Proof of the Binary Goldbach Conjecture

by

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July 2025

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Abstract

In this article the proof of the binary Goldbach conjecture is established (Any integer greater than one is the mean arithmetic of two positive primes) . To this end, Chen's weak conjecture is proved (Any even integer greater than one is the difference of two positive primes) and a "localised" algorithm is developed for the construction of two recurrent sequences of primes (U_{2n}) and (V_{2n}), ((U_{2n}) dependent of (V_{2n})) such that for any integer $n \geq 2$ their sum is equal to $2n$: (U_{2n}) and (V_{2n}) are extreme Goldbach decomponents. To form them, a third sequence of primes (W_{2n}) is defined for any integer $n \geq 3$ by $W_{2n} = \text{Sup}\,(p \in \mathcal{P} : p \leq 2n - 3)$, $\mathcal{P}$ denoting the set of positive primes. The Goldbach conjecture has been proved for all even integers $2n$ between 4 and $4.10^{18.}$ and in the neighbourhood of 10^{100} , 10^{200} and 10^{300} for intervals of amplitude 10^9 . The table of extreme Goldbach decomposants, compiled using the programs in Appendix 14 and written with the Maxima and Maple scientific computing software, as well as files from ResearchGate, Internet Archive, and the OEIS, reaches values of the order of $2n = 10^{5000}$. In addition, a global proof by strong recurrence "finite ascent and descent method" on all the Goldbach decomponents is provided by using sequences of primes (Wq_{2n}) defined by : $Wq_{2n} = \text{Sup}\,(p \in \mathcal{P} : p \leq 2n - q)$ for any odd positive prime q , and a majorization of U_{2n} by $n^{0.525}$, $0.7 \ln^{2.2}(n)$ with probability one and $5 \ln^{1.3}(n)$ on average for any integer n large enough is justified.. Finally, the Lagrange-Lemoine-Levy (3L) conjecture and its generalization called "Bachet-Bézout-Goldbach"(BBG) conjecture are proven by the same type of method.

Keywords

Prime Number Theorem, Binary Goldbach Conjecture, Chen's Weak Conjecture, Lagrange-Lemoine-Levy Conjecture, Bachet-Bézout-Goldbach Conjecture, Goldbach Decomponents, Computational Number theory, Gaps between consecutive Primes.

1 Overview

Number theory, "the queen of mathematics" studies the structures and properties defined on integers and primes (Euclid [15], Hadamard [18], Hardy, Wright [20], Landau [26], Tchebychev [44]). Numerous problems have been raised and conjectures made, the statements of which are often simple but very difficult to prove. These main components include :

- **Elementary arithmetic .**

– ␣ Operations on integers, determination and properties of primes.
(Basic operations, congruence, gcd, lcm, ...).

– Decomposition of integers into products or sums of primes
(Fundamental theorem of arithmetic, decomposition of large numbers, cryptography and Goldbach's conjecture, see Filhoa,Jaimea,de Oliveira Gouveaa,Keller Füchter, [16]).

- **Analytical number theory .**

– Distribution of primes : Prime Number Theorem, the Riemann hypothesis, (see Hadamard [18], De la Vallée-Poussin [45], Littlewood [29] and Erdos [14], ,.....).

– Gaps between consecutive primes (Bombieri,Davenport [3], Cramer [9], Baker,Harmann,Iwaniec, Pintz [4],[5],[24], Granville [17], Maynard [31], Tao [43], Shanks [40], Tchebychev [44] and Zhang [49]).

- **Algebraic, probabilistic, combinatorial and algorithmic number theories .**

– Modular arithmetic.

– Diophantine approximations and equations.

– Arithmetic and algebraic functions.

– Diophantine and number geometry.

– Computational number theory.

2 Definitions notations and background

The integers h, m,M, n, N, k, K ,p, q, Q, r,.….used in this article are always positive. (2.1)

The symbol " | " means : such as or knowing that. . (2.2)

Let $\mathcal{P}$ be the infinite set of positive primes p_k (called simply primes) (2.3)

$$(p_1 = 2 ; p_2 = 3 ; p_3 = 5 ; p_4 = 7 ; p_5 = 11 ; p_6 = 13 ; ....)$$

For any non-zero integer K $\quad \underline{\mathcal{P}}_K = \{ p \in \mathcal{P} : p \leq 2K \}$ (2.4)

Writing the large numbers calculated in Appendix 14 is simplified by defining the following constants:

$$M = 10^9 ; \ R = 4.10^8 ; G = 10^{100} ; S = 10^{500} ; T = 10^{1000} \tag{2.5}$$

$\ln(x)$ denotes the neperian logarithm of the real $x > 0$ (2.6)

Let (W_{2n}) be the sequence of primes defined by

$$\forall \ n \in \mathbb{N} + 3 \qquad W_{2n} = \text{Sup} (p \in \mathcal{P} : p \leq 2n - 3) \tag{2.7}$$

For any odd prime q , let (Wq_{2n}) be the sequence of primes defined by

$$\forall \ n \in \mathbb{N} \quad n \geq \frac{(q+3)}{2} \qquad Wq_{2n} = \text{Sup} (p \in \mathcal{P} : p \leq 2n - q) \tag{2.8}$$

Any sequence denoted by $(G_{2n}) = (U_{2n} ; V_{2n})$ verifying (2.9) is called a *Goldbach sequence***.**

$$\forall \ n \in \mathbb{N} + 2 \qquad U_{2n} , V_{2n} \in \mathcal{P} \quad \text{and} \quad U_{2n} + V_{2n} = 2n \tag{2.9}$$

U_{2n} and V_{2n} are also known as "*Goldbach partitions or Goldbach decomponents*".

Iwaniec,Pintz [24] have shown that for a sufficiently large integer n there is always a prime between $n - n^{23/42}$ and n . Baker and Harman [4],[5] concluded that there is a prime in the interval $[\, n\, ;\, n + o\,(\, n^{0.525})]$. Thus this results provides an increase of the gap between two consecutive primes p_k and p_{k+1} of the form

$$\forall\, \varepsilon > 0 \quad \exists\, k_\varepsilon \in \mathbb{N}^* \;\Big|\; \forall\, k \in \mathbb{N} \quad k \geq k_\varepsilon \qquad p_{k+1} - p_k < \varepsilon . p_k^{0.525} \qquad (2.10)$$

The results obtained on the Cramer-Granville-Maier-Nicely conjecture [1],[3],[9],[17],[30],[32] imply the following majorization.

For any real $c > 2$ and for any integer $k \geq 500$

$$p_{k+1} - p_k \leq 0.7 \ln^{c} (p_k) \qquad (\textit{with probability one}) \quad \textbf{(2.11)}$$

and

$$p_{k+1} - p_k \leq 20.\ln(p_k) \qquad (\textit{on average}) \quad (2.12)$$

The following abbreviations have been adopted :

- Lagrange-Lemoine-Levy conjecture (3L) conjecture (2.13)
- Bachet-Bézout-Goldbach conjecture (BBG) conjecture (2.14)
- (Extreme) Goldbach decomponents (E).G.D. (2.15)

3 Introduction

Chen [7], Hardy, Littlewood [21], Hegfollt, Platt [22], Ramaré, Saouter [35], Tao [43], Tchebychev [44] and Vinogradov [46] have taken important steps and obtained promising results on the Goldbach conjecture (Any integer $n \geq 2$ is the mean arithmetic of two primes). Indeed, Helfgott, Platt [22] proved the ternary Goldbach conjecture in 2013.

Silva, Herzog, Pardi [41] held the record for calculating the terms of Goldbach sequences after determining pairs of primes (U_{2n} ; V_{2n}) verifying

$$\forall\ n \in \mathbb{N} \ \Big|\ \ 4 \leq 2n \leq 4.10^{18} \qquad U_{2n} + V_{2n} = 2n \tag{3.1}$$

Goldbach's conjecture has also been verified for all even integers $2n$ satisfying

$$10^{5k} \leq 2n \leq 10^{5k} + 10^{8} \ : \qquad k = 3, 4, 5, 6,........,20$$

and

$$10^{10k} \leq 2n \leq 10^{10k} + 10^{9} \ : \qquad k = 20, 21, 22, 23, 24,.......,30$$

by Deshouillers, te Riele, Saouter [11].

In previous research work there is no explicit construction of recurrent Goldbach sequences.

In this article, for any integer n greater than two the E.G.D. U_{2n} and V_{2n} are computed iteratively using a simple and efficient "localised" algorithm.

Using Maxima and Maple scientific computing software on a personal computer Silva's record is broken and many E.G.D. are calculated up to the neighbourhood of $2n$ = 10^{500} , $10^{1000,}$ 10^{5000} and G.D. around 10^{10000} (see Sainty [37]

"In Researchgate.net, Internet Archive, and OEIS, E.G.D. files are supplied :

E.G.D. File S Around $2n = 10^{S}$ for S = 1, 2, 3,............., 10000").

The binary Goldbach conjecture can be proved globally by strong recurrence on all G.D. using (Wq_{2n}) sequences of primes in the same way via Goldbach(-) conjecture (Any even integer greater than one is the difference of two primes) demonstrated in Teorem 4.

● *Remark.*

1. **Chen conjecture:** *For any integer* $K \geq 1$ *there are infinitely many pairs of primes with a difference equal to* $2K$.

2. **De Polignac conjecture :** *Same as Chen, but with consecutive pairs of primes.*

3. **What we know :**

April 2013, Yitang Zhang [49] demonstrates that the smallest even integer $2K$ verifying the conjecture is greater than 70 million.

In 2014, James Maynard [31] then Terence Tao [43] lowered this limit to 246.

We validate Chen's weak conjecture by verifying directly in the primes tables that all even gaps from 2 to 246 are possible (see Appendix 15).

In addition, the (3L) conjectures [10],[23],[25],[28],[47] and its generalization called (BBG) conjecture are validated.

Using case disjunction reasoning we construct two recurrent E.G.D. sequences of primes (V_{2n}) and (U_{2n}) according to the sequence (W_{2n}) by the following process

Firstly,

$$U_4 = 2 \quad \text{and} \quad V_4 = 2 \tag{3.2}$$

For any integer n greater than two

● *Either*

$(2n - W_{2n})$ is a prime

then V_{2n} and U_{2n} are defined directly in terms of W_{2n} .

● *Either*

($2n$ - W_{2n}) is a composite number

then V_{2n} and U_{2n} are determined from the previous terms of the sequence (G_{2n}).

(This process can be reversed by first determining the increasing sequence of primes less than Inf (2n - W_{2k} $\in \mathcal{P}$ *:* $k \in \mathbb{N}$ *) , which saves a lot of computing time when programming).*

4 Theorem (*Chen's weak or Goldbach*(-) *conjecture*)

$$\forall\ K \in \mathbb{N}^* \quad \exists\ p, q \in \mathcal{P} \quad \Big| \quad p - q = 2K \tag{4.1}$$

$$\text{If } \ K \geq 2 \qquad 3 \leq q \leq 2K \quad \textit{and} \quad 3 + 2K \leq p \leq 4K$$

Practical method on some examples:

First of all (5 - 3 = 2), then we begin the process at (7 - 3 = 4); we will select the smallest primes for which the difference is precisely 6 (11 - 5 = 6), then 8 (11 - 3 = 8), then 10 (13 - 3 = 10),......... , then $2K$ (demonstration established by strong recurrence, by the asurd and feedback). All pairs of Goldbach(-) partitions obtained by this method for K between 2 and 123 are listed in Appendix 15 to validate it using Tao results.

Proof. An other proof can also be established by strong recurrence on the integer $K \geq 2$. Let $\mathcal{P}_{Chen}(K)$ be the following property

" $\forall\ K \in \mathbb{N}^* \quad \exists\ p, q \in \mathcal{P} \ \big|\ \ p - q = 2K \qquad 3 \leq q \leq 2K \ \ and \ \ 2K + 3 \leq p \leq 4K$ " (4.2)

► $\mathcal{P}_{Chen}(2)$ is true : 7 - 3 = 4 $\quad q = 3 \leq 4$ and $p = 7 \leq 4 \times 2 = 8$

► Let's show

$$\forall\ M \in \mathbb{N} \ \big|\ \ 2 \leq M \leq K \qquad \text{then} \qquad \mathcal{P}_{Chen}(M) \implies \mathcal{P}_{Chen}(K + 1)$$

We reason through the absurd

Let $p\,,q \in \underline{\mathcal{P}}_K \ \big|\ \ p \geq q$

$$\forall\ P\,,Q \in \mathcal{P} \ \big|\ \ P \geq Q \quad \exists\ \ h, m \in \mathbb{N} \ \big|$$

$$P = p + 2h \qquad \text{and} \qquad Q = q + 2m$$

we assume that

$$P - Q = p + 2h - q - 2m \neq 2(K + 1) \tag{4.3}$$

Therefore

$$p - q \neq 2(K + 1 - h + m) \tag{4.4}$$

You can always choose $\qquad h \geq m \qquad$ and $\qquad h - m \leq K + 1$.

The set $\{K + 1 - h + m$; $2h$ and $2m$ are any gaps between primes$\}$ contains all even integers between 2 and $2K$.

However the strong recurrence hypothesis asserts that

$$\forall\ M \in \mathbb{N} \ \big|\ \ M \leq K \quad \exists\ \ p, q \in \mathcal{P} \ \big|\ \qquad p - q = 2M \tag{4.5}$$

By choosing : $\qquad M = K + 1 - h + m$

this contradicts (4.4).

