Five Hard Problems with a Simple Solution

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Abstract. We present a collection of five nontrivial exercises in number theory (Questions 1–4) and graph theory (Question 5). These problems can be efficiently solved using insights and shortcuts derived from the author's previously published papers. This preprint invites readers to test their expertise in these fields and assess their ability to independently solve the proposed exercises. Detailed solutions are included.

Keywords: Congruence speed, Minimum-link covering trails, Tetration, Stable digits.

MSC2020: 11A07, 68R10 (Primary); 11F33 (Secondary).

1 The problems

Question 1. What is the maximum value of the positive integer n (denote it by $n_{\max}$) such that ${}^{1000000000}3333704193 \equiv {}^{31415926535897932384626433832795}3333704193 \pmod{10^n}$, where ${}^{m}3333704193$ means $3333704193^{3333704193^{\cdot^{\cdot^{\cdot}}}}$ m times (e.g., ${}^{3}3 = 3^{3^3} = 3^{27} = 7625597484987$)?

Question 2. What is the smallest positive integer n (denote it by $n_{\min}$) such that, in the decimal numeral system, ${}^{n}202520252025 \equiv {}^{n+2025}202520252025 \pmod{10^{202552022025}}$, where ${}^{n}202520252025$ is the n-th tetration of 202520252025 (i.e., $202520252025^{202520252025^{\cdot^{\cdot^{\cdot}}}}$ n times)?

Question 3. What 4-digit number is formed by juxtaposing (from left to right) the four distinct congruence classes modulo 10 of the differences between:

- the 4000000025-th rightmost digit of ${}^{1000000006}267785184193$ and the 4000000025-th rightmost digit of ${}^{1000000007}267785184193$,
- the 4000000029-th rightmost digit of ${}^{1000000007}267785184193$ and the 4000000029-th rightmost digit of ${}^{1000000008}267785184193$,
- the 4000000033-th rightmost digit of ${}^{1000000008}267785184193$ and the 4000000033-th rightmost digit of ${}^{1000000009}267785184193$,
- the 4000000037-th rightmost digit of ${}^{1000000009}267785184193$ and the 4000000037-th rightmost digit of ${}^{1000000010}267785184193$?

Question 4. What is the congruence class modulo $10^{16309690970750}$ of the following difference: ${}^{2718281828459}2922943 - {}^{3141592653589}2922943$, where ${}^{2718281828459}2922943$ means $2922943^{2922943^{\cdot^{\cdot^{\cdot}}}}$ 2718281828459 times (e.g., ${}^{3}3 = 7625597484987$)?

Question 5. Let the two grids $G_1 := \{0,1,2\}^3 \subset \mathbb{R}^3$ and $G_2 := \{4,5\}^3 \subset \mathbb{R}^3$ be given. What is the minimum number of edges that a closed polygonal chain (i.e., a circuit) must have to visit all 27 points of G_1 first and then all 8 points of G_2, returning to the starting point with its final segment?

2 The solutions

Answer 1. The value of n such that

$$^{1000000000}3333704193 \equiv {}^{31415926535897932384626433832795}3333704193 \pmod{10^n}$$

and

$$^{1000000000}3333704193 \not\equiv {}^{31415926535897932384626433832795}3333704193 \pmod{10^{n+1}}$$

is 9000000009. Hence $n_{\max} = 9000000009$.

Explanation of Answer 1. The *congruence speed* ([3, Def. 1.1]) of the tetration base 3333704193 is certainly stable from height $\nu_5(3333704193^2+1)+2 = 10+2 = 12$, since

$$3333704193^2 + 1 = 11113583646425781250 = 5^{10} \cdot 2 \cdot 42793 \cdot 13296929$$

(see [3, Def. 2.1, p. 447]). We compute the number of stable digits of $^{12}3333704193$ (i.e., the least significant digits that do not change when moving to height 13), and then use the constant congruence speed to lift up to height 10^9.

By [3, Eq. (16)], the constant congruence speed is $V(3333704193) = \nu_2(3333704193-1) = 9$. Moreover,

$$^{12}3333704193 \equiv {}^{13}3333704193 \pmod{10^{117}} \quad \text{and} \quad {}^{12}3333704193 \not\equiv {}^{13}3333704193 \pmod{10^{118}},$$

whence

$$^{10^9}3333704193 \equiv {}^{31415926535897932384626433832795}3333704193 \pmod{10^{9\cdot(10^9-12)+117}}$$

but not modulo $10^{9\cdot(10^9-12)+118}$. Therefore,

$$n_{\max} = 9\cdot(10^9-12)+117 = 9000000009.$$

Answer 2. The smallest natural number n such that $^{n}202520252025 \equiv {}^{n+1}202520252025 \pmod{10^{202552022025}}$ is 67517340674. Hence $n_{\min} = 67517340674$.

