Quantum Evidence Theory

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Abstract

A quantum evidence theory is proposed for uncertainty modeling and reasoning in both closed-world and open-world environments, referred to as QET and GQET, respectively. At the level of uncertainty representation, a series of new concepts are introduced, including (generalized) quantum basic probability amplitude function, (generalized) quantum basic probability distribution, (generalized) quantum belief function, (generalized) quantum plausibility function, and others. At the fusion level, several (generalized) quantum evidential combination rules are proposed to provide a dynamic mechanism for updating and integrating uncertain information from multiple sources, thereby flexibly accommodating diverse application requirements. At the decision-making stage, (generalized) quantum Pignistic transformations are developed to support decision-making processes. In this context, the quantum models of QET and GQET are constructed based on the quantum state representation of the (generalized) quantum basic probability amplitude function, the measurement operators for basis events, the (generalized) quantum basic probability measurements, and the (generalized) belief and

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plausibility measurements. Quantum evidence theory integrates traditional evidence theory with quantum probability theory, providing a more flexible and powerful framework for uncertainty modeling and reasoning in artificial intelligence. By leveraging the expressive capabilities of quantum state spaces and probability amplitudes, it not only handles incomplete and uncertain information inherent in classical evidence theory but also captures interference effects and non-classical correlations among pieces of information. This enables dynamic information fusion and robust decision-making in complex and uncertain environments.
Keywords: Quantum evidence theory, Uncertainty representation, Uncertainty reasoning, Information fusion, Decision-making, Closed world, Open world

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1. Introduction

Representing knowledge in uncertain environments and processing it for decision-making is a fundamental challenge in artificial intelligence (AI) systems. Dempster-Shafer evidence theory (DSET) provides a powerful mathematical framework for uncertainty reasoning by generalizing Bayesian probability theory [1, 2]. Owing to its flexibility and capability of reasoning with evidence in the absence of prior information, DSET has been extensively studied and further extended from various perspectives.

Specifically, Yang and Xu presented the evidential reasoning approach for multiple attribute decision analysis [3]. Smarandache and Dezert introduced DSmT by extending DSET to a higher-dimensional space for information fusion [4]. Deng further extended DSET to the open-world scenario and proposed the generalized evidence theory (GET) [5], which is capable of modeling not only partial or complete ignorance but also the uncertainty arising from the incompleteness of the frame of discernment. Xiao extended Dempster-Shafer evidence theory (DSET) to a high-dimensional complex plane and proposed the Complex Evidence Theory (CET) [6, 7]. Later on, Deng [8] extended DSET to random permutation set through considering the order of sets.

With the advancement of quantum information processing, increasing efforts have been made to bridge classical evidence theory with the quantum framework for addressing more complex problems. Recently, Xiao proposed the generalized quantum evidence theory, which enables uncertainty reasoning within both closed-world and open-world environments from a quantum-theoretic perspective [9, 10]. In this context, the theory is referred to as quantum evidence theory (QET) for closed-world scenario, and as generalized quantum evidence theory (GQET) for open-world scenario, in order to distinguish between the two.

The paper is organized as follows. In Section 2, some related concepts including Dempster-Shafer evidence theory, generalized evidence theory and quantum theory are reviewed. In Section 3 and Section 4, we study and explore quantum evidence theory for uncertainty reasoning for the open world and closed world, respectively.

2. Preliminaries

In this section, we review some basic concepts of Dempster-Shafer evidence theory, generalized evidence theory, and quantum probability theory.

2.1. DSET: Dempster–Shafer evidence theory

Definition 1 (Frame of discernment). Let Ω be a frame of discernment (FOD), consisting of a set of mutually exclusive and collectively nonempty events:

$$\Omega = \{h_1, ..., h_i, ..., h_g, ..., h_n\}, \tag{1}$$

where $\forall i, g = \{1, ..., n\}$, h_i and h_g are two arbitrary nonempty events and $h_i \cap h_g = \emptyset$.

Definition 2 (Power set of Ω). Let 2^Ω be the power set of Ω, denoted as:

$$2^\Omega = \{\emptyset, \{h_1\}, \{h_2\}, ..., \{h_n\}, \{h_1, h_2\}, ..., \{h_1, h_2, ..., h_i\}, ..., \Omega\}, \tag{2}$$

where $\emptyset$ is an empty set.

Definition 3 (Hypothesis or proposition). H_j is defined as a hypothesis or proposition when $H_j \subseteq \Omega$.

Definition 4 (Mass function in DSET). In FOD Ω, a mass function m in DSET is defined as a mapping:

$$m : 2^\Omega \to [0, 1], \tag{3}$$

satisfying

$$m(\emptyset) = 0 \quad \text{and} \sum_{H_j \subseteq \Omega} m(H_j) = 1, \tag{4}$$

where m is also called a basic probability assignment (BPA) or a basic belief assignment (BBA).

Definition 5 (Focal element in DSET). Let m be a BPA. $\forall H_j \subseteq \Omega$, if $m(H_j) > 0$, H_j is called a focal element in DSET.

Definition 6 (Belief function). Let H_j and H_k be two propositions such that $H_j, H_k \subseteq \Omega$. A belief function Bel for H_j, mapping from 2^Ω to $[0,1]$, is defined by

$$\mathrm{Bel}(H_j) = \sum_{H_k \subseteq H_j} m(H_k). \tag{5}$$

Definition 7 (Plausibility function). Let H_j and H_k be two propositions such that $H_j, H_k \subseteq \Omega$. A plausibility function Pl for H_j, mapping from 2^Ω to $[0,1]$, is defined by

$$\mathrm{Pl}(H_j) = \sum_{H_k \cap H_j \neq \emptyset} m(H_k) = 1 - Bel(\bar{H}_j), \tag{6}$$

where $\bar{H}_j = \Omega - H_j$.

Clearly, $\forall H_j \subseteq \Omega$, $\mathrm{Bel}(H_j) \leq \mathrm{Pl}(H_j)$ where $\mathrm{Bel}(H_j)$ and $\mathrm{Pl}(H_j)$ are the lower and upper limit functions to support H_j, respectively.

Definition 8 (Dempster's rule of combination). Let m_1 and m_2 be two independent BPAs in FOD Ω with propositions $H_k, H_h \subseteq \Omega$, respectively. Dempster's rule of combination (DRC), represented in the form $m_1 \oplus m_2$, is defined by

$$m_1 \oplus m_2(H_j) = \begin{cases} \dfrac{1}{1-K} \displaystyle\sum_{H_k \cap H_h = H_j} m_1(H_k) m_2(H_h), & H_j \neq \emptyset, \\ 0, & H_j = \emptyset, \end{cases} \tag{7}$$

with

$$K = \sum_{H_k \cap H_h = \emptyset} m_1(H_k) m_2(H_h), \tag{8}$$

where K is the conflict coefficient between m_1 and m_2.

Note that Eq. (7) is feasible under the condition $K < 1$.

2.2. GET: Generalized evidence theory

Definition 9 (Mass function in GET). In FOD Ω, a mass function m_G in GET is defined as a mapping:

$$m_G : 2^\Omega \to [0,1], \tag{9}$$

satisfying

$$\sum_{H_j \in 2^\Omega} m_G(H_j) = 1, \tag{10}$$

where m_G is also called a generalized BPA (GBPA).