So

$$\exists\ \ h, m \in \mathbb{N} \ \big|\ \qquad P - Q = p + 2h - q - 2m = 2(K + 1) \tag{4.6}$$

knowing

$$p,\ p+2h, q,\ q+2m \ \in\ \mathcal{P} \qquad\qquad h \geq m \ \text{ and } \ h - m \leq K+1$$

Thus validating the heredity of property $\mathcal{P}_{Chen}(K)$.

The property $\mathcal{P}_{Chen}(K)$ is therefore true. As a result Goldbach(-) conjecture is validated.

5 Corollary

Let (R_{2K}) *and* (Q_{2K}) *be two sequences of primes determined by*

$$R_{2K} = \text{Inf}\,(p \in \mathcal{P} :\ p - 2K \in \mathcal{P}) \quad \textit{and} \quad Q_{2K} = \text{Inf}\,(p \in \mathcal{P} :\ 2K + p \in \mathcal{P}) = R_{2K} - 2K \quad (5.1)$$

They are defined for any integer $K \in \mathbb{N}^*$ (5.2)

and satisfy

$$\lim R_{2K} = +\infty \quad (5.3)$$

$$\forall\ K \in \mathbb{N}^* \qquad R_{2K}, Q_{2K} \in \mathcal{P} \quad \textit{and} \quad R_{2K} - Q_{2K} = 2K \quad (5.4)$$

$$\forall\ K \in \mathbb{N}^* \ \big|\ 2 \leq K \leq 16 \qquad 3 \leq Q_{2K} \leq 2K \quad \text{and} \quad 2K+3 \leq R_{2K} \leq 4K \quad (5.5)$$

For any integer K *large enough*

$$3 \leq Q_{2K} \leq (2K)^{0.525} \quad \text{and} \quad 2K+3 \leq R_{2K} \leq 2K + (2K)^{0.525} \quad (5.6)$$

Proof.

(5.1) ; (5.2) : According to the previous theorem, the sequences (R_{2K}) and (Q_{2K}) are defined by strong recurrence (finite descent).

(5.3) : $R_{2K} \geq 2K \quad \Longrightarrow \quad \lim R_{2K} = +\infty$

(5.4) : By construction, these sequences thus verify : R_{2K} - Q_{2K} = $2K$

(5.5) : The property can be verified directly term-to-term by examining the sequence proposed above.

(5.6) : This property is verified up to $2K = 246$ by calculations on the previous list.

We prove this result by recurrence

First of all, we order the Goldbach(-) decomponents at a fixed prime q , so as to obtain the estimate (5.6) more easily.

Let q_r be the (r + 1)th prime :

We examine the sequences of primes $(T_r(K))_{K\in\mathbb{N}}$ satisfying :

$T_1(K) = 2K + 3$

($T_1(K)$; $2K$) → (5;2) ; (7 4) ; (11;8) ; (13;10) ; (17;14) ; (19;16) ; (23;20) ; (29;26) ; (29;28);..

$T_2(K) = 2K + 5$

($T_2(K)$; $2K$) → (7;2) ; (11;6) ; (13;8) ; (17;12) ; (19;14) ; (23;18) ; (29;24) ; (31;26) ; (37;32)................................

$T_3(K) = 2K + 7$

($T_3(K)$; $2K$) → (11;4) ; (13;6) ; (17;10) ; (19;12) ; (23;16) ; (29;22) ; (31;24) ; (37;30)...........

$T_4(K) = 2K + 11$

($T11(K)$; $2K$) → (13;2) ; (17;6) ; (19;8) ; (23;12) ; (29;18) ; (31;20) ; (37;26) ; (41;30) ; (43;34).........................

$(T13(K)\ ;\ 2K) \rightarrow (17;4)\ ;\ (19;6)\ ;\ (23;10)\ ;(29;16)\ ;\ (31;18)\ ;\ (37;24)\ ;\ (41;28)\ ;\ (43;30\ ;$

$(47;34)$..........................

....

$T_r(K) = 2K + q_r$ ($K \in \mathbb{N}^*$: $T_r(K)$ and q_r are primes) (see Appendix 16)

For any integer K satisfying $(2K)^{0.525} > q_r$ the property holds for $T_r(K)$.

Therefore it is generally validated for all $K > K_0$, since we obtain all possible cases of Chen's weak conjecture starting with $T_1(K)$, then $T_2(K)$, then $T_3(K)$ .... for $(2K)^{0.525} \leq q_r$.

(can be proved by strong recurrence using the same method as in Theorem 4 by "finite descent").

Let $a = \frac{40}{21}$ and $P_a(r)$ be the following property

"For any integer $M \mid 2M < (q_r)^a$ there exists at least a prime $q < q_r \mid 2M + q \in \mathcal{P}$ "

- ▸ $P_a(K_0)$ is true (see Appendix 16).
- ▸ Let's show : $P_a(r) \implies P_a(r+1)$

$$q_{r+1} \leq q_r + q_r^{0.525} \qquad (5.6)$$

It is assumed that $M \mid$

$T_{r+1}(K) - q_{r+1} \neq 2M$ knowing $2M < (q_{r+1})^{c_p}$

$\forall\ T_m(R), q_m \in \mathcal{P}\ \exists\ h, s \in \mathbb{N} \mid T_{r+1}(K) = T_m(R) + 2h$ and $q_{r+1} = q_m + 2s$ (5.7)

then

$$T_m(R) - q_m \neq 2(M + s - h) \qquad (5.8)$$

which is impossible according to the hypothesis of strong recurrence since $2(M + s - h)$ is less than Sup $(q_m)^a$ and that all primes $T_m(R)$, q_m satisfy the recurrence hypothesis.

We deduce that : $Pc_p(r) \implies Pc_p(r + 1)$

Thus the property (5.5) is true.

6 Lemma (*Goldbach's fundamental Lemma*)

Let q be an odd prime

For any integer $n \geq n_q$ there exists an integer s |

$$2n - Wq_{2s} \in \mathcal{P} \tag{6.1}$$

Let (Zq_{2n}) be the sequence of primes defined by

$$\forall\ n \in \mathbb{N} \quad n \geq n_q \qquad Zq_{2n} = \text{Inf}\,(2n - Wq_{2k} \in \mathcal{P} : k \in \mathbb{N}) \tag{6.2}$$

All G.D. are contains in the set $\{(2n - Zq_{2n} ; Zq_{2n})\ n \in \mathbb{N} + 3\}$

F*or any integer* $n \geq n_0$ $\qquad Z3_{2n} \leq (2n)^{0.525}$ (6.3)

$$Z3_{2n} \leq o\ (2n)^{0.525} \tag{6.4}$$

Proof. The proofs of propositions (6.1), (6.2) and (6.3) are established following the same principle of strong recurrence as in Theorem 4 and Corollary 5 by "return, absurd and finite descent"

(6.1) : For any integer $n > 3$ and for any odd primes r, q | $3 \leq r < q,$

there exists an integer M_r |

$$2n - Wq_{2k} = 2n - 2M_r - Wr_{2k} = 2(n - M_r) - Wr_{2k}$$

or

$$2(n + 1) - Wq_{2k} = 2(n + 1 - M_r) - Wr_{2k}$$

then by recurrence and the absurd the property is validated.

(*Proof to develop*).

Remark. A better estimate of the following form can be obtained by the same method with probability one or on average using the results of Bombieri [3], Cramer [9], Granville [17] , Nicely [32] and Maier [30] :

$\forall\, n \in \mathbb{N} : \; n \geq n_0$;

For any real $c > 2$ $\quad U_{2n} < 1.7 \ln(n)^c$ $\quad$ (*with probability one*) (6.5)

and

$\exists\; K' \geq 3.5 \;|$ $\quad U_{2n} < K'.\ln^{1.3}(n)$ $\quad$ (*on average*) (6.6)

7 Principle of proof

To determine the E.G.D. three sequences of primes (W_{2n}), (V_{2n}), (U_{2n}) are defined and they verify the following properties

$$\lim V_{2n} = +\infty. \tag{7.1}$$

$\forall\; n \in \mathbb{N} + 2 \quad V_{2n}$ is defined as a function of $W_{2n} = \text{Sup}\,(p \in \mathcal{P} : p \leq 2n - 3)$ (7.2)

(W_{2n}) is an increasing sequence of primes that contains all of them except $p_1 = 2$ (7.3)

$$\lim W_{2n} = +\infty \tag{7.4}$$

(U_{2n}) is a complementary sequence to (W_{2n}) of negligible primes with respect to $2n$ (7.5)

For any integer $n \geq 3$

● *If* $(2n - W_{2n})$ *is a prime*

then V_{2n} and U_{2n} are defined by

$$V_{2n} = W_{2n} \quad \text{and} \quad U_{2n} = 2n - W_{2n} \tag{7.6}$$

● *Otherwise, if* $(2n - W_{2n})$ *is a composite number*

we search for two previous terms of the sequence (G_{2n}), $U_{2(n-k)}$ and $V_{2(n-k)}$ satisfying the following conditions

$$U_{2(n-k)}, V_{2(n-k)}, [U_{2(n-k)} + 2k] \in \mathcal{P} \tag{7.7}$$

$$U_{2(n-k)} + V_{2(n-k)} = 2(n - k)$$

which is always possible (see Theorem 4 and "*Goldbach's fundamental Lemma* 6")

So by setting

$$V_{2n} = V_{2(n-k)} \quad \text{and} \quad U_{2n} = U_{2(n-k)} + 2k \tag{7.8}$$

two new primes V_{2n} and U_{2n} satisfying (4.10) are generated |

$$U_{2n} + V_{2n} = 2n \tag{7.9}$$

This process is then repeated incrementing n by one unit $(n \leftarrow n + 1)$.

● *Remark.* Using the same method as in Theorem 4, we can the following equivalent property by strong recurrence : For any integer n greater than 48

$$\mathcal{P}_{ret}(n): \text{" There exists an integer } K \text{ such that } 2K + U_{2(n-k)} \in \mathcal{P} \text{"} \tag{7.10}$$

To this end, .

▶ $\mathcal{P}_{ret}(49)$ is true.

▶ The heredity of the property $\mathcal{P}_{ret}(n)$: $\quad \mathcal{P}_{ret}(n) \implies \mathcal{P}_{ret}(n+1)$

can be proved by the absurd and returning to the previous terms by noting that

For any integer r | $r \leq n$, there is at least one integer M_r |

$$U_{2(n+1-k)} = 2\,M_r + U_{2(r+1-k)}$$

then

$$2K + U_{2(n+1-k)} = 2(K + M_r) + U_{2(r+1-k)}$$

$$= 2P + U_{2(r+1+M_r-P)} \qquad (7.11)$$

By posing : $P = K + M_r$ and $r + 1 + M_r \leq n$

Now, according to the recurrence hypothesis on $\mathcal{P}_{ret}(n)$ there exists an integer P |

$$2P + U_{2(r+1+M_r-P)} \in \mathcal{P} \qquad (7.12)$$

then there exists an integer K |

$$2K + U_{2(n+1-k)} \in \mathcal{P} \qquad (7.13)$$

In summary, the property $\mathcal{P}_{ret}(n)$ is hereditary and, as a result, verifiable.

We apply the same type of reasoning using Theorem 4 to the general case with the sequence (Wq_{2n}), showing :

For any integer $n > 2$ there exists an integer K |

$$2K + q_{2n} \in \mathcal{P}$$

8 Theorem (*Goldbach conjecture*)

(i) *There exists at least a recurrent sequence* $(G_{2n}) = (U_{2n}\,;\,V_{2n})$ *of primes satisfying the following conditions.*

For any integer $n \geq 2$

$$U_{2n}\ ,\ V_{2n}\ \in\ \mathcal{P} \qquad \text{and} \qquad U_{2n} + V_{2n} = 2n \qquad (8.1)$$

(*Any integer* $n \geq 2$ *is the mean arithmetic of two primes*)

(ii) **A***n algorithm can be used to explicitly compute any term* U_{2n} and V_{2n} (8.2)

Proof.

■ **GLOBAL STRONG RECURRENCE :**

The proof can be made using the following strong recurrence principle.

Let $P_G(n)$ be the property defined for any integer $n \geq 2$ by

$P_G(n)$: " For any integer p satisfying $2 \leq p \leq n$ there exists two primes U_{2p} and V_{2p} such their sum is equal to $2p$ ".

$$(\ \forall\ p \in \mathbb{N}\ \mid\ 2 \leq p \leq n \qquad U_{2p}\ ,\ V_{2p}\ \in\ \mathcal{P} \quad \text{and} \quad U_{2p} + V_{2p} = 2p\ )$$

Let's show by strong recurrence that $P_G(n)$ is true for any integer $n \geq 2$

▶ $P_G(2)$ is true : it suffices to choose $U_4 = V_4 = 2$.

▶ Let's show that the property $P_G(n)$ is hereditary : $P_G(n) \implies P_G(n+1)$

Assume property $P_G(n)$ is true.

