invitation to participate in this conference. Special thanks to the reviewers for the many insightful suggestions to improve this work.

APPENDIX : Spacelike, Timelike and Null intervals in Phase space

In this Appendix we shall compare the $U(1,3)$-invariant spacelike, timelike and null intervals in phase space with the $SO(1,3)$-invariant intervals in spacetime.[5] An inspection of eqs-(1.15) in the text reveals that pure force/acceleration boosts involve setting the velocity rapidity parameter to zero $\xi_v = 0$, such that $\xi = \xi_a$, and leading to $(c = 1)$

$$(dt')^2 \; - \; (dX')^2 \; = \; [(dt)^2 - (dX)^2] \; \cosh^2 \xi \; + \; \frac{1}{b^2} [(dP)^2 - (dE)^2] \; \sinh^2 \xi \; + \; \frac{1}{b} \; (dtdP + dXdE) \; \sinh(2\xi) \qquad (A1)$$

$$\frac{1}{b^2} [(dE')^2 \; - \; (dP')^2] \; = \; \frac{1}{b^2} [(dE)^2 - (dP)^2] \; \cosh^2 \xi \; + \; [(dX)^2 - (dt)^2] \; \sinh^2 \xi \; - \; \frac{1}{b} \; (dtdP + dXdE) \; \sinh(2\xi) \qquad (A2)$$

where $(d\tau)^2 \equiv (dt)^2 - (dX)^2$, and $(d\mu)^2 \equiv (dE)^2 - (dP)^2$, are the spacetime and energy-momentum infinitesimal displacement intervals, respectively. As expected, eqs-(A1,A2) furnish the $U(1,1)$ quadratic invariant in phase space

$$(dt')^2 - (dX')^2 + \frac{1}{b^2} \; [(dE')^2 \; - \; (dP')^2] \; = \; (dt)^2 \; - \; (dX)^2 + \frac{1}{b^2} \; [(dE)^2 \; - \; (dP)^2] \qquad (A3)$$

resulting from the identity $\cosh^2(\xi) - \sinh^2(\xi) = 1$.

A timelike interval in spacetime $(d\tau)^2 = (dt)^2 - (dX)^2 > 0$ is associated to a subluminal particle moving at speeds less than light. It is known that a non-inertial observer (in an accelerated frame of reference) assigns a pseudo-force acting on the particle. The centrifugal force is an example of a pseudo-force pointing in the opposite direction to the centripetal force. Hence a free particle from the point of view of a non-inertial observer will experience a pseudo-force. One could then envision that when the force/acceleration boost rapidity parameter tends to infinity $\xi \rightarrow \infty$ the particle's velocity relative to the accelerated frame of reference may reach the speed of light, and even surpass it. Namely, there could be a transition from a subluminal $(d\tau)^2 > 0$ to a superluminal regime $(d\tau')^2 < 0$. When $\xi \rightarrow \infty$ one has that $\cosh^2(\xi) \simeq \sinh^2(\xi)$, and $\sinh(2\xi) = 2\sinh(\xi)\cosh(\xi) \simeq 2\cosh^2(\xi)$, and eq-(A1) becomes

$$(d\tau')^2 \; \simeq \; \left((d\tau)^2 \; - \; \frac{1}{b^2}(d\mu)^2 \; + \; \frac{2}{b} \; (dtdP + dXdE) \right) \cosh^2 \xi \qquad (A4)$$

---

[5] We are indebted to one referee for stressing the key importance of performing the rigorous analysis described in this Appendix

At first glance, if one wishes to exclude the possibility that there is a crossover from the subluminal $(d\tau)^2 > 0$ to superluminal regime $(d\tau')^2 < 0$, and to a null regime $(d\tau')^2 = 0$, then one must have that $b >> 1$ in Planck units such that the leading term in eq-(A4) becomes

$$(d\tau')^2 \simeq \left((d\tau)^2 \ + \ O(\frac{1}{b}) \right) \cosh^2 \xi \ > \ 0, \ \ with \ \ (d\tau)^2 > 0, \ \ (d\mu)^2 < 0 \qquad (A5)$$