Definition 10 (Focal element in GET). Let m_G be a GBPA. $\forall H_j \in 2^\Omega$, if $m_G(H_j) > 0$, H_j is called a focal element in GET.

Definition 11 (Generalized belief function in GET). Let H_j and H_k be two propositions such that $H_j, H_k \in 2^\Omega$. A generalized belief function GBel for H_j in GET, mapping from 2^Ω to $[0,1]$, is defined by

$$\mathrm{GBel}(H_j) = \sum_{H_k \subseteq H_j} m_G(H_k), \tag{11}$$

$$\mathrm{GBel}(\emptyset) = m_G(\emptyset). \tag{12}$$

Definition 12 (Generalized plausibility function in GET). Let H_j and H_k be two propositions such that $H_j, H_k \in 2^\Omega$. A generalized plausibility function GPl for H_j in GET, mapping from 2^Ω to $[0,1]$, is defined by

$$\mathrm{GPl}(H_j) = \sum_{H_k \cap H_j \neq \emptyset} m_G(H_k), \tag{13}$$

$$\mathrm{GPl}(\emptyset) = m_G(\emptyset). \tag{14}$$

Definition 13 (Generalized combination rule in GET). Let m_{G_1} and m_{G_2} be two independent GBPAs in FOD Ω with propositions $H_k, H_h \in 2^\Omega$, respectively. The generalized combination rule (GCR), represented in the form $m_{G_1} \oplus m_{G_2}$, is defined by

$$m_{G_1} \oplus m_{G_2}(H_j) = \begin{cases} \dfrac{1 - m_{G_1}(\emptyset)m_{G_2}(\emptyset)}{1 - K_G} \displaystyle\sum_{H_k \cap H_h = H_j} m_{G_1}(H_k)m_{G_2}(H_h), & H_j \neq \emptyset, \\ m_{G_1}(\emptyset)m_{G_2}(\emptyset), & H_j = \emptyset, \end{cases} \tag{15}$$

with

$$K_G = \sum_{H_k \cap H_h = \emptyset} m_{G_1}(H_k)m_{G_2}(H_h), \tag{16}$$

where $m_G(\emptyset) = 1$ if and only if $K_G = 1$.

2.3. QPT: Quantum probability theory

Definition 14 (Hilbert space). Let $|\phi_g\rangle$ be a basis vector representing a distinguishable event. A Hilbert space is spanned by a set of orthonormal basis vectors:

$$\mathcal{H} = \{|\phi_1\rangle, ..., |\phi_g\rangle, ..., |\phi_n\rangle\}. \tag{17}$$

Definition 15 (Quantum state). A quantum state, also called a superposition state is defined as:

$$|\Psi\rangle = \sum_g \alpha_g |\phi_g\rangle, \tag{18}$$

where α_g denotes the probability amplitude expressed by a complex number, $\Pr(|\phi_g\rangle) = |\alpha_g|^2$ representing the probability of $|\phi_g\rangle$, and $\sum_g |\alpha_g|^2 = 1$.

Definition 16 (Quantum interference). Quantum interference from the union of n mutually exclusive events $\{|\phi_1\rangle, ..., |\phi_f\rangle, ..., |\phi_g\rangle, ..., |\phi_n\rangle\}$ is defined as:

$$\Pr(|\Psi_1\rangle \cup ... \cup |\Psi_g\rangle \cup ... \cup |\Psi_n\rangle) = |\alpha_1 + ... + \alpha_g + ... + \alpha_n|^2$$
$$= \sum_g |\alpha_g|^2 + \mathrm{Int}_{\Pr}(|\Psi_1\rangle \cup ... \cup |\Psi_g\rangle \cup ... \cup |\Psi_n\rangle), \tag{19}$$

which composes of classical probability $\sum_g |\alpha_g|^2$, and quantum interference:

$$\mathrm{Int}_{\Pr}(|\Psi_1\rangle \cup ... \cup |\Psi_g\rangle \cup ... \cup |\Psi_n\rangle) = 2\sum_{f=1}^{n-1}\sum_{g=f+1}^{n} |\alpha_f||\alpha_g| \cos(\theta_f - \theta_g), \tag{20}$$

where $-1 \leq \cos(\theta_f - \theta_g) \leq 1$.

When $\cos(\theta_f - \theta_g) = 0$, the interference term becomes zero, and Eq.(19) converges to classical probability, such that:

$$\Pr(|\Psi_1\rangle \cup ... \cup |\Psi_g\rangle \cup ... \cup |\Psi_n\rangle) = \sum_g |\alpha_g|^2. \tag{21}$$

3. Quantum evidence theory for an open world

Recently, Xiao extended the DSET to Hilbert space and proposed the generalized quantum evidence theory (GQET), which provides a promising way to handle uncertainty in a new perspective of quantum framework in an open world [9, 10]. The main concepts and knowledge of GQET are introduced below.

3.1. GQET: Generalized quantum evidence theory

3.1.1. Basic concepts of the generalized quantum basic probability amplitude function

Definition 17 (Quantum frame of discernment). Let Φ be a quantum frame of discernment (QFOD), consisting of a set of mutually exclusive and collectively nonempty events, each of which is expressed as an orthonormal basis ϕ_g in a Hilbert space:

$$\Phi = \{\phi_1, ..., \phi_g, ..., \phi_n\}. \tag{22}$$

Definition 18 (Power set of Φ). Let 2^Φ be the power set of Φ, denoted as:

$$2^\Phi = \{\emptyset, \{\phi_1\}, \{\phi_2\}, ..., \{\phi_n\}, \{\phi_1\phi_2\}, ..., \{\phi_1\phi_2...\phi_g\}, ..., \Phi\}, \tag{23}$$

where $\emptyset$ represents an unknown event or unknown events. Eq. (23) can be simply represented as:

$$2^\Phi = \{\emptyset, \phi_1, \phi_2, ..., \phi_n, \phi_{12}, ..., \phi_{12...g}, ..., \phi_{12...n}\}. \tag{24}$$

Definition 19 (Quantum hypothesis or proposition). ψ_j is defined as a quantum hypothesis or proposition when $\psi_j \in 2^\Phi$.

Definition 20 (Generalized quantum basic probability amplitude function). A generalized quantum basic probability amplitude (GQBPA) function $\mathbb{Q}_\mathbb{M}$ in QFOD Φ, also referred to as a generalized quantum mass function (GQMF), is defined as a mapping:

$$\mathbb{Q}_\mathbb{M} : 2^\Phi \to \mathbb{C}, \tag{25}$$

satisfying

$$\begin{aligned} &\mathbb{Q}_\mathbb{M}(\psi_j) = \varphi(\psi_j)e^{i\theta(\psi_j)}, \ \psi_j \in 2^\Phi, \\ &\sum_{\psi_j \in 2^\Phi} |\mathbb{Q}_\mathbb{M}(\psi_j)|^2 = 1, \end{aligned} \tag{26}$$

in which $i = \sqrt{-1}$; $\varphi(\psi_j) \in [0, 1]$ denotes the modulus of $\mathbb{Q}_\mathbb{M}(\psi_j)$; $\theta(\psi_j)$ denotes a phase term of $\mathbb{Q}_\mathbb{M}(\psi_j)$; $\mathbb{Q}_\mathbb{M}(\psi_j)$ denote a generalized quantum basic probability amplitude for ψ_j; and $|\mathbb{Q}_\mathbb{M}(\psi_j)|^2$ denotes the modulus squared of $\mathbb{Q}_\mathbb{M}(\psi_j)$.