Explanation of Answer 2. We apply the constancy of the *congruence speed* (see [3, Def. 1.1]) for the base 202520252025 to prove $n_{\min} = 67517340674$.

From [3, Eq. (16), line 5] we obtain the exact increase in the number of stable digits of $^{m}202520252025$ at any height $m \geq \nu_2(202520252025^2-1)-1+2$. Here,

$$(\nu_2(202520252025^2-1)-1)+2 = (4-1)+2 = 5,$$

and the last 30 digits of $^{1}202520252025, \ldots, {}^{5}202520252025$ are:

$$000000000000000000202520252 0\mathbf{25} \text{ (height 1)},$$
$$409532443620264530181\mathbf{884765625} \text{ (height 2)},$$
$$3475736537948250770\mathbf{056884765625} \text{ (height 3)},$$
$$69167842064052 8\mathbf{202056884765625} \text{ (height 4)},$$
$$451657668687\mathbf{403202056884765625} \text{ (height 5)}.$$

Thus, the congruence speed is 2 at height 1, 7 at height 2, and 3 for every height ≥ 3. Therefore,

$$n_{\min} = \left\lceil \frac{202552022025 - 9}{3} \right\rceil + 2 = \frac{202552022025 - 9}{3} + 2 = 67517340674.$$

Answer 3. The correct answer is 2684.

Explanation of Answer 3. The constant congruence speed of the base 267785184193 equals 4 (since $267785184193 \equiv 13 \pmod{20}$ and $\frac{84193-04193}{10^4} \neq 5$, [3, Eq. (16), line 13] gives $V(267785184193) = \nu_5(267785184193^2 + 1) = 4$).

Using the sufficient condition $\nu_5(267785184193^2 + 1) + 2 = 6$, set $\bar{b}(267785184193) = 6$. This guarantees that, for every $k \geq 0$, the congruence class modulo 10 of the difference between the rightmost non-stable digit of ${}^{\bar{b}+4k}267785184193$ and that of ${}^{\bar{b}+1+4k}267785184193$ is constant across k. Since these classes form a 4-step cycle for any base not a multiple of 10 (see the 2011 book *La strana coda della serie* $n^{n^{\cdot^{\cdot^{\cdot^{n}}}}}$), it suffices to inspect heights $6, 7, 8, 9, 10$.

The last 50 digits of ${}^{1}267785184193, \ldots, {}^{10}267785184193$ are:

$$00000000000000000000000000000000000002677851840\mathbf{193} \text{ (height 1)},$$
$$727467747873102307860513270556648730013131\mathbf{89344193} \text{ (height 2)},$$
$$6970554517069889275015781013215417815 1\mathbf{469989344193} \text{ (height 3)},$$
$$46930577055807235383561110643145 37\mathbf{0215469989344193} \text{ (height 4)},$$
$$2985003507212416357496415566 82\mathbf{000902154699893 44193} \text{ (height 5)},$$
$$865720834243388447908912 20\mathbf{6538000902154699893 44193} \text{ (height 6)},$$
$$87160656879247250823 38\mathbf{990865380009021546998934 4193} \text{ (height 7)},$$
$$8566215479996430 24\mathbf{8162990865380009021546998934 4193} \text{ (height 8)},$$
$$9748384960774 6\mathbf{49768162990865380009021546998934 4193} \text{ (height 9)},$$
$$394465326 9\mathbf{664249768162990865380009021546998934 4193} \text{ (height 10)}.$$

Hence, the cycle $[2, 6, 8, 4]$ arises for the four differences at heights $(6, 7)$, $(7, 8)$, $(8, 9)$, $(9, 10)$, respectively.

Therefore, the juxtaposed number is 2684.

Answer 4. The answer is $5 \cdot 10^{16309690970749}$, since

$${}^{2718281828459}2922943 \equiv {}^{3141592653589}2922943 \pmod{10^{16309690970749}}$$

(the congruence speed for the base 2922943 is 0 at height 1, 7 from heights 2 to 8, and stabilizes to the constant value $V(2922943) = 6$ for every hyperexponent $\bar{b} \geq 9$ [3]).

Moreover, for every $\bar{b} \geq 9$, the absolute value of the difference between the rightmost non-stable digit at heights $\bar{b}$ and $\bar{b} + 1$ is 5.

Explanation of Answer 4. See [4].

Answer 5. The minimum-link closed polygonal chain in $\mathbb{R}^3$ that visits all points of G_1 and then all points of G_2 consists of 18 connected line segments.

Explanation of Answer 5. A constructive proof is given: for each $k \in \mathbb{N} \setminus \{0, 1\}$, the provided upper bound matches the trivial lower bound obtained by combining the general solution $3 \cdot 2^{k-2}$ for any $\{0,1\}^k$ grid [5] with $\frac{3^k-1}{2}$ for any $\{0,1,2\}^k$ grid [1].