However, a more rigorous study reveals that one should factor out the $(d\tau)^2$ in eq-(A4) leading to

$$(d\tau')^2 \ \simeq \ (d\tau)^2 \ \left(1 \ + \ \frac{F^2}{b^2} \ + \ \frac{2}{b} \ (\frac{dt}{d\tau}\frac{dP}{d\tau} + \frac{dX}{d\tau}\frac{dE}{d\tau}) \right) \cosh^2 \xi \qquad (A5)$$

Eq-(A5) results after invoking the relations : when $(d\tau)^2 > 0 \Rightarrow (d\mu)^2 < 0$; and when $(d\tau)^2 < 0 \Rightarrow (d\mu)^2 > 0$ such that $1 - \frac{1}{b^2}\frac{(d\mu)^2}{(d\tau)^2} = 1 + \frac{F^2}{b^2}$. The first two terms inside the parenthesis in eq-(A5) are positive. This leaves the analysis of the last term inside the parenthesis. Let us evaluate this last term in the case of hyperbolic (Rindler) trajectories associated with a particle moving with a uniform proper acceleration g and proper force $F = mg$. The equations of motion in $c = 1$ units lead to

$$t \ = \ \frac{1}{g}\sinh(g\tau); \ \ X \ = \ \frac{1}{g}\cosh(g\tau); \ \ P \ = \ \gamma m \frac{dx}{dt} \ = \ m \ \cosh(g\tau) \ tanh(g\tau) \ =$$

$$m \ \sinh(g\tau); \qquad (A6a)$$

$$E \ = \ m\gamma \ = \ m \ \cosh(g\tau); \ \ \frac{dt}{d\tau} \ = \ \cosh(g\tau); \ \ \frac{dX}{d\tau} \ = \ \sinh(g\tau);$$

$$\frac{dP}{d\tau} \ = \ mg \ \cosh(g\tau); \ \ \frac{dE}{d\tau} \ = \ mg \ \sinh(g\tau) \qquad (A6b)$$

γ above is the Lorentz dilation factor $(1 - v^2)^{-1/2} = \cosh(g\tau)$. Hence, the last term inside the parenthesis in eq-(A5) turns out to be *positive* for all values of τ,

$$\frac{2}{b} \ (\frac{dt}{d\tau}\frac{dP}{d\tau} + \frac{dX}{d\tau}\frac{dE}{d\tau}) \ = \ \frac{2mg}{b} \ [\cosh^2(g\tau) \ + \ \sinh^2(g\tau)] \ > 0 \qquad (A7)$$

Therefore, all the terms inside the parenthesis in eq-(A5) are positive, so that if $(d\tau)^2 > 0 \Rightarrow (d\tau')^2 > 0$; and if $(d\tau)^2 < 0 \Rightarrow (d\tau')^2 < 0$, consequently there is no crossover in the spacetime intervals. Furthermore, due to the invariance $(d\omega)^2 = (d\tau)^2 + \frac{1}{b^2}(d\mu)^2$ one also has that if $(d\mu)^2 < 0 \Rightarrow (d\mu')^2 < 0$, and if $(d\mu)^2 > 0 \Rightarrow (d\mu')^2 > 0$.

The pending question is what happens when $\xi \rightarrow -\infty$? In that case there is a crucial sign change due $\sinh(\xi) < 0$ when $\xi < 0$, and the terms inside the parenthesis become

$$\left(1 + \frac{F^2}{b^2} - \frac{2mg}{b} [\cosh^2(g\tau) + \sinh^2(g\tau)] \right), \quad F = mg \qquad (A8a)$$

Due to the minus sign of the third term in eq-(A8a), there will be a point in proper time τ when the parenthesis flips sign from positive to negative, and there will be a crossover in the spacetime intervals. In the particular instance when the proper force reaches its maximum value $F = mg = b$, and when $\tau = 0$, eq-(A8) turns out to be $1 + 1 - 2 = 0$, and such $(d\tau')^2 = 0$, and the crossover occurs at $\tau = 0$.