The $\mathbb{Q}_{\mathbb{M}}(\psi_j)$, which is the generalization of generalized basic probability assignment in GET, is called a generalized quantum basic probability amplitude, and can be represented as Algebraic form:

$$\mathbb{Q}_{\mathbb{M}}(\psi_j) = x_j + y_j i, \quad x_j^2 + y_j^2 \in [0, 1], \tag{27}$$

or alternatively, using Polar form:

$$\mathbb{Q}_{\mathbb{M}}(\psi_j) = \varphi(\psi_j)\big(\cos\theta(\psi_j) + i\sin\theta(\psi_j)\big). \tag{28}$$

Its modulus is expressed as:

$$|\mathbb{Q}_{\mathbb{M}}(\psi_j)| = \varphi(\psi_j) = \sqrt{x_j^2 + y_j^2}. \tag{29}$$

Definition 21 (Quantum focal element in GQET). Let $\mathbb{Q}_{\mathbb{M}}$ be a GQBPA function. $\forall \psi_j \in 2^{\Phi}$, if $|\mathbb{Q}_{\mathbb{M}}(\psi_j)|$ or $\varphi(\psi_j) > 0$, ψ_j is called a focal element in GQET.

Definition 22 (Bayesian GQBPA function). When the quantum focal element of $\mathbb{Q}_{\mathbb{M}}$ are singletons, such that $\forall \psi_j \in 2^{\Phi}$, $|\psi_j| > 1 \Rightarrow |\mathbb{Q}_{\mathbb{M}}(\psi_j)|^2 = 0$, $\mathbb{Q}_{\mathbb{M}}$ is called a Bayesian GQBPA function.

Definition 23 (Vacuous GQBPA function). In GQET, when $|\mathbb{Q}_{\mathbb{M}}(\Phi)|^2 = 1$, $\mathbb{Q}_{\mathbb{M}}$ is called a vacuous GQBPA function.

As $|\mathbb{Q}_{\mathbb{M}}(\emptyset)|^2 > 0$ indicates an open world, a GQBPA is effective for uncertainty reasoning from the view of the quantum framework in an open world. The physical meaning of $|\mathbb{Q}_{\mathbb{M}}(\emptyset)|^2 = 0$ will be discussed in the next section.

3.1.2. Basic concepts of the generalized quantum basic probability distribution

Definition 24 (Generalized quantum basic probability distribution). The generalized quantum basic probability distribution (GQBPD) of $\mathbb{Q}_{\mathbb{M}}$, is defined as:

$$\mathrm{M}: 2^{\Phi} \to [0,1], \tag{30}$$

and satisfies:

$$\begin{aligned} &\mathrm{M}(\psi_j) = |\mathbb{Q}_{\mathbb{M}}(\psi_j)|^2, \quad \psi_j \in 2^{\Phi}, \\ &\sum_{\psi_j \in 2^{\Phi}} \mathrm{M}(\psi_j) = 1, \end{aligned} \tag{31}$$

where $|\mathbb{Q}_{\mathbb{M}}(\psi_j)|^2 = \mathbb{Q}_{\mathbb{M}}(\psi_j)\widehat{\mathbb{Q}}_{\mathbb{M}}(\psi_j) = \varphi^2(\psi_j) = x_j^2 + y_j^2$, in which $\widehat{\mathbb{Q}}_{\mathbb{M}}(\psi_j)$ is the complex conjugate of $\mathbb{Q}_{\mathbb{M}}(\psi_j)$, e.g., $\widehat{\mathbb{Q}}_{\mathbb{M}}(\psi_j) = x_j - y_j i$.

Definition 25 (Bayesian GQBPD). When the generalized quantum basic probabilities are only assigned to singleton states, M is called a Bayesian GQBPD, and $\mathrm{M}(\phi_g)$ is called a Bayesian GQBP. Mathematically, when $\mathrm{M}(\emptyset) = 0$, a Bayesian GQBPD is just a quantum probability distribution.

Definition 26 (Vacuous GQBPD). In GQET, when $\mathrm{M}(\Phi) = 1$, M is called a vacuous GQBPD.

Definition 27 (Generalized quantum basic probability). In GQET, $\mathrm{M}(\psi_j)$ ($\psi_j \in 2^{\Phi}$) is called generalized quantum basic probability (GQBP), which represents the degree of belief or support to quantum proposition ψ_j.

The GQET inherits the merits of GET and has the following attractive characteristics:

- The $\mathbb{Q}_{\mathbb{M}}$ in GQET can be expressed by not only complex numbers but also positive real numbers, while m_G can only be expressed by positive real numbers in GET.

- In contrast to GET, $\forall \psi_j \in 2^{\Phi}$, the value of $|\mathbb{Q}_{\mathbb{M}}(\psi_j)|^2$ or $\varphi^2(\psi_j)$ represents the degree of belief or support to ψ_j.

- When $\mathrm{M} = m_G$, the GQBPD M of GQET is in accordance with the classical GBPA m_G in GET.

3.1.3. Generalized quantum belief and plausibility functions

Taking into account application scenarios involving interference, the generalized quantum interference belief and plausibility functions in GQET are defined as follows.

Definition 28 (Generalized quantum interference belief function in GQET). Let $\mathbb{Q}_{\mathbb{M}}$ be a GQBPA with proposition $\psi_j \in 2^{\Phi}$. A generalized quantum interference belief function GQIBel for ψ_j in GQET, mapping from 2^{Φ} to $[0,1]$, is defined by:

$$\mathrm{GQIBel}(\psi_j) = \begin{cases} \max\limits_{\psi_p} \left| \sum\limits_{\psi_p \subseteq \psi_j} \mathbb{Q}_{\mathbb{M}}(\psi_p) \right|^2, & \psi_j \neq \emptyset, \\ |\mathbb{Q}_{\mathbb{M}}(\psi_j)|^2, & \psi_j = \emptyset. \end{cases} \tag{32}$$

According to Eq. (32), for $\psi_j \neq \emptyset$, we obtain:

$$\Lambda_A = \arg\max_{\psi_p} \left| \sum_{\psi_p \subseteq \psi_j} \mathbb{Q}_{\mathbb{M}}(\psi_p) \right|^2, \tag{33}$$

where Λ_A includes essential subsets, i.e., $\{\psi_p \subseteq \psi_j\}$, to achieve the maximal modulus squared of Eq. (32).