● *If* $(\ 2(n+1) - W_{2(n+1)})$ *is a prime*

then $V_{2(n+1)}$ and $U_{2(n+1)}$ are defined by

$$V_{2(n+1)} = W_{2(n+1)} \qquad \text{and} \qquad U_{2(n+1)} = 2(n+1) - W_{2(n+1)} \qquad (8.3)$$

● *Otherwise, if* $(\ 2(n+1) - W_{2(n+1)})$ *is a composite number* there exists an integer k to obtain two terms $U_{2(n+1-k))}$ and $V_{2(n+1-k)}$ satisfying the following conditions

$$U_{2(n+1-k)}, V_{2(n+1-k)} \quad \text{and} \quad U_{2(n+1-k)} + 2k \quad \in \mathcal{P} \tag{8.4}$$

$$U_{2(n+1-k)} + V_{2(n+1-k)} = 2(\, n+1- k\,)$$

we use the previous terms of the sequence (G_{2n}).

For any integer q | $1 \leq q \leq n$ - 3 we have

$$3 \leq U_{2(n-q)} \leq n\,.$$

Then there exists an integer k $1 \leq k \leq n$ - 3 |

$$R_{2n} = U_{2(n-k)} + 2k \in \mathcal{P} \tag{8.5}$$

following the Bertrand principle and Theorem 4 since all primes smaller than $(2n)^{0.525}$ are in the set $\{ U_{2k} : k \leq n \}$

(If there were no such primes, we would have a contradiction with the Theorem 4 or with *Goldbach's fundamental Lemma 6*) . In fact, in an equivalent way (see the previous remark) we can copy the proof of Teorem 4 by performing a similar strong recurrence "finite descent feedback and absurd" directly on the set $\{ U_{2k} : k \leq n \}$ |

$$R_{2n} = U_{2(n-k)} + 2k \in \mathcal{P} \tag{8.6}$$

The smallest integer k | $R_{2n} \in \mathcal{P}$ is denoted by k_n .

So by setting

$$U_{2n} = U_{2(n-k_n)} + 2k_n \quad \text{and} \quad V_{2n} = V_{2(n-k_n)} \in \mathcal{P} \tag{8.7}$$

(*These two terms are primes*)

In the previous steps two primes $U_{2(n-k_n)}$ and $V_{2(n-k_n)}$ whose sum is equal to $2(n - k_n)$ were determined.

$$U_{2(n-k_n)} + V_{2(n-k_n)} = 2(n - k_n) \tag{8.8}$$

By adding the term $2k_n$ to each member of the equality (8.6) it follows

$$U_{2(n-k_n)} + 2k_n + V_{2(n-k_n)} = 2(n - k_n) + 2k_n \tag{8.9}$$

$$\Leftrightarrow \quad [\, U_{2(n-k_n)} + 2k_n \,] + V_{2(n-k_n)} = 2n \tag{8.10}$$

$$\Leftrightarrow \quad U_{2n} + V_{2n} = 2n \tag{8.11}$$

Two new primes $V_{2(n+1)}$ and $U_{2(n+1)}$ satisfying ($U_{2(n+1)} + V_{2(n+1)} = 2(n+1)$) are generated.

It follows that $P_G(n+1)$ is true. Then the property $P_G(n)$ is hereditary :

$$P_G(n) \implies P_G(n+1).$$

Therefore for any integer $n \geq 2$ the property $P_G(n)$ is true.

It follows

$\forall\, n \in \mathbb{N} + 2$ there are two primes U_{2n} and V_{2n} and such their sum is $2n$: $U_{2n} + V_{2n} = 2n$

■ **ALGORITHM :**

For any integer $n \geq 3$

● *If* $(2n - W_{2n})$ *is a prime*

then V_{2n} and U_{2n} are defined by

$$V_{2n} = W_{2n} \quad \text{and} \quad U_{2n} = 2n - W_{2n} \tag{8.12}$$

● *Otherwise, if* $(2n - W_{2n})$ *is a composite number*

we use the previous terms of the sequence (G_{2n}).

For any integer q | $1 \leq q \leq n - 3$ we have

$$3 \leq U_{2(n-q)} \leq n .$$

Then there exists an integer k $1 \leq k \leq n - 3$ |

$$R_{2n} = U_{2(n-k)} + 2k \in \mathcal{P} \tag{8.13}$$

following Theorem 4 since all primes smaller than $(2n)^{0.525}$ are in the set $\{ U_{2k} : k \leq n \}$

(If there were no such primes, we would have a contradiction with the Theorem 4 or with *Goldbach's fundamental Lemma 6*) . In fact, in an equivalent way (see the previous remark) we can copy the proof of Teorem 4 by performing a similar strong recurrence ''finite descent return and absurd" directly on the set $\{ U_{2k} : k \leq n \}$ |

$$R_{2n} = U_{2(n-k)} + 2k \in \mathcal{P} \tag{8.14}$$

The smallest integer k | $R_{2n} \in \mathcal{P}$ is denoted by k_n .

So

$$U_{2n} = U_{2(n-k_n)} + 2k_n \quad \text{and} \quad V_{2n} = V_{2(n-k_n)} \in \mathcal{P} \tag{8.15}$$

(*These two terms are primes*)

In the previous steps two primes $U_{2(n-k_n)}$ and $V_{2(n-k_n)}$ whose sum is equal to $2(n - k_n)$ were

determined.

$$U_{2(n-k_n)} + V_{2(n-k_n)} = 2(n - k_n) \tag{8.16}$$

By adding the term $2k_n$ to each member of the equality (8.16) it follows

$$U_{2(n-k_n)} + 2k_n + V_{2(n-k_n)} = 2(n - k_n) + 2k_n \tag{8.17}$$

$$\Leftrightarrow \quad [\, U_{2(n-k_n)} + 2k_n \,] + V_{2(n-k_n)} = 2n \tag{8.18}$$

$$\Leftrightarrow \quad U_{2n} + V_{2n} = 2n \tag{8.19}$$

Finally, for any integer $n \geq 3$ this algorithm determines two sequences of primes (U_{2n}) and (V_{2n}) verifying Goldbach's conjecture.

9 Lemma

The sequence (U_{2n}) verifies the following majorization

For any integer $n \geq 65$

$$U_{2n} \leq (2n)^{0.525} \tag{9.1}$$

and

$$U_{2n} = o\,((2n)^{0.525}) \tag{9.2}$$

Proof. According to the programm 12.2 and Appendix 13 the majorization (9.1) is verified for any integer $n \mid 65 \le n \le 2000$.

For any integer $n > 2000$ the proof is established by recurrence. For this purpose let $P_{bhip}(n)$ be the following property

$$P_{bhip}(n): \text{ " } U_{2n} \le (2n)^{0.525} \text{ " }. \tag{9.3}$$

▶ $P_{bhip}(2000)$ is true according to program 12.2 and the table in appendix 13.

▶ For any integer $n \ge 2000$ let's show that $P_{bhip}(n)$ is hereditary :

$$P_{bhip}(n) \implies .P_{bhip}(n+1)$$

Assume that $P_{bhip}(n)$ is true : then

● *If* $(2(n+1) - W_{2(n+1)})$ *is a prime*

then $V_{2(n+1)}$ and $U_{2(n+1)}$ are defined

by

$$V_{2(n+1)} = W_{2(n+1)} \quad \text{and} \quad U_{2(n+1)} = 2(n+1) - W_{2(n+1)} \tag{9.4}$$

According to the results in [4],[5],[24] (see Lemma 9) there is a constant $K > 0$ such that

$$2(n+1) - K.[2(n+1)]^{0.525} < W_{2(n+1)} < 2(n+1)$$

$$\implies \quad U_{2(n+1)} = 2(n+1) - W_{2(n+1)} < K.[2(n+1)]^{0.525}$$

$$\implies \quad U_{2(n+1)} \le K.[2(n+1)]^{0.525}$$

● *Otherwise, if* $(2(n+1) - W_{2(n+1)})$ *is a composite number*

$$\exists\, p \in \mathbb{N}^* \mid \quad U_{2(n+1)} = U_{2(n+1-p)} + 2p \tag{9.5}$$

According to [4],[5],[24]

$U_{2(n+1)} = 2p + U_{2(n+1-p)} = 2p + 2(n + 1 - p) - W_{2(n+1-p)} = 2(n + 1) - W_{2(n+1-p)}$ (9.6)

Via " *Goldbach's fundamental Lemma 6* " it follows that

$$U_{2(n+1)} < K.[2(n+1)]^{0.525} \quad (9.7)$$

$P_{bhip}(n + 1)$ is true then $P_{bhip}(n)$ is hereditary.

So for any integer $n \geq 2000$ the property $P_{bhip}(n)$ is true.

Finally $$U_{2(n+1)} \leq [2(n+1)]^{0.525}$$

● *Remark.* A more precise estimate can be obtained using the Cipolla or Axler frames [8],[2].

10 Proposition

A log-exp relationship is determined by linking the sum and product of primes via Goldbach's conjecture and the fundamental theorem of arithmetic. since if G.D. of 2n are p' and q', and if 2n decomposes into factors P " and Q " |

(p', q' ∈ P | p' ≫ q' and P " ≫ Q ") ; then,

$$2n = P''.Q'' = p' + q'$$

$$ln(P''.Q'') = ln(P'') + ln(Q'')$$

$$= ln(p' + q') = ln(p'(1 + q'/p'))$$

$$\approx ln(p') + q'/p'$$

By choosing p' = next or prevprime(P ") (P " = p' +/- a) *we obtain a q' localization of the form*

$$q' \approx [p'.\ln(Q'')] \approx [p'.\ln(2n / P'')] \approx [p'.\ln(2n / p')]$$

.then

$$2n \approx p'(1 + \ln(2n / p'))$$

$$2n \approx p'(1 + \ln(2n / P'')) \approx p'(1 + \ln(2n / (p' +/- a))$$

$$2n \approx p'(1 + \ln(2n / (p'(1 +/- a/p')))$$

$$2n \approx p'(1 + \ln((2n / p').(1 / (1 +/- a / p'))))$$

$$2n \approx p'(1 + \ln(2n/p') + \ln(1/(1 +/- a/p')))$$

$$2n \approx p'(1 + \ln(2n/p')) -/+ a$$

You can solve equations like these using Maple,

solve(2n +/- a = x(1 + ln(2n / x)), x)*

to locate p' and proceed by successive next or prevprime to determine two G.D. of 2n, (programming possible in Algorithm 13). This procedure appears to generalise Pocklington's theorem, and we observe that the G.D. and their number G(E) are related to the number of prime factors in the decomposition of 2n .

Examples :

- *evalf(solve([90 = x*(1 + ln(96/x)), x < 96], x));*

  {x = 64.12418697} ; p' = 67 q' = 29

- *evalf(solve([1000 = x*(1 + ln(1100/x)), x < 1100], x));*

  {x = 665.6361412}

prevprime(665); *661*

isprime(1100 - 661); *true ; p' = 661 q' = 439*

- *evalf(solve([9700 = x*(1 + ln(10000/x)), x < 10000], x));*

{x = 7652.697929}

prevprime(7652); *7649*

isprime(10000 - 7649); *true ; p' = 7649 q' = 4351*

- *evalf(solve([99950 = x*(1 + ln(100000/x)), x < 100000], x));*

{x = 96854.43333}

a := prevprime(96799); *a := 96797* *# Obtenu au bout de 3 ou 4 prevprime( )*

isprime(100000 - a); *true ; p' = 96799 q' = 3201*

Solutions are :

x_0 = *Re(- (2n +/- a)/LambertW(-1,-(2n +/- a) / (2n.e)))*

and

x_1= *Re(- (2n +/- a)/LambertW(-(2n +/- a) / (2n.e)))*

Remark. The G.D. of any even composite number greater than four are different from its prime factors.

10 Theorem

For any integer $n \geq 3$ *it is easy to check*

$$(W_{2n}) \text{ is a positive increasing sequence of primes} \quad (10.1)$$

$$\{ W_{2n} : n \in \mathbb{N} + 3 \} \cup \{ 2 \} = \mathcal{P} \quad (10.2)$$

$$\lim W_{2n} = +oo \quad (10.3)$$

(U_{2n}) and (V_{2n}) *are sequences of primes and the set* $\{ U_{2k} : k \leq n \}$ (10.4) *contains all primes less than* $\ln(n)$

$$n \leq V_{2n} \leq W_{2n} \tag{10.5}$$

$$3 \leq 2n - W_{2n} \leq U_{2n} \leq n \tag{10.6}$$

$$\lim V_{2n} = +oo \tag{10.7}$$

Proof.

(10.1) : For any integer $n \geq 2$ $\mathcal{P}_n \subset \mathcal{P}_{n+1}$. Therefore, $W_{2n} \leq W_{2(n+1)}$. So the sequence (W_{2n}) is increasing.

(10.2) : Any prime except $p_1 = 2$ is odd, hence the result.

(10.3) : $\lim W_{2n} = \lim p_k = +oo$

(10.4) : By definition $V_{2n} = W_{2n}$ or there exits an integer $k \leq n - 2 \mid V_{2n} = V_{2(n-k)}$.

So the terms of the sequence (V_{2n}) are primes.

(10.5) : According to Lemma 9, for any integer $n \geq 65$

$$U_{2n} < (2n)^{0.525}$$

therefore

$$U_{2n} < (2n)^{0.55} < n$$

and

$$V_{2n} = 2n - U_{2n} > 2n - n > n$$

For any integer $n \mid 3 \leq n \leq 65$ verification is carried out according to the computer program in paragraph 12.2 and the table in appendix 13.