Here $k = 3$, so we need at least $\frac{3^3-1}{2} + 3 \cdot 2^{3-1} - 1$ segments to join all vertices of $G_1 \cup G_2$ with a single polygonal chain (since $\{0,1\}^3$ has no more than two collinear points and the last segment covering G_1 can be used to fit two additional points of G_2), leaving 6 points of G_2. Then Lemma 1 of [5] ensures that no covering trail for $G_1 \cup G_2$ exists with fewer than 18 segments.

A closed polygonal chain achieving this bound is

$$P_{18} := (0,1,0) - (0,3,0) - (3,0,3) - (0,0,0) - (0,0,3) - (3,3,0) - (0,0,0) - (0,3,3) - (3,0,0) - (0,3,0) - (0,0,3) - (3,0,0) - (0,0,0) - (6,6,6) - (2,4,4) - (5,4,\tfrac{11}{2}) - (5,4,3) - (5,6,5) - (0,1,0),$$

and $\{0,1,2\}^3 \cup \{4,5\}^3 \subset P_{18}$.

Therefore, the minimum number is 18 (see Figures 1 and 2).

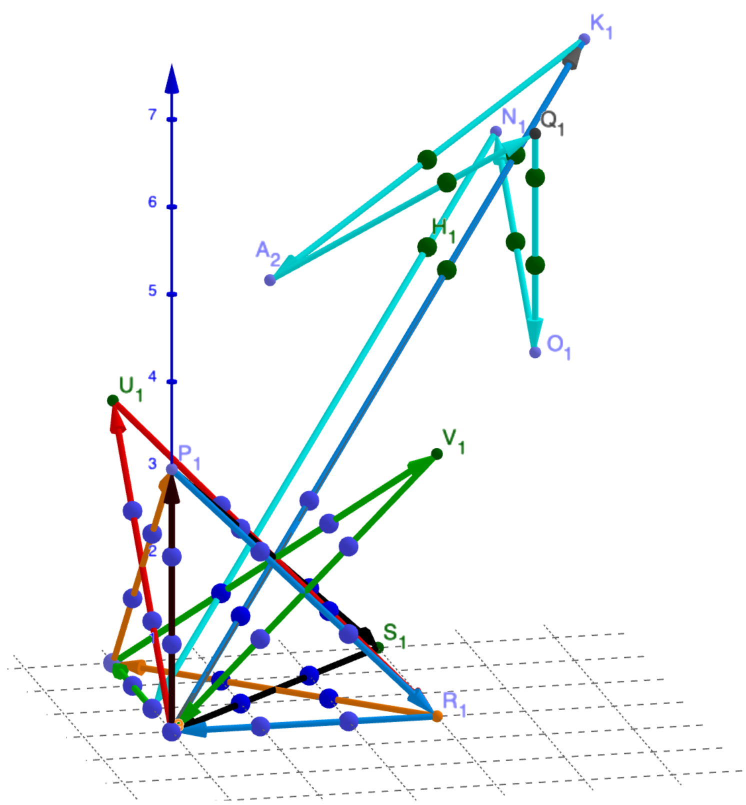

Figure 1. P_{18}, perspective 1.

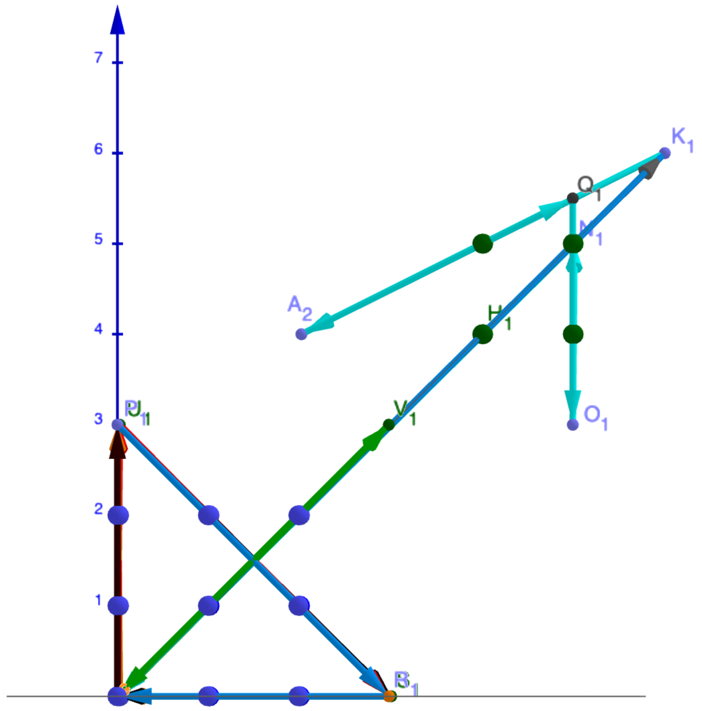

Figure 2. P_{18}, perspective 2.

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