Consider $\psi_s, \psi_t \subseteq \Lambda_A = \{\psi_p \subseteq \psi_j\}$, and $\psi_s \neq \psi_t$. According to Feynman's rule, and $|\mathbb{Q}_{\mathbb{M}}|^2 = \varphi^2 = \mathrm{M}$, $\mathrm{GQIBel}(\psi_j)$ is calculated as:

$$\begin{aligned}\mathrm{GQIBel}(\psi_j) &= \sum_{\psi_s \subseteq \Lambda_A} |\mathbb{Q}_{\mathbb{M}}(\psi_s)|^2 + 2 \sum_{\psi_s \subseteq \Lambda_A} \sum_{\substack{\psi_t \subseteq \Lambda_A \\ \psi_s \neq \psi_t}} |\mathbb{Q}_{\mathbb{M}}(\psi_s)||\mathbb{Q}_{\mathbb{M}}(\psi_t)| \cos(\theta_s - \theta_t) \\ &= \sum_{\psi_s \subseteq \Lambda_A} \mathrm{M}(\psi_s) + 2 \sum_{\psi_s \subseteq \Lambda_A} \sum_{\substack{\psi_t \subseteq \Lambda_A \\ \psi_s \neq \psi_t}} \sqrt{\mathrm{M}(\psi_s)}\sqrt{\mathrm{M}(\psi_t)} \cos(\theta_s - \theta_t),\end{aligned} \tag{34}$$

where $-1 \leq \cos(\theta_s - \theta_t) \leq 1$.

Definition 29 (Interference effect in GQIBel function within GQET). Let $\mathbb{Q}_{\mathbb{M}}$ be a GQBPA with proposition $\psi_j \in 2^{\Phi}$. Let $\Lambda_A = \{\psi_p \subseteq \psi_j \mid \psi_j \neq \emptyset\}$ denote essential subsets to achieve the maximal modulus squared of GQIBel, and $\psi_s, \psi_t \subseteq \Lambda_A$ with $\psi_s \neq \psi_t$. The interference effect in the GQIBel function is given by:

$$\begin{aligned}\mathrm{Int}_{\mathrm{GQIBel}}(\psi_j) &= 2 \sum_{\psi_s \subseteq \Lambda_A} \sum_{\substack{\psi_t \subseteq \Lambda_A \\ \psi_s \neq \psi_t}} |\mathbb{Q}_{\mathbb{M}}(\psi_s)||\mathbb{Q}_{\mathbb{M}}(\psi_t)| \cos(\theta_s - \theta_t) \\ &= 2 \sum_{\psi_s \subseteq \Lambda_A} \sum_{\substack{\psi_t \subseteq \Lambda_A \\ \psi_s \neq \psi_t}} \sqrt{M(\psi_s)}\sqrt{M(\psi_t)} \cos(\theta_s - \theta_t),\end{aligned} \tag{35}$$

where $-1 \leq \cos(\theta_s - \theta_t) \leq 1$.

When $\mathrm{Int}_{\mathrm{GQIBel}} = 0$, Eq. (34) becomes:

$$\mathrm{GQIBel}(\psi_j) = \sum_{\psi_s \subseteq \Lambda_A} |\mathbb{Q}_{\mathbb{M}}(\psi_s)|^2 = \sum_{\psi_s \subseteq \Lambda_A} \mathrm{M}(\psi_s).$$

Under the situation that $\mathrm{Int}_{\mathrm{GQIBel}} = 0$, Eq. (32) turns into:

$$\mathrm{GQIBel}(\psi_j) = \begin{cases} \sum_{\psi_p \subseteq \psi_j} \mathrm{M}(\psi_p), & \psi_j \neq \emptyset, \\ \mathrm{M}(\psi_j), & \psi_j = \emptyset. \end{cases} \tag{36}$$

Therefore, when $\text{Int}_{\text{GQIBel}} = 0$ and $\text{M} = m_G$, Eq. (36) becomes:

$$\text{GQIBel}(\psi_j) = \begin{cases} \sum_{\psi_p \subseteq \psi_j} m_G(\psi_p), & \psi_j \neq \emptyset, \\ m_G(\psi_j), & \psi_j = \emptyset, \end{cases} \tag{37}$$

which is consistent with the classical GBel in GET [5].

Definition 30 (Generalized quantum interference plausibility function in GQET). Let $\mathbb{Q}_{\mathbb{M}}$ be a GQBPA with proposition $\psi_j \in 2^{\Phi}$. A generalized quantum interference plausibility function GQIPl for ψ_j in GQET, mapping from 2^{Φ} to [0,1] is defined by:

$$\text{GQIPl}(\psi_j) = \begin{cases} \max_{\psi_p} \left| \sum_{\psi_p \cap \psi_j \neq \emptyset} \mathbb{Q}_{\mathbb{M}}(\psi_p) \right|^2, & \psi_j \neq \emptyset, \\ |\mathbb{Q}_{\mathbb{M}}(\psi_j)|^2, & \psi_j = \emptyset. \end{cases} \tag{38}$$

According to Eq. (38), for $\psi_j \neq \emptyset$, we have:

$$\Lambda_B = \arg\max_{\psi_p} \left| \sum_{\psi_p \cap \psi_j \neq \emptyset} \mathbb{Q}_{\mathbb{M}}(\psi_p) \right|^2, \tag{39}$$

where Λ_B includes essential subsets to achieve the maximal modulus squared of Eq. (38).

Consider $\psi_u, \psi_v \subseteq \Lambda_B = \{\psi_p \cap \psi_j \neq \emptyset\}$, and $\psi_u \neq \psi_v$. According to Feynman's rule, and $|\mathbb{Q}_{\mathbb{M}}|^2 = \varphi^2 = \text{M}$, $\text{GQIPl}(\psi_j)$ is given by:

$$\begin{aligned} \text{GQIPl}(\psi_j) &= \sum_{\psi_u \subseteq \Lambda_B} |\mathbb{Q}_{\mathbb{M}}(\psi_u)|^2 + 2 \sum_{\psi_u \subseteq \Lambda_B} \sum_{\substack{\psi_v \subseteq \Lambda_B \\ \psi_u \neq \psi_v}} |\mathbb{Q}_{\mathbb{M}}(\psi_u)||\mathbb{Q}_{\mathbb{M}}(\psi_v)| \cos(\theta_u - \theta_v) \\ &= \sum_{\psi_u \subseteq \Lambda_B} \text{M}(\psi_u) + 2 \sum_{\psi_u \subseteq \Lambda_B} \sum_{\substack{\psi_v \subseteq \Lambda_B \\ \psi_u \neq \psi_t}} \sqrt{\text{M}(\psi_u)}\sqrt{\text{M}(\psi_v)} \cos(\theta_u - \theta_v), \end{aligned} \tag{40}$$

where $-1 \leq \cos(\theta_u - \theta_v) \leq 1$.

Definition 31 (Interference effect in GQIPl function within GQET). Let $\mathbb{Q}_{\mathbb{M}}$ be a GQBPA with proposition $\psi_j \in 2^{\Phi}$. Let $\Lambda_B = \{\psi_p \subseteq \psi_j \mid \psi_j \neq \emptyset\}$ denote essential subsets to achieve the maximal modulus squared of GQIPl, and $\psi_u, \psi_v \subseteq \Lambda_B$ with $\psi_u \neq \psi_v$. The interference effect in the GQIPl function is given by:

$$\begin{aligned} \mathrm{Int}_{\mathrm{GQIPl}}(\psi_j) &= 2 \sum_{\psi_u \subseteq \Lambda_B} \sum_{\substack{\psi_v \subseteq \Lambda_B \\ \psi_u \neq \psi_v}} |\mathbb{Q}_{\mathbb{M}}(\psi_u)||\mathbb{Q}_{\mathbb{M}}(\psi_v)| \cos(\theta_u - \theta_v) \\ &= 2 \sum_{\psi_u \subseteq \Lambda_B} \sum_{\substack{\psi_v \subseteq \Lambda_B \\ \psi_u \neq \psi_v}} \sqrt{M(\psi_u)}\sqrt{M(\psi_v)} \cos(\theta_u - \theta_v), \end{aligned} \tag{41}$$

where $-1 \leq \cos(\theta_u - \theta_v) \leq 1$.