We can also see that by construction $V_{2n} \geq U_{2n}$ because if we assume the opposite then V_{2n} is not the largest prime number verifying

$$\frac{1}{2}\,(\,U_{2n} + V_{2n}\,) = n\ .$$

So

$$V_{2n} \geq n$$

According to (10.5) $\quad n \leq V_{2n} \quad \Longrightarrow \quad U_{2n} = 2n - V_{2n} \leq 2n - n \leq n \qquad (10.6)$

$$V_{2n} \leq W_{2n} \quad \Longrightarrow \quad 2n - W_{2n} \leq 2n - V_{2n} = U_{2n} \qquad (10.7)$$

By (10.5) for any integer $n \geq 2$: $\quad n \leq V_{2n}$

$$\lim\ V_{2n} = +oo\ .$$

11 Lemma

We dissociate the following cases mod 6 for any even integer 2n $n \geq 3$: $p + q = 2n \quad p, q \in \mathcal{P}$

1. *If* $2n = 6m$ *then* $(p\ ;\ q) = (6r + 5\ ;\ 6(m - r - 1) + 1)$ *or* $(6r+1\ ;\ 6(m - 1 - r) + 5)$
2. *If* $2n = 6m + 2$ *then* $(p\ ;\ q) = (\,6r + 1\ ;\ 6(m - r) + 1)$
3. *If* $2n = 6m + 4$ *then* $(p\ ;\ q) = (6r + 5\ ;\ 6(m- 1 - r) + 5)$

Table : Sum of integers 1, 5 modulo 6 (in ℤ/6ℤ).

$p + q$ mod 6	1	5
1	2	0
5	0	4

(*To adapt with* $2n = 30m + k$)

Table : Sum of integers 1, 7, 11, 13, 17, 19, 23, 29 modulo 30 (in $\mathbb{Z}/30\mathbb{Z}$).

+ mod 30	1	7	11	13	17	19	23	29
1	2	8	12	14	18	20	24	0
7	8	14	18	20	24	26	0	6
11	12	18	22	24	28	0	4	10
13	14	20	24	26	0	2	6	12
17	18	24	28	0	4	6	10	16
19	20	26	0	2	6	8	12	18
23	24	0	4	6	10	12	16	22
29	0	6	10	12	16	18	22	28

Proof.

12 Properties

For any integer $k \geq 2$ *there are infinitely many integers* $n \mid U_{2n} = p_k$ (12.1)

$$V_{2n} \sim 2n \qquad (n \to +oo) \qquad (12.2)$$

For any integer $n \geq 5000$

$$U_{2n} \ll V_{2n} \quad \textit{and} \quad \lim \left(\frac{U_{2n}}{V_{2n}}\right) = 0 \qquad (12.3)$$

The smallest integer $n \mid U_{2n} \neq 2n - W_{2n}$ *is obtained for* $n = 49$ *and* $G_{98} = (79 ; 19)$ (12.4)

(*This type of terms increases in the Goldbach sequence* (G_{2n}) *as* n *increases in the sense* of the *Schnirelmann density and there are an infinite number of them; their proportion per interval can be computed using the results given in* [39]).

The sequence (G_{2n}) *is "extremal" in the sense that for any integer* $n \geq 2$ (12.5)

V_{2n} and U_{2n} *are the largest and smallest possible primes* $\mid$ $U_{2n} + V_{2n} = 2n.$

The Cramer-Granville-Maier-Nicely conjecture [9],[17],[30],[32] *is verified with probability one. It leads to the following majorization*

For any integer $p \geq 500$

$$U_{2p} \leq 0.7\ [\ln(2p)]^{\left(2.2 - \frac{1}{p}\right)} \qquad (\textit{with probability one}) \quad (12.7)$$

The proof is similar to that of Lemma 9 *and is validated by the studying functions of the type*

$f : x \to a\,.g\,(x) + b[\ln(g(x))]^c \quad (\,a,b > 0\ ;\ c > 2)$ *with*

$g : x \to 0.7\ [\ln(x)]^{\left(c - \frac{1}{x}\right)}$ *and* $h : x \to 0.7\ [\ln(x)]^{\left(2.2 - \frac{1}{x}\right)}$ *by using Maple software.*

A better estimate can be obtained via [29],[31],[30] .

According to Bombieri [3] *and using the same method as in the proof of Lemma* 9,

we obtain the following estimate of U_{2n}

$$\forall\ \varepsilon > 0 \qquad U_{2n} = \boldsymbol{O}\,(\,\ln^{1.3+\varepsilon}(n)) \qquad (\textit{on average}) \quad (12.8)$$

13 Algorithm

13.1 Algorithm written in natural language.

Inputs :

Input four integer variables : *k, N, n, P*

Input : $p_1 = 2$, $p_2 = 3$, $p_3 = 5$, $p_4 = 7$, ..., p_N the first N primes.

: $n \leftarrow 3$

: $P = M, R, G, S$ or T as indicated in paragraph **2**

Algorithm body :

A) Compute : $W_{2n} = \text{Sup}(\, p \in \mathcal{P} : \, p \leq 2n - 3 \,)$

If $T_{2n} = (2n - W_{2n})$ is a prime

$$U_{2n} \leftarrow T_{2n} \quad \text{and} \quad V_{2n} \leftarrow W_{2n} \tag{13.1.1}$$

otherwise

B) If T_{2n} is a composite number

Let : $k = 1$

B.1) While $U_{2(n-k)} + 2k$ is a composite number

assign to k the value $k + 1$ ($k \longleftarrow k + 1$).

return to **B1)**

End while

Assign to k the value k_n ($k_n \longleftarrow k$)

Let :

$$U_{2n} = U_{2(n-k_n)} + 2k_n \quad \text{and} \quad V_{2n} = V_{2(n-k_n)} \tag{13.1.2}$$

Assign to n the value $n + 1$ ($n \longleftarrow n + 1$ and return to **A)**

End :

Outputs for integers less than 10^4 :

Print ($2n$ = ● ; $2n$ - 3 = ● ; W_{2n} = ● ; T_{2n} = ● ; V_{2n} = ● ; U_{2n} = ●)

Outputs for large integers :

Print ($2n$ - P = ● ; $2n$ - 3 - P = ● ; W_{2n} - P = ● ; T_{2n} = ● ; $V_{2n} - P$ = ● ; U_{2n} = ●)

13.2 Program written with Maxima software for $2n$ around 10^{1000}

```
c : 10**1000 ; for  n :c + 40000  step 2  thru  c + 40100  do

( b:2,test : 0 , b : next_prime(b) ,e :n - b ,

if primep(e )

then print( n - c , b , e - c )

else   while  test = 0  do ( e :n -b , if  primep(e )

then  test:1 , print( n - c , b  , e - c )

else  test : 0 ,b:next-prime(b)) ;
```

13.3 Program written with Maplesoft Maple for $2n$ around 10^{1000}

```
G :=  10^1000:
V :=  [1, 11, 13, 17, 19, 23, 29]:
A :=  G + 500000:
B :=  A + 59:
b:=2:
st := time( ):
for  q  from  A  by  6  to B  do  #   Program modulo 30 .using the results of Lemma 11
                                      Possibility of inverting the two loops or defining three
                                      similar structures with s := 0, 1, 2.
    for  s  from  0  to  2  do
 n :=  q + s  + s:
   b :=  trunc(0.59b - 20); # Improving computation time: the idea is to recognise that for
                              any integer n large enough there exists a Goldbach
                              decomponent  p'_n and a successor  p'_n+1  such that
```

(E): $|p'_{n+1} - p'_n| < k \ln^s(n)$ *; this reduces the number of 'nextprime(•)' operations which take up the most computing time.* (*If* $G = 10^{500}$: *Computingtime is around* 10 *sec for thirty terms;The algorithm can be refined by exploiting frame* (E). *Cesàro averages can also be used to determine the initial condition for b*).

```
t:= 0:
R := [[1, 5], [1], [5]]: Q := [[1, 7, 11, 13, 17, 19, 23, 29], [1, 13, 19], [11, 17, 23], [7, 13, 17,
19, 23, 29], [1, 7, 19], [11, 17, 23, 29], [1, 11, 13, 19, 23, 29], [1, 7, 13], [17, 23, 29], [1, 7, 11,
17, 19, 29], [1, 19, 7, 13], [11, 23, 29], [1, 23, 7, 17, 11, 13], [7, 19, 13], [11, 17, 29]]:
        while t = 0 do .
           b := nextprime(b + 100); # Additional test possible by improving Lemma 11.
                          (with V mod 30).
                          # Possibility of replacing nextprime with a faster
                          procedure ( see Sainty [37]).( the computation time is
                          greatly reduced by replacing with b:=nextprime(b +
                          k(b,G)), k(b,G) constant around 150 for G=10^1000,
                          k(b,G) chosen randomly with the rand procedure or very
                          slowly increasing as a function of b and G ), but in
                          general we don't obtain the E.G.D. but any Goldbach
                          decomponents.
        e := n - b;
        K := e mod 6;
              if K in R[s+1] then
                  if isprime(e)
                  Then t := 1;
                  print(n - G, b, e - G);
              end if;
           end if;
        end do:
     end do:
end do:
Computingtime:= time( ) - st;
```

RESULTS :

$G = 10^{1000}$

b:= nextprime(b+rand(100..150))	***b:= nextprime(b+100)***	***b:= nextprime(b+150)***
n - G b n - G - b		
500000, 54133, 445867	500000, 139387, 360613	500000, 361069, 138931
500002, 40693, 459309	500002, 40693, 459309	500002, 40693, 459309
500004, 422393, 77611	500004, 731447, -231443	500004, 535637, -35633
500006, 49157, 450849	500006, 54139, 445867	500006, 277789, 222217
500008, 222991, 277017	500008, 205651, 294357	500008, 205651, 294357
500010, 259451, 240559	500010, 100109, 399901	500010, 138959, 361051
500012, 521981, -21969	500012, 40693, 459319	500012, 40693, 459319
500014, 622561, -22547	500014, 261823, 238191	500014, 145501, 354513
500016, 342929, 157087	500016, 82913, 417103	500016, 198659, 301357
500018, 25097, 474921	500018, 300889, 199129	500018, 26309, 473709
500020, 95083, 404937	500020, 12583, 487437	500020, 77347, 422673
500022, 201821, 298201	500022, 233591, 266431	500022, 160709, 339313
500024, 226337, 273687	500024, 159871, 340153	500024, 162553, 337471
500026, 255859, 244167	500026, 106087, 393939	500026, 106087, 393939
500028, 8147, 491881	500028, 608459, -108431	500028, 263009, 237019
500030, 83833, 416197	500030, 30347, 469683	500030, 151813, 348217
500032, 43261, 456771	500032, 43261, 456771	500032, 24049, 475983
500034, 162251, 337783	500034, 201833, 298201	500034, 400031, 100003
500036, 179203, 320833	500036, 186859, 313177	500036, 145037, 354999
500038, 12601, 487437	500038, 95101, 404937	500038, 854257, -354219
500040, 608471,-108431	500040, 121763, 378277	500040, 121763, 378277
500042, 157103, 342939	500042, 9029, 491013	500042, 8161, 491881
500044, 145531, 354513	500044, 148663, 351381	500044, 145987, 354057
500046, 440303, 59743	500046, 304847, 195199	500046, 304847, 195199
500048, 162577, 337471	500048, 157109, 342939	500048, 12611, 487437
500050, 258637, 241413	500050, 40459, 459591	500050, 163729, 336321
500052, 111791, 388261	500052, 8171, 491881	500052, 100151, 399901
500054, 139661, 360393	500054, 223037, 277017	500054, 155291, 344763
500056, 126397, 373659	500056, 49207, 450849	500056, 126397, 373659
500058, 40739, 459319	500058, 301349, 198709	500058, 208277, 291781
500060, 106121, 393939		500060, 67547, 432513
om***Computtime***:= **179.343 sec**	***Computtime***:= **188.250 sec**	***Computime***:= **163.828 sec**

b:= nextprime(b+rand(150..175))   ***b:= nextprime(b+rand(140..160))***

n-G b n-b-G	***n-G b n-b-G***	
500000, 139387, 360613	500000, 112429,-387571	Record : 116 sec; see in researchgate files PDFGOLDBACHTEST4,10 (*For n from G*+5000000 *to* 5000058 *by* 2), [37].
500002, 90481, 409521	500002, 40693, 459309	
500004, 422393, 77611	500004, 277787, 222217	
500006, 145007, 354999	500006, 82903, 417103	
500008, 604339, -104331	500008, 148627, 351381	
500010, 138959, 361051	500010, 139397, 360613	
500012, 221021, 278991	500012, 40693, 459319	
500014, 334843, 165171	500014, 145501, 354513	