What went wrong ?? We have to go back to eq-(1.16), with $\xi_v = 0$, $\xi_a = \xi$ and $tanh(\xi_a) = tanh(\xi) = \frac{F}{b}$. When $\xi = -\infty \Rightarrow tanh(-\infty) = -1 = \frac{F}{b} \Rightarrow F = -b$. And one learns that one has to choose a minus sign $F = -mg$ after replacing $g \to -g$. This is consistent with taking the negative sign under the square root $-g = -\sqrt{|(d^2t/d\tau^2)^2 - (d^2X/d\tau^2)^2|}$. Therefore, one must replace g for $-g$ in eq-(A8a), and in doing so, one arrives correctly at

$$\left(1 + \frac{F^2}{b^2} + \frac{2mg}{b} [\cosh^2(g\tau) + \sinh^2(g\tau)] \right) > 0 \qquad (A8b)$$

and there will not be a crossover of the spacetime intervals when $\xi \to -\infty$ for all values of τ.

Reversing the sign of $F = mg$ has also an analogy in special relativity. The relation between the velocity boost parameter ξ_v and the velocity is $tanh(\xi_v) = \frac{v}{c}$. When the observer moves in the positive, negative x-direction, one has $v > 0, v < 0$ and $\xi_v >, \xi_v < 0$, respectively. Furthermore, one can also rewrite the terms in eq-(2.1) $(dtdP + dXdE) \sinh(2\xi)$ as $[(dt)^2(dP/dt) + (dX)^2(dE/dX)] \sinh(2\xi)$. Since $(dP/dt), (dE/dX)$ correspond to a force f, to compensate a reversal in the sign of ξ one must reverse the signs of f.

It is important to remark that a free particle will not experience a crossover. This can be verified by rewriting the third term inside the parenthesis of eq-(A5), after some straightforward algebra, as

$$\frac{2}{b} \gamma \frac{dP}{d\tau} (1 + v^2) \to 0 \qquad (A9)$$

This is due to $\frac{dP}{d\tau} = 0$ for a free particle. It stays at rest or it moves with uniform velocity. In the most general case there is no crossover in the spacetime intervals under acceleration boosts, in the asymptotic limit $\xi \to \infty$, if the following condition is satisfied for all values of proper time τ during the motion of a particle

$$1 + \frac{F^2(\tau)}{b^2} + \frac{2}{b} \gamma(\tau) \frac{dP(\tau)}{d\tau} (1 + v^2(\tau)) \geq 0, \quad f(\tau) \equiv \frac{dP(\tau)}{d\tau} \qquad (A10)$$

Eq-(A10) restricts the dynamics of the particle, namely one is looking for trajectories with $(dP(\tau)/d\tau) \geq 0$. We have studied above two examples where eq-(A10) is obeyed. Naturally, setting $b \to \infty$ yields $(d\omega)^2 \simeq (d\tau)^2$ and the invariance $U(1,3)$

group effectively "contracts" to the $SO(1,3)$ group and BRRT "reduces" to special relativity and no crossover will occur.

Finally, if one wants to preserve the null like conditions $(d\tau)^2 = 0, (d\mu)^2 = 0$ in eqs-(A1,A2) one must have $dtdP + dXdE = 0$[6] which is only satisfied in two cases out of $four$ branches resulting from the relations $dt = \pm dX; dP = \pm dE$, and which in turn, are a consequence of the null like conditions $(dt)^2 - (dX)^2 = 0; (dE)^2 - (dP)^2 = 0$. One finds that there are two cases where $dtdP + dXdE \neq 0$, namely when $dt = dX, dP = dE$, and $dt = -dX, dP = -dE$. And two cases where $dtdP + dXdE = 0$, namely when $dt = dX, dP = -dE$, and $dt = -dX, dP = dE$. The former two branches do *not* lead to $(d\tau')^2 = 0$ in eq-(A1), while the latter two branches do lead to $(d\tau')^2 = 0$ in eq-(A1). Consequently, if one wishes, one may discard those two branches which do not retain the null conditions. However this is not necessary because the condition $(d\omega)^2 = (d\tau)^2 + \frac{1}{b^2}(d\mu)^2 = 0$ is still valid : $(d\tau')^2 + \frac{1}{b^2}(d\mu')^2 = 0$ despite that each individual piece $(d\tau')^2$, $\frac{1}{b^2}(d\mu')^2$ may cease to be null. If one is positive, the other is negative, and vice versa, they cancel each other. See cases **3b, 3c** below. There is also the trivial solution $dP = dE = 0$ which automatically retains the null like conditions.