When $\mathrm{Int}_{\mathrm{GQIPl}} = 0$, Eq. (40) becomes:

$$\mathrm{GQIPl}(\psi_j) = \sum_{\psi_u \subseteq \Lambda_B} |\mathbb{Q}_{\mathbb{M}}(\psi_u)|^2 = \sum_{\psi_u \subseteq \Lambda_B} \mathrm{M}(\psi_u). \tag{42}$$

Under the situation that $\mathrm{Int}_{\mathrm{GQIPl}} = 0$, Eq. (38) turns into:

$$\mathrm{GQIPl}(\psi_j) = \begin{cases} \sum\limits_{\psi_p \cap \psi_j \neq \emptyset} \mathrm{M}(\psi_p), & \psi_j \neq \emptyset, \\ \mathrm{M}(\psi_j), & \psi_j = \emptyset. \end{cases} \tag{43}$$

Therefore, when $\mathrm{Int}_{\mathrm{GQIPl}} = 0$ and $\mathrm{M} = m_G$, Eq. (43) becomes:

$$\mathrm{GQIPl}(\psi_j) = \begin{cases} \sum\limits_{\psi_p \cap \psi_j \neq \emptyset} m_G(\psi_p), & \psi_j \neq \emptyset, \\ m_G(\psi_j), & \psi_j = \emptyset, \end{cases} \tag{44}$$

which is consistent with the classical GPl in GET [5].

The functions of GQIBel and GQIPl in GQET have the following properties:

- Similar to those in GET, $\mathrm{GQIBel}(\psi_j)$ and $\mathrm{GQIPl}(\psi_j)$ in GQET are the lower and upper limit probabilities for ψ_j, respectively.

- When the GQBPD M is in accordance with the GBPA m_G, and there are no interferences involving in the functions of GQIBel and GQIPl, GQIBel and GQIPl in GQET degrade into the classical GBel and GPl in GET, respectively.

Moreover, in scenarios where interference is absent, the alternative generalized quantum belief and plausibility functions in GQET are defined as follows.

Definition 32 (Generalized quantum belief function in GQET). Let $\mathbb{Q}_{\mathbb{M}}$ be a GQBPA with proposition $\psi_j \in 2^\Phi$. A generalized quantum belief function GQBel for ψ_j in GQET, mapping from 2^Φ to $[0,1]$, is defined by:

$$\mathrm{GQBel}(\psi_j) = \begin{cases} \sum\limits_{\psi_p \subseteq \psi_j} \left|\mathbb{Q}_{\mathbb{M}}(\psi_p)\right|^2, & \psi_j \neq \emptyset, \\ |\mathbb{Q}_{\mathbb{M}}(\psi_j)|^2, & \psi_j = \emptyset. \end{cases} \tag{45}$$

According to Eq. (31), Eq. (45) can also be represented as:

$$\mathrm{GQBel}(\psi_j) = \begin{cases} \sum\limits_{\psi_p \subseteq \psi_j} \mathrm{M}(\psi_p), & \psi_j \neq \emptyset, \\ \mathrm{M}(\psi_j), & \psi_j = \emptyset, \end{cases} \tag{46}$$

which is consistent with the GQIBel in Eq. (36) in the case without interference.

Therefore, when $\mathrm{M} = m_G$, Eq. (46) becomes:

$$\mathrm{GQBel}(\psi_j) = \begin{cases} \sum_{\psi_p \subseteq \psi_j} m_G(\psi_p), & \psi_j \neq \emptyset, \\ m_G(\psi_j), & \psi_j = \emptyset, \end{cases} \tag{47}$$

which is consistent with the classical GBel in GET [5].

Definition 33 (Generalized quantum plausibility function in GQET). Let $\mathbb{Q}_{\mathbb{M}}$ be a GQBPA with proposition $\psi_j \in 2^{\Phi}$. A generalized quantum plausibility function GQIPl for ψ_j in GQET, mapping from 2^{Φ} to [0,1] is defined by:

$$\mathrm{GQPl}(\psi_j) = \begin{cases} \sum_{\psi_p \cap \psi_j \neq \emptyset} \left| \mathbb{Q}_{\mathbb{M}}(\psi_p) \right|^2, & \psi_j \neq \emptyset, \\ |\mathbb{Q}_{\mathbb{M}}(\psi_j)|^2, & \psi_j = \emptyset. \end{cases} \tag{48}$$

According to Eq. (31), Eq. (48) can also be represented as:

$$\mathrm{GQPl}(\psi_j) = \begin{cases} \sum_{\psi_p \cap \psi_j \neq \emptyset} \mathrm{M}(\psi_p), & \psi_j \neq \emptyset, \\ \mathrm{M}(\psi_j), & \psi_j = \emptyset, \end{cases} \tag{49}$$

which is consistent with the GQIPl in Eq. (43) in the case without interference.

Therefore, when $\mathrm{M} = m_G$, Eq. (49) becomes:

$$\mathrm{GQPl}(\psi_j) = \begin{cases} \sum_{\psi_p \cap \psi_j \neq \emptyset} m_G(\psi_p), & \psi_j \neq \emptyset, \\ m_G(\psi_j), & \psi_j = \emptyset, \end{cases} \tag{50}$$

which is consistent with the classical GPl in GET [5].

The functions of GQBel and GQPl in GQET have the following properties:

- When considering scenarios in the absence of interference, GQBel(ψ_j) and GQPl(ψ_j) in GQET are the lower and upper probabilities for ψ_j, respectively.

- In GQET, when there are no interferences involving in the function of GQIBel and GQIPl, GQIBel and GQIPl degrade into the GQBel and GQPl, respectively.

- When GQBPD M are in accordance with GBPA m_G, GQBel and GQPl in GQET degrade into the classical GBel and GPl in GET, respectively.