500016, 297779, 202237 500018, 167267, 332751 500020, 54577, 445443 500022, 139409, 360613 500024, 336491, 163533 500026, 12589, 487437 500028, 263009, 237019 500030, 145517, 354513 500032, 334861, 165171 500034, 163697, 336337 500036, 318979, 181057 500038, 221047, 278991 500040, 761591, -261551 500042, 178691, 321351 500044, 54601, 445443 500046, 174989, 325057 500048, 84229, 415819 500050, 163729, 336321 500052, 159899, 340153 500054, 155291, 344763 500056, 166183, 333873 500058, 151841, 348217 ***Computtime*:= 174.438 *sec***	500016, 388313, 111703 500018, 258329, 241689 500020, 77347, 422673 500022, 453683, 46339 500024, 67511, 432513 500026, 221197, 278829 500028, 263009, 237019 500030, 112459, 387571 500032, 178681, 321351 500034, 208253, 291781 500036, 274019, 226017 500038, 14071, 485967 500040, 162257, 337783 500042, 361111, 138931 500044, 52903, 447141 500046, 582299, -82253 500048, 8167, 491881 500050, 67537, 432513 500052, 111791, 388261 500054, 126641, 373413 500056, 126397, 373659 500058, 40739, 459319 ***Computtime*:= 138.578 *sec***	

500000, 9473, 490527
500002, 24019, 475983
500004, 8123, 491881
500006, 9479, 490527
500008, 25087, 474921
500010, 57917, 442093
500012, 8999, 491013
500014, 9001, 491013
500016, 40697, 459319
500018, 9491, 490527
500020, 9007, 491013
500022, 139409, 360613
500024, 9011, 491013
500026, 9013, 491013
500028, 8147, 491881
500030, 26321, 473709
500032, 24049, 475983
500034, 54167, 445867

500036, 57943, 442093
500038, 9511, 490527
500040, 57947, 442093
500042, 8161, 491881
500044, 24061, 475983
500046, 162263, 337783
500048, 8167, 491881
500050, 12613, 487437
500052, 8171, 491881
500054, 9041, 491013
500056, 9043, 491013
500058, 40739, 459319

Computingtime : 343.453 sec

$$G = 10^{2000}$$

n - G	*n - b - G*	*b*
40000,	39957,	43
40002,	39091,	911
40004,	39957,	47
40006,	39549,	457
40008,	25369,	14639
40010,	39957,	53
40012,	39549,	463
40014,	17737,	22277
40016,	39957,	59
40018,	39957,	61
40020,	39091,	929
40022,	39141,	881
40024,	39957,	67
40026,	35443,	4583
40028,	39957,	71
40030,	39957,	73
40032,	39091,	941

n - G	*b*	*n - b - G*
40050,	86117,	-46067
40052,	503,	39549
40054,	97,	39957
40056,	89393,	-49337
40058,	101,	39957
40060,	103,	39957
40062,	971,	39091
40064,	107,	39957
40066,	109,	39957
40068,	977,	39091
40070,	113,	39957
40072,	523,	39549
40074,	983,	39091
40076,	16937,	23139
40078,	937,	39141
40080,	4637,	35443
40082,	941,	39141

40034,	35443,	4591
40036,	39957,	79
40038,	39091,	947
40040,	39957,	83
40042,	23139,	16903
40044,	39091,	953
40046,	39957,	89
40048,	39549,	499

40084, 127, 39957
40086, 4643, 35443
40088, 131, 39957
40090, 541, 39549
40092, 4649, 35443
40094, 137, 39957
40096, 139, 39957
40098, 31991, 8107
40100, 1009, 39091

$$\mathbf{G = 10^{3000}}$$

n - G ***b*** ***n - b - G***

100000, 36529, 63471
100002, 77069, 22933
100004, 22717, 77287
100006, 181873, -81867
100008, 12239, 87769
100010, 4547, 95463
100012, 4549, 95463
100014, 22727, 77287
100016, 59497, 40519
100018, 24847, 75171
100020, 12251, 87769
100022, 12253, 87769
100024, 4561, 95463
100026, 22739, 77287
100028, 22741, 77287
100030, 4567, 95463
100032, 12263, 87769
100034, 36563, 63471
100036, 42649, 57387
100038, 12269, 87769
100040, 23143, 76897
100042, 36571, 63471
100044, 43973, 56071

100046, 4583, 95463
100048, 24877, 75171
100050, 12281, 87769

$G = 10^{5000}$

n - G   b   n - b - G

100000, 31147, 68853
100002, 309371, -209369

100004, 31151, 68853
100006, 31153, 68853
100008, 12569, 87439

100010, 13553, 86457
100012, 31159, 68853
100014, 108923, -8909
100016, 12577, 87439
100018, 592237, -492219
100020, 104987, -4967
100022, 12583, 87439
100024, 13567, 86457
100026, 18287, 81739
100028, 12589, 87439
100030, 31177, 68853
100032, 61871, 38161
100034, 13577, 86457
100036, 31183, 68853
100038, 108947, -8909
10038, 108947, -8909
100040, 12601, 87439
100042, 31189, 68853

n - G   b   n - b - G

100050, 12611, 87439
100052, 12613, 87439

100054, 13597, 86457
100056, 105023, -4967
100058, 12619, 87439

100060, 54151, 45909
100062, 108971, -8909
100064, 103091, -3027
100066, 87943, 12123
100068, 18329, 81739
100070, 13613, 86457
100072, 31219, 68853
100074, 264881, -
100076, 12637, 87439
100078, 107971, -7893
100080, 12641, 87439
100082, 76913, 23169
100084, 13627, 86457
100086, 12647, 87439
100088, 61927, 38161
100090, 13633, 86457
100092, 12653, 87439

n - G   b   n - b - G

100100, 31247, 68853
100102, 31249, 68853
100104, 105071, -4967
100106, 13649, 86457
100108, 640669, -540561
100110, 12671, 87439
100112, 31259, 68853
100114, 87991, 12123
100116, 122033, -21917
100118, 18379, 81739

100044, 457091, -357047
100046, 18307, 81739
100048, 13591, 86457

100094, 61933, 38161
100096, 87973, 12123
100098, 12659, 87439

100120, 31267, 68853
100122, 61961, 38161

100124, 31271, 68853

100126, 13669, 86457
100128, 12689, 87439
100130, 31277, 68853
100132, 76963, 23169
100134, 122051, -21917
100136, 12697, 87439
100138, 13681, 86457
100140, 18401, 81739
100142, 12703, 87439
100144, 13687, 86457
100146, 152993, -52847
100148, 13691, 86457
100150, 13693, 86457

1000000, 35509, 964491
1000002, 113, 999889
1000004, 69193, 930811
1000006, 95233, 904773
1000008, 69197, 930811
1000010, 31873, 968137
1000012, 35521, 964491

1000014, 69203, 930811
1000016, 127, 999889
1000018, 35527, 964491
1000020, 131, 999889

Maple program corrected and improved by the online code checker (IA) : CODEGPT , Yihao.com and AI CLAUDE, (see Sainty [37]).

14 Appendix

Application of Algorithm 13 : Table of extreme Goldbach partitions U_{2n} and V_{2n} computed from program 13.2 ($2 \le 2n \le 10^{1000} + 4020$).

The ** sign in the table below indicates the results given by the algorithm 13 in case **B)** of return to the previous terms of the sequence (G_{2n}).

WATCH OUT !

To simplify the display of large numbers n ($2n > 10^9$) the results are entered as follows :

$$2n - P\ ,\ (2n - 3) - P\ ,\ W_{2n} - P\ ,\ T_{2n}\ ,\ V_{2n} - P \text{ and } U_{2n}$$

with

$P = M, R, G, S,$ or T constants defined in (2.3)

$2n$	$2n - 3$	W_{2n}	$T_{2n}=2n - W_{2n}$	V_{2n}	U_{2n}
4	1	X	X	**2**	**2**
6	3	3	3	**3**	**3**

8	5	5	3	5	3
10	7	7	3	7	3
12	9	7	5	7	5
14	11	11	3	11	3
16	13	13	3	13	3
18	15	13	5	13	5
20	17	17	3	17	3
22	19	19	3	19	3
24	21	19	5	19	5
26	23	23	3	23	3
28	25	23	5	23	5
30	27	23	7	23	7
32	29	29	3	29	3
34	31	31	3	31	3
36	33	31	5	31	5
38	35	31	7	31	7
40	37	37	3	37	3
80	77	73	7	73	7
82	79	79	3	79	3
84	81	79	5	79	5
86	83	83	3	83	3
88	85	83	5	83	5
90	87	83	7	83	7
92	89	89	3	89	3
94	91	89	5	89	5
96	93	89	7	89	7
**98	95	89	9	79	19
100	97	97	3	97	3
120	117	113	7	113	7
**122	119	113	9	109	13
124	121	113	11	113	11
126	123	113	13	113	13
**128	125	113	15	109	19
130	127	127	3	127	3
132	129	127	5	127	5
134	131	131	3	131	3
136	133	131	5	131	5
138	135	131	7	131	7
140	137	137	3	137	3

**500	497	491	9	487	13
502	499	499	3	499	3
504	501	499	5	499	5
506	503	503	3	503	3
508	505	503	5	503	5
510	507	503	7	503	7
1000	997	997	3	997	3
1002	999	997	5	997	5
1004	1001	997	7	997	7
**1006	1003	997	9	983	23
1008	1005	997	11	997	11
1010	1007	997	13	997	13
1012	1009	1009	3	1009	3
1014	1011	1009	5	1009	5
1016	1013	1013	3	1013	3
1018	1015	1013	5	1013	5
10002	9999	9973	29	9973	29
10004	10001	9973	31	9973	31
**10006	10003	9973	33	9923	83
**10008	10005	9973	35	9967	41
10010	10007	10007	3	10007	3
10012	10009	10009	3	10009	3
10014	10011	10009	5	10009	5
10016	10013	10009	7	10009	7
**10018	10015	10009	9	10007	11
10020	10017	10009	11	10009	11
$2n - M$	$(2n - 3) - M$	$W_{2n} - M$	$T_{2n} = 2n - W_{2n}$	$V_{2n} - M$	U_{2n}
+1000	+997	+993	7	+993	7
**+1002	+999	+993	9	+931	71
+1004	+1001	+993	11	+993	11
+1006	+1003	+993	13	+993	13
**+1008	+1005	+993	15	+919	89
+1010	+1007	+993	17	+993	17
+1012	+1009	+993	19	+993	19
+1014	+1011	+1011	3	+1011	3
+1016	+1013	+1011	5	+1011	5
+1018	+1015	+1011	7	+1011	7
**+1020	+1017	+1011	9	+931	89

2n - R	*(2n - 3) - R*	W_{2n} - *R*	T_{2n} = *2n* - W_{2n}	V_{2n} - *R*	U_{2n}
**+1000	+997	+979	21	+903	97
+1002	+999	+979	23	+979	23
**+1004	+1001	+979	25	+951	53
**+1006	+1003	+979	27	+903	103
+1008	+1005	+979	29	+979	29
+1010	+1007	+979	31	+979	31
**+1012	+1009	+979	33	+951	61
**+1014	+1011	+979	35	+ 781	233
+1016	+1013	+979	37	+979	37
**+1018	+1015	+979	39	+951	67
+1020	+1017	+1017	3	+1017	3

2n - G	*(2n - 3) - G*	W_{2n} - *G*	T_{2n} = *2n* - W_{2n}	V_{2n} - *G*	U_{2n}
**+10000	+9997	+9631	369	+7443	2557
**+10002	+9999	+9631	371	+9259	743
+10004	+10001	+9631	373	+9631	373
**+10006	+10003	+9631	375	+8583	1423
**+10008	+ 10005	+9631	377	+6637	3371
+10010	+10007	+9631	379	+9631	379
**+10012	+10009	+9631	381	+8583	1429
+10014	+10011	+9631	383	+9631	383
**+10016	+10013	+9631	385	+9259	757
**+10018	+10015	+9631	387	+4491	5527
+10020	+10017	+9631	389	+9631	389

2n-S	*(2n-3)-S*	W_{2n} - *S*	T_{2n} = *2n* - W_{2n}	V_{2n} - *S*	U_{2n}
**+20000	+19997	+18031	1969	+17409	2591
**+20002	+19999	+18031	1971	+ 17409	2593
+20004	+20001	+18031	1973	+18031	1973
**+20006	+20003	+18031	1975	+16663	3343
**+20008	+20005	+18031	1977	+16941	3067
+20010	+20007	+18031	1979	+18031	1979
**+20012	+20009	+18031	1981	+5671	14341
**+20014	+20011	+18031	1983	+4101	15913
**+20016	+20013	+18031	1985	+3229	16787
+20018	+20015	+18031	1987	+18031	1987
**+20020	+20017	+18031	1989	+16941	3079

2n-T	*(2n-3)-T*	W_{2n}-*T*	T_{2n} = *2n* - W_{2n}	$V_{2n} - T$	U_{2n}
**+40000	+39997	+29737	10263	+ 21567	18433
**+40002	+39999	+29737	10265	+ 22273	17729

+40004	+40001	+29737	10267	+29737	10267
**+40006	+40003	+29737	10269	+21567	18439
+40008	+40005	+29737	10271	+29737	10271
+40010	+ 40007	+29737	10273	+29737	10273
**+40012	+40009	+29737	10275	+10401	29611
**+40014	+40011	+29737	10277	-56003	96017
**+40016	+40013	+29737	10279	+27057	12959
**+40018	+40015	+29737	10281	+25947	14071
**+40020	+40017	+29737	10283	+24493	15527