Let us proceed now with the study of spacelike, timelike and null intervals in phase space taking into account the above findings in eqs-(A1-A10) under force/acceleration boosts transformations. It would be interesting to find, if possible, if there is a particular $subgroup$ of $U(1,1)$ involving both velocity and force/acceleration boosts preserving $(d\tau)^2 > 0$, for example. An analogous situation occurs with the Lorentz group which is not compact, nor connected. The subgroup of all Lorentz transformations in four dimensions preserving both orientation and direction of time is called the proper, orthochronous Lorentz group or restricted Lorentz group, and is denoted by $SO^+(1,3)$.

Given the $U(1,3)$ invariant interval in phase space $(d\omega)^2 = (d\tau)^2 + \frac{1}{b^2}(d\mu)^2$, when $(d\tau)^2 \neq 0$, it allows the $factorization$ $(d\omega)^2 = (d\tau)^2[1 + \frac{1}{b^2}\frac{(d\mu)^2}{(d\tau)^2}]$. Before proceeding it is very important to emphasize once again that there are *no* factors of $(1 + F^2/b^2)$ appearing in the factorization process because in the superluminal case one has $m^2a^2 < 0$, despite that $a^2 > 0$, because $m^2 < 0$ due to the imaginary mass of tachyons. Hence we always have the factor $(1 - F^2/b^2)$ as expected. This is a consequence of the fact that if $(d\tau)^2 > 0$, then $(d\mu)^2 < 0$, and vice versa, if $(d\tau)^2 < 0$, then $(d\mu)^2 > 0$.

The above factorization leads to the following 2 cases to explore :

Case **1** : The timelike interval $(d\omega)^2 > 0$ (in phase space) leads to the following two sub-cases

$$\mathbf{1a} \ : \ (d\tau)^2 > 0, \ \ 1 - \frac{F^2}{b^2} > 0 \tag{A11}$$

and

$$\mathbf{1b} \ : \ (d\tau)^2 < 0, \ \ 1 - \frac{F^2}{b^2} < 0 \tag{A12}$$

[6] The combination $dtdP + dXdE$ is Lorentz invariant

The case **1b** must be disregarded because it implies that F is $larger$ than b violating the maximal force postulate, in addition to having a superluminal particle (tachyon). Therefore, this leaves the case **1a** where the timelike interval $(d\omega)^2 > 0$ has also a correspondence with the special relativistic timelike interval $(d\tau)^2 > 0$ (subluminal velocities) and with the maximal force condition $F^2 < b^2$. As we have shown above, one can assure that force/acceleration boosts transformations will not lead to a crossover from case **1a** to the unphysical case **1b** in the case of hyperbolic trajectories; for free particles and when $f(\tau) = (dP(\tau)/d\tau) \geq 0$.

Case **2** : The spacelike interval $(d\omega)^2 < 0$ (in phase space) leads to the following two sub-cases

$$\mathbf{2a} \ : \ (d\tau)^2 < 0, \ \ 1 - \frac{F^2}{b^2} > 0 \qquad (A13)$$

and

$$\mathbf{2b} \ : \ (d\tau)^2 > 0, \ \ 1 - \frac{F^2}{b^2} < 0 \qquad (A14)$$

The case **2a** involves the (spacetime) spacelike interval $(d\tau)^2 < 0$ corresponding to superluminal velocities, and to $F^2 < b^2$ obeying the maximal force postulate. Whereas, one finds in case **2b** that despite that $(d\tau)^2 > 0$ involving subluminal speeds, F^2 is $larger$ than b^2 leading to a $violation$ of the maximal force postulate.