3.1.4. Generalized quantum evidential combination rules

Definition 34 (Generalized quantum evidential combination rule). Let $\{\mathbb{Q}_{\mathbb{M}_1}, \ldots, \mathbb{Q}_{\mathbb{M}_h}, \ldots, \mathbb{Q}_{\mathbb{M}_k}\}$ be a set of independent GQBPAs with proposition ψ_j in QFOD Φ. The generalized quantum evidential combination rule (GQECR), denoted as $\mathbb{Q}_{\mathbb{M}_1} \oplus \cdots \oplus \mathbb{Q}_{\mathbb{M}_h} \oplus \cdots \oplus \mathbb{Q}_{\mathbb{M}_k}$, is defined as:

$$\mathbb{Q}_{\mathbb{M}_1} \oplus \cdots \oplus \mathbb{Q}_{\mathbb{M}_h} \oplus \cdots \oplus \mathbb{Q}_{\mathbb{M}_k}(\psi_j) = \frac{\left| \sum_{\cap \psi_p = \psi_j} \prod_{1 \leq h \leq k} \mathbb{Q}_{\mathbb{M}_h}(\psi_p) \right|^2}{\sum_{\psi_v \subseteq \Phi} \left| \sum_{\cap \psi_p = \psi_v} \prod_{1 \leq h \leq k} \mathbb{Q}_{\mathbb{M}_h}(\psi_p) \right|^2 + \left| \prod_{1 \leq h \leq k} \mathbb{Q}_{\mathbb{M}_h}(\emptyset) \right|^2}, \tag{51}$$

$$\mathbb{Q}_{\mathbb{M}_1} \oplus \cdots \oplus \mathbb{Q}_{\mathbb{M}_h} \oplus \cdots \oplus \mathbb{Q}_{\mathbb{M}_k}(\emptyset) = \frac{\left| \prod_{1 \leq h \leq k} \mathbb{Q}_{\mathbb{M}_h}(\emptyset) \right|^2}{\sum_{\psi_v \subseteq \Phi} \left| \sum_{\cap \psi_p = \psi_v} \prod_{1 \leq h \leq k} \mathbb{Q}_{\mathbb{M}_h}(\psi_p) \right|^2 + \left| \prod_{1 \leq h \leq k} \mathbb{Q}_{\mathbb{M}_h}(\emptyset) \right|^2}, \tag{52}$$

Using Feynman's rule, due to $|\mathbb{Q}_{\mathbb{M}_h}|^2 = \varphi_h^2 = \mathrm{M}_h$, Eqs. (51) and (52) can be rewritten as:

$$\mathbb{Q}_{\mathbb{M}_1} \oplus \cdots \oplus \mathbb{Q}_{\mathbb{M}_h} \oplus \cdots \oplus \mathbb{Q}_{\mathbb{M}_k}(\psi_j) = \frac{\sum\limits_{\cap \psi_p = \psi_j} \prod\limits_{1 \le h \le k} \mathrm{M}_h(\psi_p) + \mathrm{Int}_{\mathrm{GQECR}}}{\sum\limits_{\psi_v \subseteq \Phi} \left(\sum\limits_{\cap \psi_p = \psi_v} \prod\limits_{1 \le h \le k} \mathrm{M}_h(\psi_p) + \mathrm{Int}_{\mathrm{GQECR}} \right) + \prod\limits_{1 \le h \le k} \mathrm{M}_h(\emptyset)}, \tag{53}$$

$$\mathbb{Q}_{\mathbb{M}_1} \oplus \cdots \oplus \mathbb{Q}_{\mathbb{M}_h} \oplus \cdots \oplus \mathbb{Q}_{\mathbb{M}_k}(\emptyset) = \frac{\prod\limits_{1 \le h \le k} \mathrm{M}_h(\emptyset)}{\sum\limits_{\psi_v \subseteq \Phi} \left(\sum\limits_{\cap \psi_p = \psi_v} \prod\limits_{1 \le h \le k} \mathrm{M}_h(\psi_p) + \mathrm{Int}_{\mathrm{GQECR}} \right) + \prod\limits_{1 \le h \le k} \mathrm{M}_h(\emptyset)}, \tag{54}$$

where the interference term $\mathrm{Int}_{\mathrm{GQECR}}$ is given by:

$$\mathrm{Int}_{\mathrm{GQECR}} = 2 \sum_{\cap \psi_p = \psi_j} \sum_{\substack{\cap \psi_q = \psi_j \\ \psi_p \neq \psi_q}} \prod_{1 \le h \le k} \sqrt{\mathrm{M}_h(\psi_p)} \prod_{1 \le h \le k} \sqrt{\mathrm{M}_h(\psi_q)} \cos(\theta_p - \theta_q), \tag{55}$$

with $-1 \le \cos(\theta_p - \theta_q) \le 1$.

Definition 35 (Interference effect in GQECR function). Let $\{\mathbb{Q}_{\mathbb{M}_1}, \ldots, \mathbb{Q}_{\mathbb{M}_h}, \ldots, \mathbb{Q}_{\mathbb{M}_k}\}$ be a set of independent GQBPAs with propositions ψ_p and ψ_q ($\psi_p \neq \psi_q$) in QFOD Φ. The interference effect in GQECR function is defined as:

$$\begin{aligned} \mathrm{Int}_{\mathrm{GQECR}} &= 2 \sum_{\cap \psi_p = \psi_j} \sum_{\substack{\cap \psi_q = \psi_j \\ \psi_p \neq \psi_q}} \prod_{1 \le h \le k} |\mathbb{Q}_{\mathbb{M}_h}(\psi_p)| \prod_{1 \le h \le k} |\mathbb{Q}_{\mathbb{M}_h}(\psi_q)| \cos(\theta_p - \theta_q) \\ &= 2 \sum_{\cap \psi_p = \psi_j} \sum_{\substack{\cap \psi_q = \psi_j \\ \psi_p \neq \psi_q}} \prod_{1 \le h \le k} \sqrt{M_h(\psi_p)} \prod_{1 \le h \le k} \sqrt{M_h(\psi_q)} \cos(\theta_p - \theta_q), \end{aligned} \tag{56}$$

where $-1 \le \cos(\theta_p - \theta_q) \le 1$.

When $\mathrm{Int}_{\mathrm{GQECR}} = 0$ and $\mathrm{M_h} = m_{G_h}$, Eqs. (53) and (54) become:

$$\mathbb{Q}_{\mathbb{M}_1} \oplus \cdots \oplus \mathbb{Q}_{\mathbb{M}_h} \oplus \cdots \oplus \mathbb{Q}_{\mathbb{M}_k}(\psi_j) = \frac{\sum\limits_{\cap\psi_p=\psi_j} \prod\limits_{1\le h\le k} m_{G_h}(\psi_p)}{\sum\limits_{\psi_v\subseteq\Phi}\left\{\sum\limits_{\cap\psi_p=\psi_v}\prod\limits_{1\le h\le k} m_{G_h}(\psi_p)\right\} + \prod\limits_{1\le h\le k} m_{G_h}(\emptyset)} = \frac{\sum\limits_{\cap\psi_p=\psi_j} \prod\limits_{1\le h\le k} m_{G_h}(\psi_p)}{1-K_G}, \tag{57}$$

$$\mathbb{Q}_{\mathbb{M}_1} \oplus \cdots \oplus \mathbb{Q}_{\mathbb{M}_h} \oplus \cdots \oplus \mathbb{Q}_{\mathbb{M}_k}(\emptyset) = \frac{\prod\limits_{1\le h\le k} m_{G_h}(\emptyset)}{\sum\limits_{\psi_v\subseteq\Phi}\left\{\sum\limits_{\cap\psi_p=\psi_v}\prod\limits_{1\le h\le k} m_{G_h}(\psi_p)\right\} + \prod\limits_{1\le h\le k} m_{G_h}(\emptyset)} = \frac{\prod\limits_{1\le h\le k} m_{G_h}(\emptyset)}{1-K_G}, \tag{58}$$

in which

$$K_G = \sum_{\substack{\cap|\psi_p=\emptyset \\ \cup|\psi_p\neq\emptyset}} \prod_{1\le h\le k} m_{G_h}(\psi_p). \tag{59}$$

From Eqs. (57)-(59), it is learned that in the case of $\mathrm{M}_h = m_{G_h}$ and $\mathrm{Int}_{\mathrm{GQECR}} = 0$, GQECR degrades into GCR of GET [5].