15 Appendix

7-3=4	11-5=6	11-3=8	13-3=10	17-5=12	17-3=14	19-3=16	23-5=18
23-3=20	29-7=22	29-5=24	29-3=26	31-3=28	37-7=30	37-5=32	37-3=34
41-5=36	41-3=38	43-3=40	47-5=42	47-3=44	53-7=46	53-5=48	53-3=50
59-7=52	59-5=54	59-3=56	61-3=58	67-7=60	67-5=62	67-3=64	71-5=66
71-3=68	73-3=70	79-7=72	79-5=74	79-3=76	83-5=78	83-3=80	89-7=82
89-5=84	89-3=86	101-13=88	97-7=90	97-5=92	97-3=94	101-5=96	101-3=98
103-3=100	107-5=102	107-3=104	109-3=106	113-5=108	113-3=110	131-19=112	127-13=114
127-11=116	131-13=118	127-7=120	127-5=122	127-3=124	131-5=126	131-3=128	137-7=130
137-5=132	137-3=134	139-3=136	149-11=138	151-11=140	149-7=142	149-5=144	149-3=146
151-3=148	157-7=150	157-5=152	157-3=154	163-7=156	163-5=158	163-3=160	167-5=162
167-3=164	173-7=166	173-5=168	173-3=170	179-7=172	179-5=174	179-3=176	181-3=178
191-11=180	193-11=182	191-7=184	191-5=186	191-3=188	193-3=190	197-5=192	197-3=194
199-3=196	211-13=198	211-11=200	233-31=202	211-7=204	211-5=206	211-3=208	223-13=210
229-17=212	227-13=214	223-7=216	223-5=218	223-3=220	227-5=222	227-3=224	229-3=226
233-5=228	233-3=230	239-7=232	239-5=234	239-3=236	241-3=238	251-11=240	271-29=242
251-7=244	251-5=246						

16 Appendix

$$T_r(K)$$

	$q_1 = 3$	$q_2 = 5$	$q_3 = 7$	$q_4 = 11$	$q_5 = 13$	$q_6 = 17$	$q_7 = 19$	$q_8 = 23$	$q_9 = 29$	$q_{10} = 31$	$q_{11} = 37$
$2K = 2$	5	7		13		19			31		
$2K = 4$	7		11		17		23				41

$2K = 6$		11	13	17	19	23		29		37	43
$2K = 8$	11	13		19				31	37		
$2K = 10$	13				23		29			41	47
$2K = 12$		17	19	23		29	31		41	43	
$2K = 14$	17	19				31		37	43		
$2K = 16$	19		23		29					47	59
$2K = 18$		23		29	31		37	41	47		61
$2K = 20$	23			31		37		43			67
$2K=22$			29				41			53	
$2K=24$		29	31		37	41	43	47	53		71
$2K=26$	29	31		37		43					73
$2K=28$	31				41		47			59	
$2K=30$			37	41	43	47		53	59	61	
$2K=32$		37		43					61		79
$2K=34$	37		41		47		53				
$2K=36$		41	43	47		53		59		67	83
$2K=38$	41	43						61	67		
$2K=40$	43		47		53		59			71	
$2K=42$		47		53		59	61		71	73	89
$2K=44$	47					61		67	73		
$2K=46$			53		59						
2K=48		53		59	61		67	71		79	
2K=50	53			61		67		73	79		97
2K=52			59				71			83	
2K=54		59	61		67	71	73		83		
2K=56	59	61		67		73		79			
2K=58	61				71					89	
2K=60			67	71	73		79	83	89		

17 Perspectives and generalizations

17.1 Other Goldbach sequences (G'_{2n}) independent of (G_{2n}) may be studied using the increasing sequences of primes (W'_{2n}) defined by

For any integer $n \geq 3$

$$W'_{2n} = \text{Sup} \left(p \in \mathcal{P} : p \leq f(n) \right) \quad (17.1.1)$$

f is a function defined on the interval $I = [3 \,;\, +\infty[$ and satisfying the following conditions

● f is strictly increasing on the interval I

● f (3) = 3 and $\lim_{x \to +\infty} f(x) = +\infty$

● $\forall\, x \in I \quad f(x) \leq 2x - 3$

For example, one of the following functions defined on I can be selected.

■ $f: x \to a\,x + 3 - 3a$ $\qquad (a \in \mathbb{R}: \; 0 < a \leq 2)$

■ $g: x \to [\, 4\sqrt{3x} - 9 \,]$ $\qquad$ ($[x]$ is the integer part of the real x)

■ $h: x \to 6 \ln\left(\frac{x}{3}\right) + 3$

17.2 Using this method it would be interesting to study the Schnirelmann density [39] of primes 3 , 5 , 7, 11 ,......... ... in the sequence (U_{2n}) on variable intervals and the Caesaro sums of U_{2n} E.D.G.'s with a view to more efficient programming for their calculation.

17.3 It is possible to exceed the values shown in the table of $2n = 10^{1000}$ (Many E.G.D have been calculated for values of $2n$ in the order of 10^{2000} , 10^{5000} (and G.D. in the order of 10^{10000} Sainty [37]) by perfecting this algorithm, exploiting the fact that one of Goldbach's decomponents can be chosen equal to $4p + 3$, (G.D. are primes of the form $6m + 1$ or $6m + 5$ and can be expressed more precisely using primes of the form $30m + r$: $r \in$ [1,7,11,13,17,19,23,29] (see Table mod 30, Lemma 11) ,by using De Pocklington Theorem [6],[34],[36] , Primality tests [37], Cipolla-Axler-Dusart type functions and improvment of primes frames [2],[8],[12],[13],[37] via a new Prime number Theorem to better identify the terms of (G_{2n}), supercomputers and more efficients software as C++, or Assembleur compilation.

17.4 Any Goldbach decomponent of order $2n = 10^{10000}$ can be determined more quickly by replacing the instruction b:=2 by b:=trunc($c.b + d$) and b := *nextprime*(b) with b := *nextprime*(b + k(b, G)), where k(b, G) is a constant of around 150 for $G = 10^{1000}$ and is chosen randomly using the rand procedure or increases very slowly as a function of b and G. An increasing sequence of primes, b_k, can also be determined in stages by replacing the initial value b:=2 by b:=trunc($k_0.b$ - $k_1.\ln^s(n)$ - k_2) and by setting c := trunc($a.\ln^d(b)$), $1 \le d \le 2$ and $b := b + c$ for each stage, followed by b := *nextprime*(b) until the next stage, (see Sainty [37]); Note that for any even integer $2n$ large enough there exists G.D. $p'_n , p'_{n+1} , q'_n , q'_{n+1}$ | $p'_n + q'_n = 2n$ and $p'_{n+1} + q'_{n+1} = 2(n + 1)$ with $p'_{n+1} - p'_n$ and $q'_{n+1} - q'_n < k.\ln^s(n)$). It is therefore advisable to develop adaptive algorithms based on this model using A.I., as a function of the program's G parameter.

17.5 Diophantine equations and conjectures of the same nature ((3L) conjecture [9],[21],[23],[26],[27],[44]) can be processed using similar reasoning and algorithms.

■ To validate the (3L) conjecture we study the following sequences of primes (Wl_{2n}), (Vl_{2n}) and (Ul_{2n}) defined by

For any integer $n \geq 3$ $$Wl_{2n} = \text{Sup} (p \in \mathcal{P} : p \leq n - 1) \quad (17.5.1)$$

- If $Tl_{2n} = (2n + 1 - 2\, Wl_{2n})$ is a **prime**

then let

$$Vl_{2n} = Wl_{2n} \quad \text{and} \quad Ul_{2n} = Tl_{2n} \quad (17.5.2)$$

- If Tl_{2n} is a **composite number**

then there exists an integer k $1 \leq k \leq n - 3$ |

$$Ul_{2(n-k)} + 2k \in \mathcal{P} \quad (17.5.3)$$

then let

$$Vl_{2n} = Vl_{2(n-k)} \quad \text{and} \quad Ul_{2n} = Ul_{2(n-k)} + 2k \qquad (17.5.4)$$

■ Using the same type of reasoning a generalization, the (BBG) conjecture of the following form can be validated

- Let K and Q be two odd integers prime to each other :

For any integer n | $2n \geq 3(K+Q)$ there exist two primes Ub_{2n} and Vb_{2n} verifying

$$K.\, Ub_{2n} + Q.\, Vb_{2n} = 2n \qquad (17.5.5)$$

- Let K and Q be two integers of different parity prime to each other :

For any integer n | $2n \geq 3(K+Q)$ there are two primes Ub_{2n} and Vb_{2n} verifying

$$K.Ub_{2n} + Q.Vb_{2n} = 2n + 1 \qquad (17.5.6)$$

17.6 Remark.

GOLDBACH (-) :

$R_{2K} = \text{Inf}\,(p \in \mathcal{P} : p - 2K \in \mathcal{P})$ and $\underline{Q_{2K}} = \text{Inf}\,(p \in \mathcal{P} : 2K + p \in \mathcal{P}) = R_{2K} - 2K$

GOLDBACH (+) :

$V_{2K} = \text{Sup}\,(p \in \mathcal{P} : 2K - p \in \mathcal{P})$ and $\underline{U_{2K}} = \text{Inf}\,(p \in \mathcal{P} : 2K - p \in \mathcal{P}) = 2K - V_{2K}$

(It is possible to envisage symmetries in the Goldbach triangle).

17.7 The sequences (Wq_{2n}) generate all the G.D. and may enable us to better estimate the values of distribution function G of the Goldbach's Comet, probably of type :

$$a_1.E / \ln^2(E) < G(E) < a_2.E / \ln^2(E) \qquad , \text{ see Woon [48] .}$$

18 Conclusion

18.1 A recurrent and explicit Goldbach sequence $(G_{2n}) = (U_{2n} ; V_{2n})$ verifying

$$\forall n \in \mathbb{N}+2 \qquad U_{2n} , V_{2n} \in \mathcal{P} \quad \text{and} \quad U_{2n} + V_{2n} = 2n$$

has been developed using an simple and efficient "localised" algorithm. The Goldbach conjecture has been proved by strong recurrence (absurd and finite descent), and a relation (Proposition 10) is established between the fundamental theorem of arithmetic and the Goldbach conjecture (sum and product of primes), allowing fast computation of G.D. of very large even integers using a generalized Pocklington-type algorithm and perhaps another demonstration of Goldbach's conjecture via the decomposition of $2n$ into prime factors.

18.2 The records of Silva [41] and Deshouillers, te Riele, Saouter [11] are beaten on a personal computer. Hundreds E.G.D. U_{2n} and V_{2n} are obtained for values around $2n = 10^{1000}$, twenty-six around $2n = 10^{2000}$, seventy-five around $2n = 10^{5000}$ and G.D. around $2n = 10^{10000}$ for a computation time of less than three hours (see Sainty [37]).

18.3 For a given integer $n \geq 49$ the evaluation of the terms U_{2n} and V_{2n} does not require the computing of all previous terms U_{2k} and $V_{2k} \;\big|\; 1 \leq k < n - 1$. we will only consider those that verify :

$$U_{2k} \leq 5.\ln^{1.3}(2n) \quad \text{and} \quad 2n - 5.\ln^{1.3}(2n) \leq V_{2k} \leq 2n \qquad (\textit{on average}) \quad \textbf{(}18.3.1)$$

This property allows any E.G.D U_{2n} and V_{2n} to be calculated quite quickly, the upper limit being defined by the scientific software and the computer's ability to determine the largest prime preceding $2n$ - 2 (*next or prevprime*($2n$ - 2) function).

18.4 Therefore the (BBG), the (3L) and the binary Goldbach(- / +) conjectures "Any even integer greater than three is the sum and difference of two primes" are true.

In fact these two conjectures are intertwined.