Case **3a** : The null case $(d\omega)^2 = (d\tau)^2 + \frac{1}{b^2}(d\mu)^2 = 0$ with $(d\tau)^2 = 0$, and $(d\mu)^2 = 0$ corresponds to the null lines of a massless particle.

Case **3b** : The null case $(d\omega)^2 = 0$ with $(d\tau)^2 > 0 \Rightarrow (d\omega)^2 = (d\tau)^2(1 - \frac{F^2}{b^2}) = 0 \Rightarrow F = b$ involves a subluminal particle experiencing the maximal proper force $F = F_{max} = b$.

Case **3c** : The null case $(d\omega)^2 = 0$ with $(d\tau)^2 < 0 \Rightarrow (d\omega)^2 = (d\tau)^2(1 - \frac{F^2}{b^2}) = 0 \Rightarrow F^2 = b^2$, involves a superluminal particle (tachyon) experiencing the maximal proper force.

Concluding, out of all these cases, only three cases **1a**, **3a** , **3b** are physically viable under force (acceleration) boosts and also trivially so under Lorentz transformations. So far we have studied the $flat$ Born geometry. A more rigorous treatment involves the curved geometry of the phase space (cotangent space) which requires the tools of Finsler geometry. The Born interval in an $8D$ curved phase space (cotangent space) is given by

$$(d\omega)^2 \ = \ g_{\mu\nu}(x,p)\, dx^\mu\, dx^\nu + h_{ab}(x,p)\, (dp^a + A^a_\mu(x,p)\, dx^\mu)\, (dp^b + A^b_\nu(x,p)\, dx^\nu) \qquad (A15)$$

$g_{\mu\nu}(x,p)$ is the horizontal base spacetime metric; $\mu,\nu = 0,1,2,3$. $h_{ab}(x,p)$ is the vertical space (fiber) metric; $a,b = 0,1,2,3$. $A^a_\mu(x,p)$ is the *nonlinear* connection. The flat space limit occurs when $g_{\mu\nu} = \eta_{\mu\nu}$; $h_{ab} = \frac{1}{b^2}\eta_{ab}$; $A^a_\mu = 0$. See [15] and references therein.

Competing Interests
The author has no conflicts of interest to declare that are relevant to the content of this chapter.