Definition 36 (Generalized quantum evidential conflict coefficient). The generalized quantum evidential conflict coefficient (GQECC) among GQBPAs $\{\mathbb{Q}_{\mathbb{M}_1}, \cdots, \mathbb{Q}_{\mathbb{M}_h}, \cdots, \mathbb{Q}_{\mathbb{M}_k}\}$, denoted as $\mathbb{K}_{\mathbb{Q}}$, is defined by:

$$\mathbb{K}_{\mathbb{Q}} = \left| \sum_{\substack{\cap\psi_p=\emptyset \\ \cup\psi_p\neq\emptyset}} \prod_{1\le h\le k} \mathbb{Q}_{\mathbb{M}_h}(\psi_p) \right|^2. \tag{60}$$

According to Feynman's rule, since $|\mathbb{Q}_{\mathbb{M}_h}|^2 = \varphi_h^2 = \mathrm{M}_h$, Eq. (60) can also be expressed as:

$$\begin{aligned}
\mathbb{K}_{\mathbb{Q}} &= \sum_{\substack{\cap\psi_p=\emptyset\\ \cup\psi_p\neq\emptyset}} \prod_{1\leq h\leq k} |\mathbb{Q}_{\mathbb{M}_h}(\psi_p)|^2 \\
&\quad + 2 \sum_{\substack{\cap\psi_p=\emptyset\\ \cup\psi_p\neq\emptyset}} \sum_{\substack{\cap\psi_q=\emptyset\\ \cup\psi_q\neq\emptyset}} \prod_{1\leq h\leq k} |\mathbb{Q}_{\mathbb{M}_h}(\psi_p)| \prod_{1\leq h\leq k} |\mathbb{Q}_{\mathbb{M}_h}(\psi_q)| \cos(\theta_p-\theta_q) \\
&= \sum_{\substack{\cap\psi_p=\emptyset\\ \cup\psi_p\neq\emptyset}} \prod_{1\leq h\leq k} \mathrm{M}_h(\psi_p) \\
&\quad + 2 \sum_{\substack{\cap\psi_p=\emptyset\\ \cup\psi_p\neq\emptyset}} \sum_{\substack{\cap\psi_q=\emptyset\\ \cup\psi_q\neq\emptyset}} \prod_{1\leq h\leq k} \sqrt{\mathrm{M}_h(\psi_p)} \prod_{1\leq h\leq k} \sqrt{\mathrm{M}_h(\psi_q)} \cos(\theta_p-\theta_q), \qquad (61)
\end{aligned}$$

in which $-1 \leq \cos(\theta_p - \theta_q) \leq 1$.

Definition 37 (Interference effect in GQECC function). The interference effect involved in GQECC function among GQBPAs $\{\mathbb{Q}_{\mathbb{M}_1}, \cdots, \mathbb{Q}_{\mathbb{M}_h}, \cdots, \mathbb{Q}_{\mathbb{M}_k}\}$, denoted as $\mathrm{Int}_{\mathbb{K}_{\mathbb{Q}}}$, is defined by:

$$\begin{aligned}
\mathrm{Int}_{\mathbb{K}_{\mathbb{Q}}} &= 2 \sum_{\substack{\cap\psi_p=\emptyset\\ \cup\psi_p\neq\emptyset}} \sum_{\substack{\cap\psi_q=\emptyset\\ \cup\psi_q\neq\emptyset}} \prod_{1\leq h\leq k} |\mathbb{Q}_{\mathbb{M}_h}(\psi_p)| \prod_{1\leq h\leq k} |\mathbb{Q}_{\mathbb{M}_h}(\psi_q)| \cos(\theta_p-\theta_q) \\
&= 2 \sum_{\substack{\cap\psi_p=\emptyset\\ \cup\psi_p\neq\emptyset}} \sum_{\substack{\cap\psi_q=\emptyset\\ \cup\psi_q\neq\emptyset}} \prod_{1\leq h\leq k} \sqrt{\mathrm{M}_h(\psi_p)} \prod_{1\leq h\leq k} \sqrt{\mathrm{M}_h(\psi_q)} \cos(\theta_p-\theta_q), \qquad (62)
\end{aligned}$$

in which $-1 \leq \cos(\theta_p - \theta_q) \leq 1$.

When $\mathrm{Int}_{\mathbb{K}_{\mathbb{Q}}} = 0$ and $\mathrm{M_h} = m_{G_h}$, Eq. (61) becomes:

$$\mathbb{K}_{\mathbb{Q}} = \sum_{\substack{\cap\psi_p=\emptyset\\ \cup\psi_p\neq\emptyset}} \prod_{1\leq h\leq k} m_{G_h}(\psi_p), \qquad (63)$$

which is consistent with the conflict coefficient K_G in GET [5].

GQECR has the following characteristics:

- When the GQBPDs of GQET are in accordance with the GBPAs of GET, and there are no interferences involving in GQECR, GQECR reduces to the GCR in GET.

- When the GQBPDs of GQET are in accordance with the GBPAs of GET, and there are no interferences involving in GQECR, GQECC $\mathbb{K}_{\mathbb{Q}}$ reduces to the conflict coefficient K_G in GET.

- When the GQBPDs of GQET are in accordance with the BPAs of DSET, and there are no interferences involving in GQECR, GQECR reduces to the DRC in DSET.

- When the GQBPDs of GQET are in accordance with the BPAs of DSET, and there are no interferences involving in GQECR, GQECC $\mathbb{K}_{\mathbb{Q}}$ reduces to the conflict coefficient K in DSET.

- If the sum of the GQBPDs of all nonempty sets is zero or GQECC is equal to 1, the whole belief is reallocated to $\emptyset$.

To support a wide range of applications, an alternative generalized progressive quantum evidential combination rule is proposed to enable the continuous, progressive, and incremental fusion of GQBPAs.