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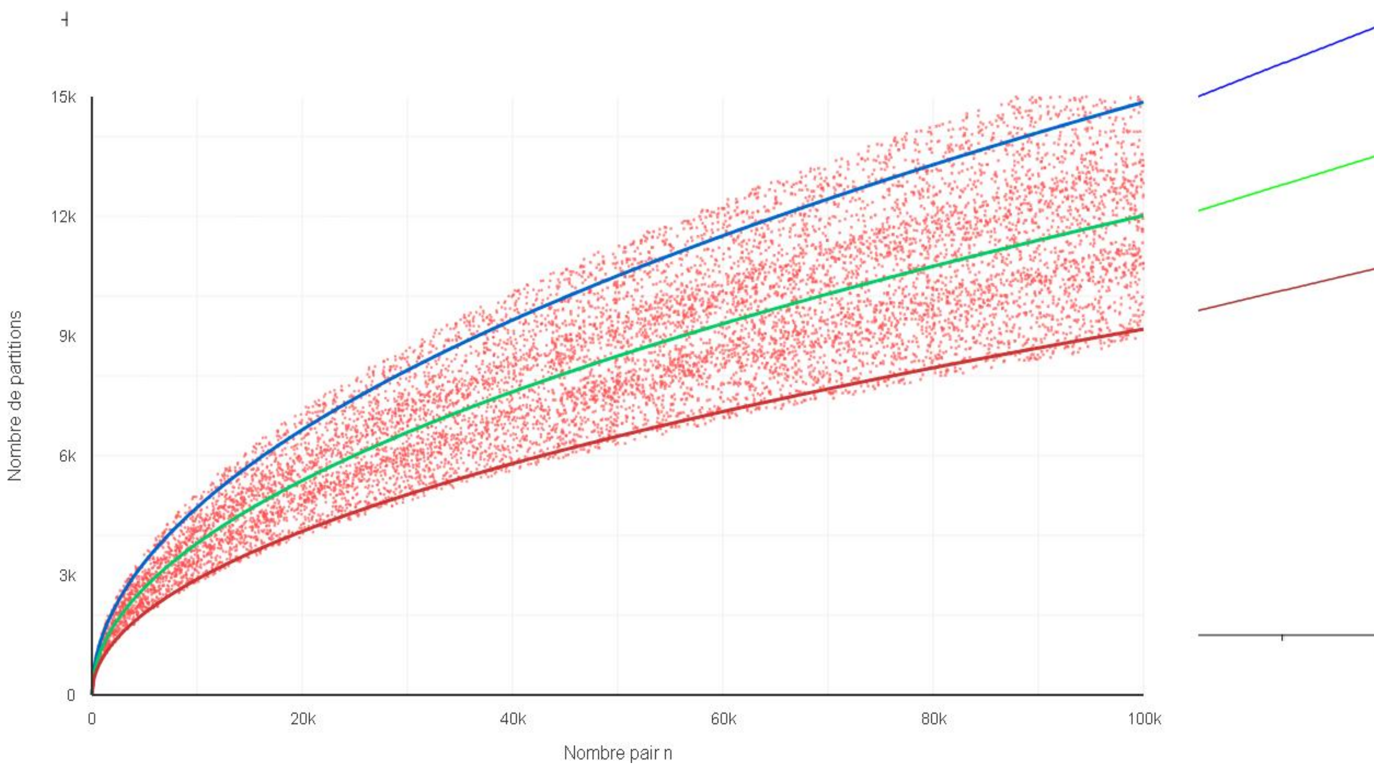
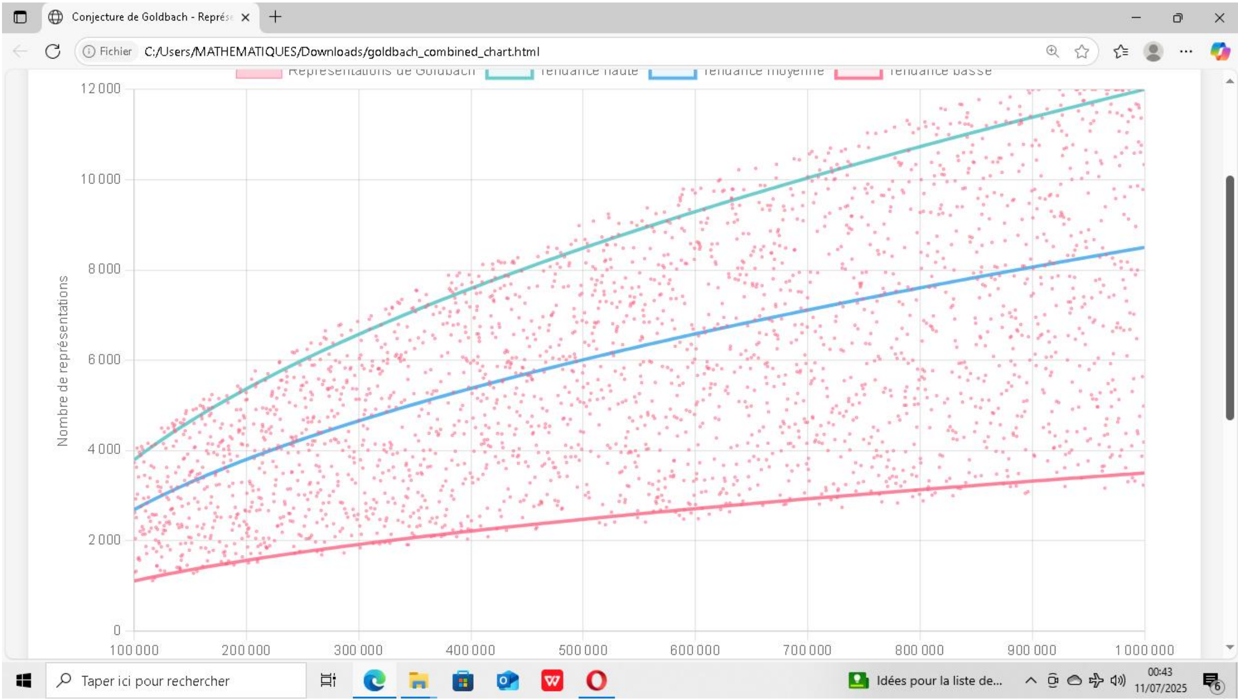


Framing (via AI CLAUDE) and mean value of the Goldbach comet by functions of the type f : x -> a.x / $\ln^2(x)$, (to be specified).


Conjecture de Goldbach - Représ
Fichier C:/Users/MATHEMATIQUES/Downloads/goldbach_combined_chart.html
Représentations de Goldbach
Tendance haute
Tendance moyenne
Tendance basse
Nombre de représentations
12 000
10 000
8 000
6 000
4 000
2 000
0
100 000
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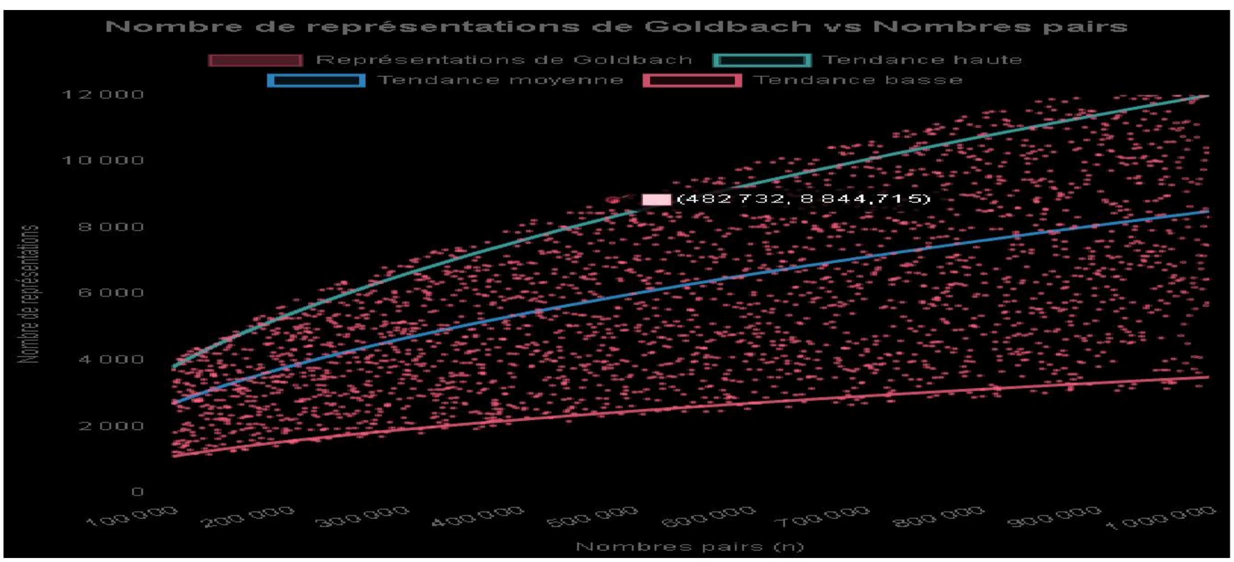
Nombre de représentations de Goldbach vs Nombres pairs
Représentations de Goldbach
Tendance haute
Tendance moyenne
Tendance basse
Nombre de représentations
12 000
10 000
8 000
6 000
4 000
2 000
0
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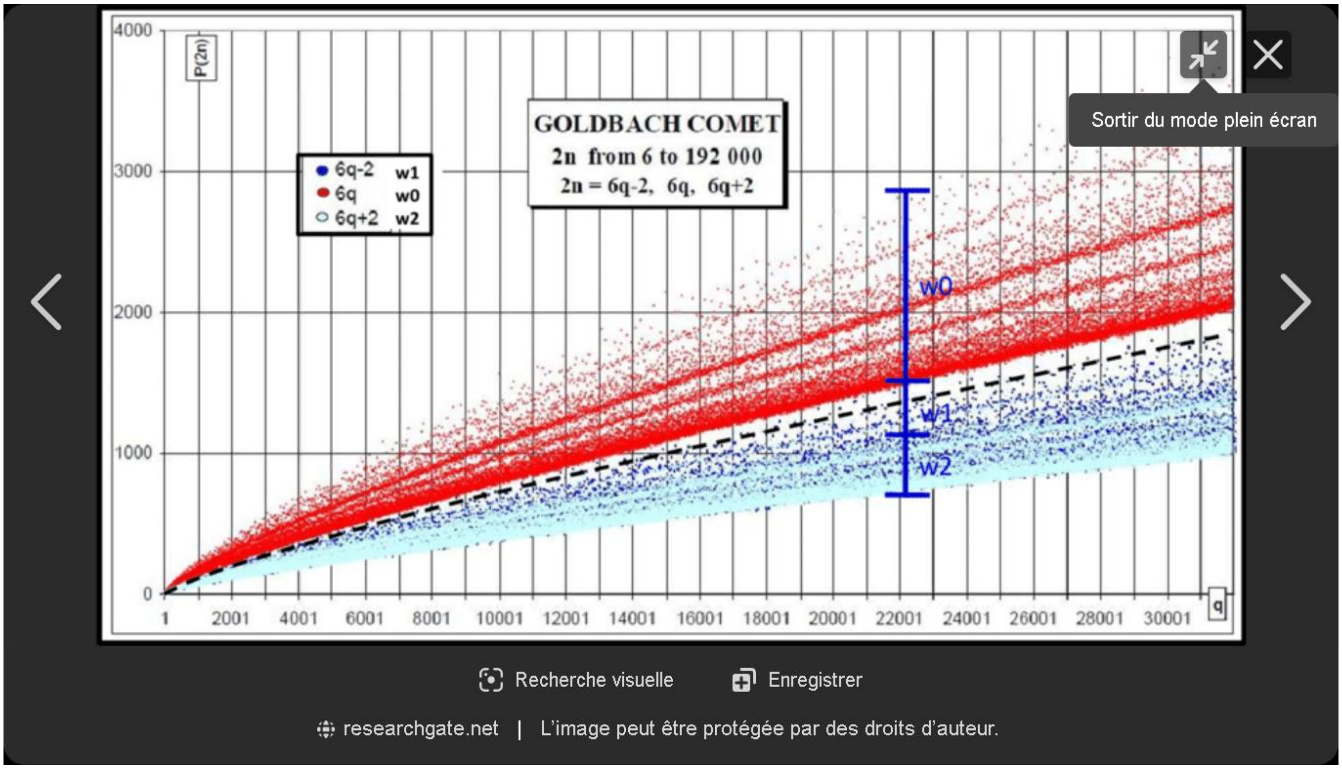
1 000 000
Nombres pairs (n)