References

1. I. Bars, "Relativistic Harmonic Oscillator Revisited" arXiv : 0810.2075.
2. M. Born, "Elementary Particles and the Principle of Reciprocity". Nature **163** (1949), p. 207
3. S.Low, " Canonically relativistic quantum mechanics: Casimir field equations of the Quaplectic group", arXiv : math-ph/0404077
4. S. Low, " $U(3, 1)$ Transformations with Invariant Symplectic and Orthogonal Metrics" , *Il Nuovo Cimento* **B 108** (1993) 841.
   S. Low, "Representations of the canonical group, (the semi-direct product of the unitary and Weyl-Heisenberg groups, acting as a dynamical group on noncommutative extended phase space", J. Phys. **A** : Math. Gen., **35** (2002) 5711.
5. C. Castro, "Is Dark Matter and Black-Hole Cosmology an Effect of Born Reciprocal Relativity Theory ?" Canadian Journal of Physics. Published on the web 29 May 2018, https://doi.org/10.1139/cjp-2018-0097.
   C. Castro, "Some consequences of Born Reciprocal Relativity in Phase Spaces" *Foundations of Physics* **35**, no.6 (2005) 971.
6. E.R. Cainiello, "Is there a Maximal Acceleration", Il. Nuovo Cim., **32** (1981) 65.
   H. Brandt, "Finslerian Fields in the Spacetime Tangent Bundle" *Chaos, Solitons and Fractals* **10** (2-3) (1999) 267.
   H. Brandt, "Maximal proper acceleration relative to the vacuum" Lett. Nuovo Cimento **38** (1983) 522.
   M. Toller, "The geometry of maximal acceleration" Int. J. Theor. Phys **29** (1990) 963.
7. L. C.B Crispino, A. Higuchi and G.E.A. Matsas, "The Unruh effect and its applications", arXiv : 0710.5373.
8. J.D Jarvis and S. O. Morgan, "Born reciprocity and the granularity of space-time", Found. Phys. Lett. **19** (2006) 501-517.
9. C. Castro Perelman, "On Thermal Relativity, Modified Hawking Radiation, and the Generalized Uncertainty Principle " International Journal of Geometric Methods in Modern Physics, **Vol 16**, Issue 10, id. 1950156-3590 (2019).
   C. Castro Perelman, "Further Insights into Thermal Relativity Theory and Black Hole Thermodynamics ", Foundations of Physics, **vol 51**, no. 99 (2021)
10. A. Popov, "Klein-Gordon oscillators, RQM and quantum time", arXiv : 2405.14349
11. Jan Govaerts, Peter D. Jarvis, Stuart O. Morgan, Stephen G. Low, "World-line Quantisation of a Reciprocally Invariant System", arXiv : 0706.3736
    P.D. Jarvis and S.O Morgan, "Constraint quantisation of a worldline system invariant under reciprocal relativity. II" arXiv : 0806.4794.
12. M Moshinsky and A Szczepaniak, "The Dirac oscillator", Journal of Physics **A**: Mathematical and General **Vol 22**, no. 17 (1989) L817.
13. R. Gallego Torrome, "Some consequences of theories with maximal acceleration in laser-plasma acceleration", Mod. Phys. Letts **A 34** (2019) 1950118.
    R. Gallego Torrome, "On the effect of the maximal proper acceleration in the inertia" Proc. R. Soc. **A 480** : (2024) 20230876.
    R. Gallego Torrome, "Maximal acceleration and black hole evaporation", IJGMMP **vol 19**, no. 5, 2250064 (2022).
    R. Gallego Torrome, "On the Unruh effect for Hyperbolic Observers in Spacetimes with Maximal Acceleration", arXiv : 2410.18155.
14. I. Gutierrez-Sagredo, A. Ballesteros, G. Gubitosi, and F.J. Herranz, "Quantum groups, noncommutative Lorentzian spacetimes and curved momentum spaces" "Spacetime Physics 1907 - 2017". C. Duston and M. Holman (Eds). Minkowski Institute Press, Montreal (2019), pp. 261-290. arXiv : 1907.07979.
    A. Ballesteros, G. Gubitosi, I. Gutierrez-Sagredo, and F. J. Herranz"Curved momentum spaces from quantum (Anti-)de Sitter groups in (3+1) dimensions", Phys. Rev. **D 97** (2018), 106024.
15. C. Castro Perelman, "Born Reciprocal Relativity Theory, Curved Phase Space, Finsler Geometry and the Cosmological Constant", Annals of Physics **416**, (May 2020) 168143.

16. C. Castro Perelman, "On Born Deformed Reciprocal Complex Gravitational Theory and Noncommutative Gravity", Phys. Lett. **B 668** (2008) 442-446.
C. Castro Perelman, "On Born Reciprocal Relativity, Algebraic Extensions of the Yang and Quaplectic Algebra, and Noncommutative Curved Phase Spaces", Universe **9** (3) (2003) 144.
17. C. Castro Perelman, "On Maximal Acceleration, Strings with Dynamical Tension, and Rindler Worldsheets", Phys. Letts. **B** Published online https://doi.org/10.1016/j.physletb.2022.137102, April 2022.
18. G. Gibbons, Foundations of Physics **32** (2002). 1891.
C. Schiller, Phys. Rev. **D 104** (2021) 124079.
S. Hod, "Maximum force conjecture in curved spacetimes of stable self-gravitating matter configurations", Physical Review **D 110**, 104040 (2024).