GOLDBACH COMET
2n from 6 to 192 000
2n = 6q-2, 6q, 6q+2
6q-2 w1
6q w0
6q+2 w2
P(2n)
q
w0
w1
w2
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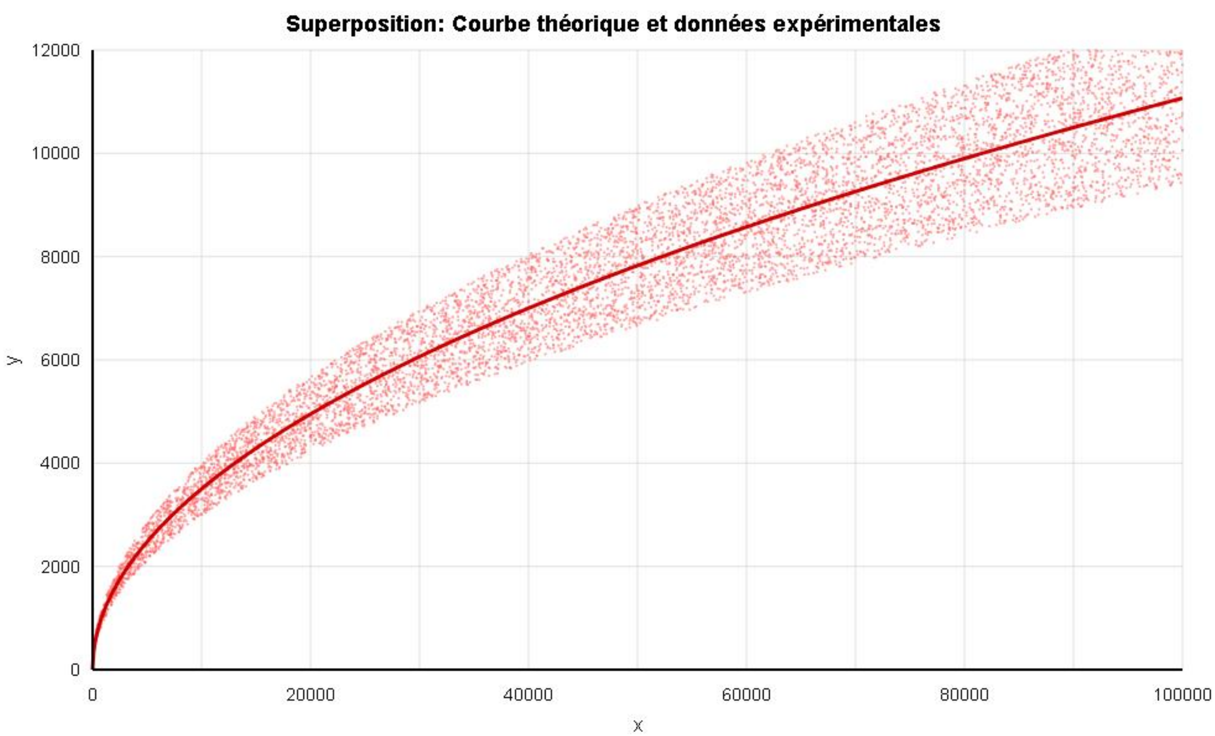
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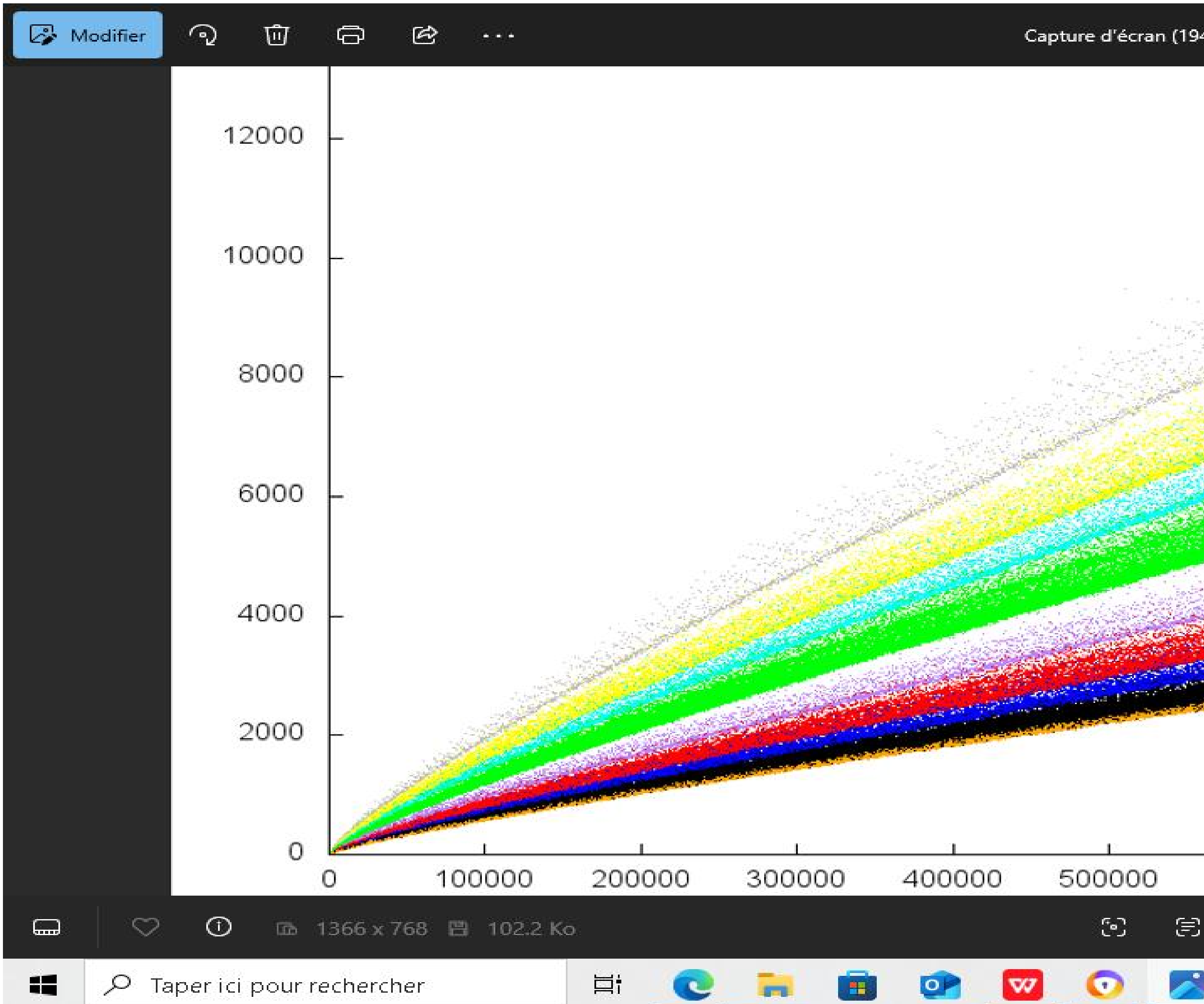
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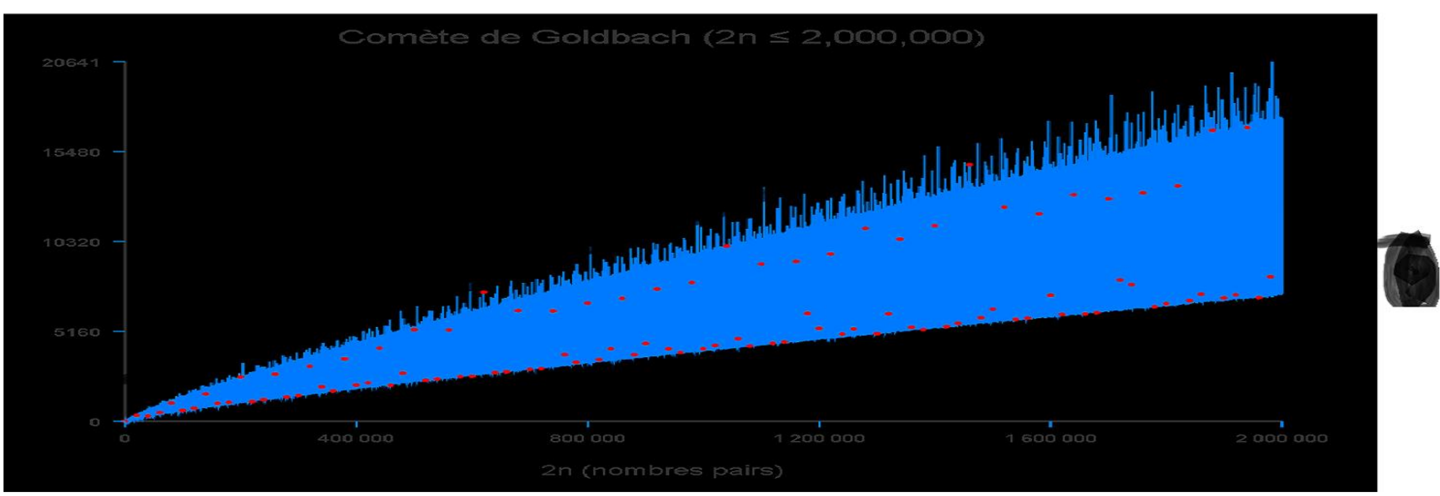
Comète de Goldbach (2n ≤ 2,000,000)
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15480
10320
5160
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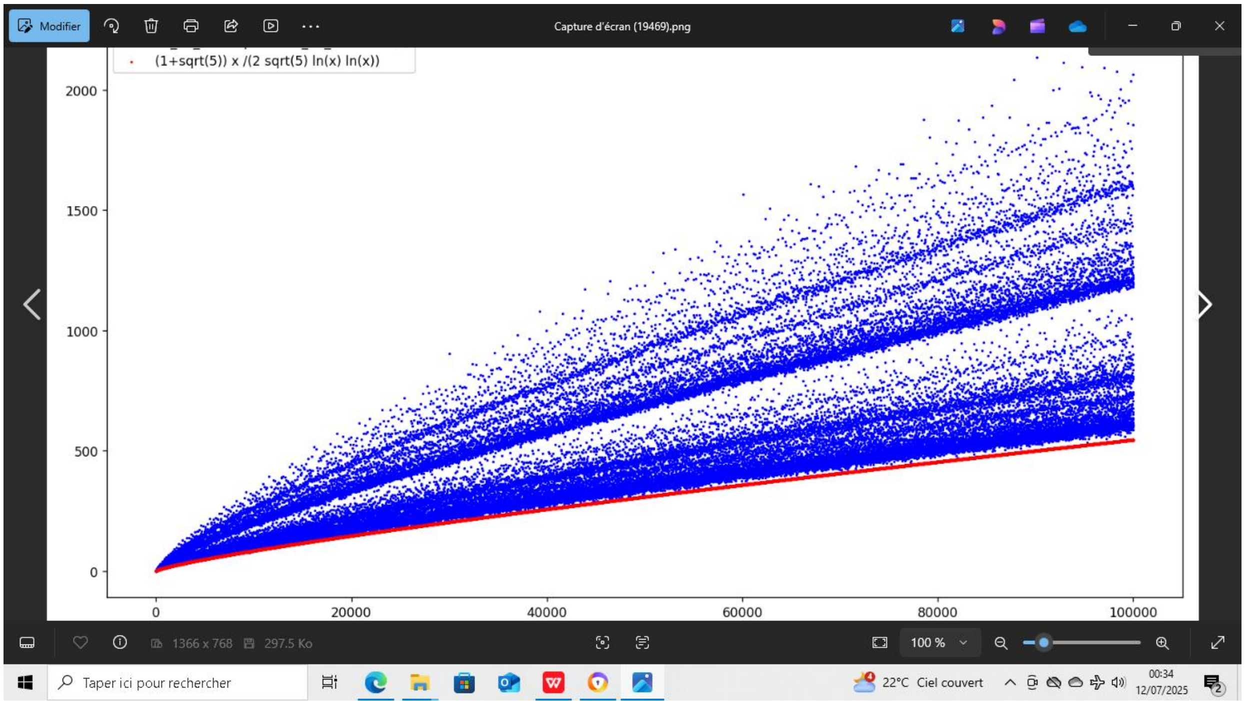
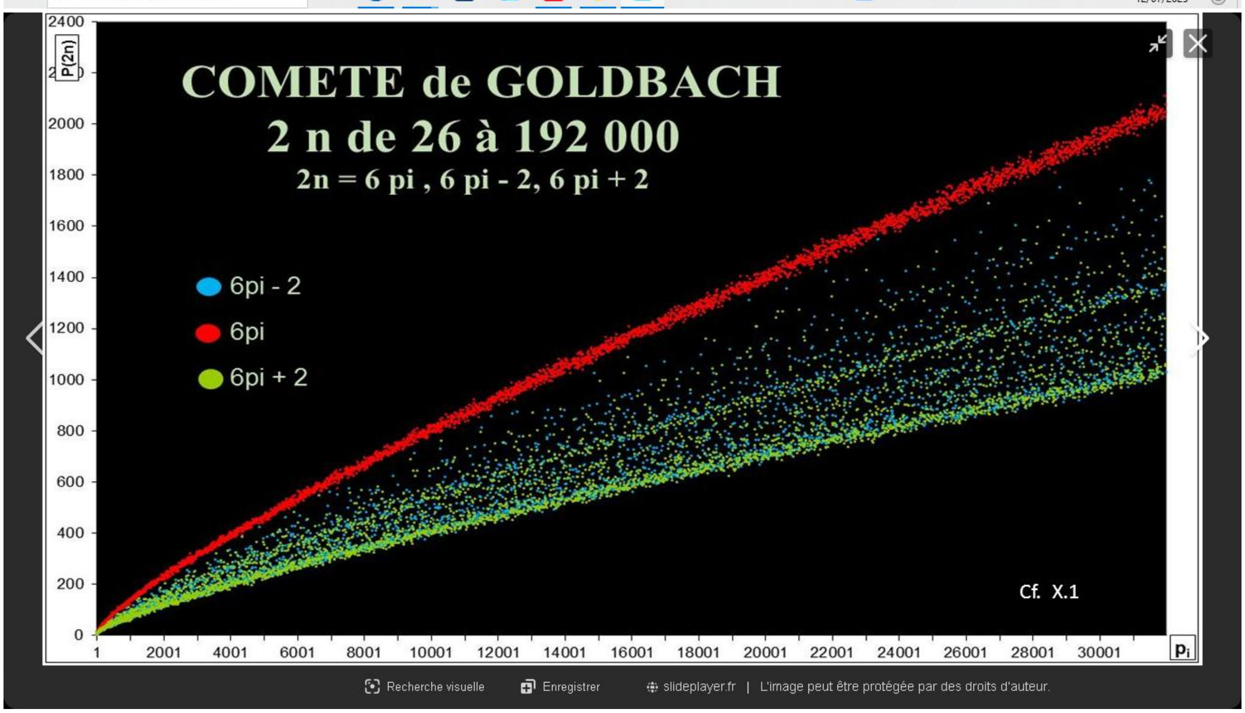
2n (nombres pairs)





Comments :

The majority of mathematicians believe Goldbach's conjecture to be true, mainly,, based on statistical reasoning centred on the distribution of primes. The larger the number, the more ways there are to decompose it into a sum of two or three other primes. A crude heuristic approach to this argument (for the Binary Goldbach Conjecture) is to consider the prime number theorem, this states that a randomly chosen integer ***m*** has a probability of being prime equal to **1/ln(*m*).**

. Therefore, if n is a large even integer and m is a number between **3** and n, the probability that both m and $(n - m)$ are primes is approximately $1/(\ln(n).\ln(n - m))$. Although this heuristic argument is imperfect for several reasons, such as the lack of consideration of correlations between the probabilities of m and $(n - m)$ being primes, it nevertheless indicates that the total number of ways of writing a large even integer n as the sum of two odd primes is approximately proportional to $n / \ln^2(n)$.