Definition 38 (Generalized progressive quantum evidential combination rule). Let $\{\mathbb{Q}_{\mathbb{M}_1}, \ldots, \mathbb{Q}_{\mathbb{M}_h}, \ldots, \mathbb{Q}_{\mathbb{M}_k}\}$ be a set of independent QBPAs with proposition ψ_j in QFOD Φ. The generalized progressive quantum evidential combination rule (GPQECR), denoted as $\mathbb{Q}_{\mathbb{M}_1} \oplus \cdots \oplus \mathbb{Q}_{\mathbb{M}_h} \oplus$

$\cdots \oplus \mathbb{Q}_{\mathbb{M}_k}$, is defined as:

$$\mathbb{Q}_{\mathbb{M}_1} \oplus \cdots \oplus \mathbb{Q}_{\mathbb{M}_h} \oplus \cdots \oplus \mathbb{Q}_{\mathbb{M}_k}(\psi_j) = \frac{\sum\limits_{\cap \psi_p = \psi_j} \prod\limits_{1 \leq h \leq k} \mathbb{Q}_{\mathbb{M}_h}(\psi_p)}{\sum\limits_{\psi_v \subseteq \Phi} \sum\limits_{\cap \psi_p = \psi_v} \prod\limits_{1 \leq h \leq k} \mathbb{Q}_{\mathbb{M}_h}(\psi_p) + \prod\limits_{1 \leq h \leq k} \mathbb{Q}_{\mathbb{M}_h}(\emptyset)}, \tag{64}$$

$$\mathbb{Q}_{\mathbb{M}_1} \oplus \cdots \oplus \mathbb{Q}_{\mathbb{M}_h} \oplus \cdots \oplus \mathbb{Q}_{\mathbb{M}_k}(\emptyset) = \frac{\prod\limits_{1 \leq h \leq k} \mathbb{Q}_{\mathbb{M}_h}(\emptyset)}{\sum\limits_{\psi_v \subseteq \Phi} \sum\limits_{\cap \psi_p = \psi_v} \prod\limits_{1 \leq h \leq k} \mathbb{Q}_{\mathbb{M}_h}(\psi_p) + \prod\limits_{1 \leq h \leq k} \mathbb{Q}_{\mathbb{M}_h}(\emptyset)}. \tag{65}$$

Definition 39 (Generalized progressive quantum evidential conflict coefficient). The generalized progressive quantum evidential conflict coefficient (GPQECC) among QBPAs $\{\mathbb{Q}_{\mathbb{M}_1}, \cdots, \mathbb{Q}_{\mathbb{M}_h}, \cdots, \mathbb{Q}_{\mathbb{M}_k}\}$, denoted as $\mathbb{K}_{\mathbb{Q}}$, is defined by:

$$\mathbb{K}_{\mathbb{Q}} = \sum_{\substack{\cap \psi_p = \emptyset \\ \cup \psi_p \neq \emptyset}} \prod_{1 \leq h \leq k} \mathbb{Q}_{\mathbb{M}_h}(\psi_p). \tag{66}$$

GPQECR satisfies the following properties:

- **Commutativity:** Ensures that the fusion result of multisource uncertain information is independent of the order of information.

$$\mathbb{Q}_{\mathbb{M}_1} \oplus \mathbb{Q}_{\mathbb{M}_2} = \mathbb{Q}_{\mathbb{M}_2} \oplus \mathbb{Q}_{\mathbb{M}_1}.$$

- **Associativity:** Grants the system the capability of flexibly structured fusion of multisource uncertain information.

$$(\mathbb{Q}_{\mathbb{M}_1} \oplus \mathbb{Q}_{\mathbb{M}_2}) \oplus \mathbb{Q}_{\mathbb{M}_3} = \mathbb{Q}_{\mathbb{M}_1} \oplus (\mathbb{Q}_{\mathbb{M}_2} \oplus \mathbb{Q}_{\mathbb{M}_3}).$$

- **Identity:** Ensures that the fusion result remains unchanged when no new effective information is added.

$$\text{Let } \mathbb{Q}_{\mathbb{M}_0}(\Phi) = 1, \quad \mathbb{Q}_{\mathbb{M}_1} \oplus \mathbb{Q}_{\mathbb{M}_0} = \mathbb{Q}_{\mathbb{M}_1}.$$

- **Focalization:** Enables the fusion of consistent quantum evidence to reduce the quantum focal element, guiding the fusion result toward convergence.

GPQECR has the following characteristics:

- When the GQBPDs of GQET are in accordance with the GBPAs of GET, GPQECR reduces to the GCR in GET.
- When the GQBPDs of GQET are in accordance with the GBPAs of GET, GPQECC $\mathbb{K}_{\mathbb{Q}}$ reduces to the conflict coefficient K_G in GET.
- When the GQBPDs of GQET are in accordance with the BPAs of DSET, GPQECR reduces to the DRC in DSET.
- When the GQBPDs of GQET are in accordance with the BPAs of DSET, GPQECC $\mathbb{K}_{\mathbb{Q}}$ reduces to the conflict coefficient K in DSET.
- If the sum of the GQBPDs of all nonempty sets is zero or GPQECC is equal to 1, the whole belief is reallocated to $\emptyset$.

3.1.5. Generalized quantum Pignistic transformations

When considering the interference during the process of decision-making, a generalized quantum interference Pignistic transformation is defined to transform a GQBPA into a generalized quantum interference Pignistic probability function.

Definition 40 (Generalized quantum interference Pignistic transformation). Let $\mathbb{Q}_{\mathbb{M}}$ be a GQBPA function on QFOD Φ and ψ_j be a quantum proposition with $\psi_j \in \Phi$. Its generalized quantum interference Pignistic probability function, denoted as BetGQIP, is defined as:

$$\mathrm{BetGQIP}(\psi_j) = \left| \sum_{\psi_p \subseteq \Phi} \frac{\mathbb{Q}_{\mathbb{M}}(\psi_p)}{1 - \mathbb{Q}_{\mathbb{M}}(\emptyset)} \cdot \frac{|\psi_p \cap \psi_j||}{|\psi_p||} \right|^2, \tag{67}$$

where $|\psi|$ represents the cardinality of ψ. The transformation between $\mathbb{Q}_{\mathbb{M}}$ and BetGQIP is called a generalized quantum interference Pignistic transformation.

$\forall \phi_g \in \Phi$, Eq. (67) can be expressed as:

$$\mathrm{BetQIP}(\phi_g) = \left| \sum_{\phi_p \subseteq \Phi} \frac{\mathbb{Q}_{\mathbb{M}}(\psi_p)}{1 - \mathbb{Q}_{\mathbb{M}}(\emptyset)} \cdot \frac{|\psi_p \cap \phi_g||}{|\psi_p||} \right|^2. \tag{68}$$

When considering scenarios in the absence of interference, a generalized quantum Pignistic transformation is defined to transform a GQBPA into a generalized quantum Pignistic probability function.

Definition 41 (Generalized quantum Pignistic transformation). Let $\mathbb{Q}_{\mathbb{M}}$ be a GQBPA on QFOD Φ and ψ_j be a quantum proposition with $\psi_j \in \Phi$. Its generalized quantum Pignistic probability function, denoted as BetGQP, is defined as:

$$\mathrm{BetGQP}(\psi_j) = \sum_{\psi_p \subseteq \Phi} \frac{|\mathbb{Q}_{\mathbb{M}}(\psi_p)|^2}{1 - |\mathbb{Q}_{\mathbb{M}}(\emptyset)|^2} \cdot \frac{|\psi_p \cap \psi_j||}{|\psi_p||}, \tag{69}$$

where $|\psi|$ represents the cardinality of ψ. The transformation between $\mathbb{Q}_{\mathbb{M}}$ and BetGQP is called a generalized quantum Pignistic transformation.

Since $|\mathbb{Q}_{\mathbb{M}}|^2 = \varphi^2 = \mathrm{M}$, Eq. (69) can be expressed as:

$$\mathrm{BetGQP}(\psi_j) = \sum_{\psi_p \subseteq \Phi} \frac{\mathrm{M}(\psi_p)}{1 - \mathrm{M}(\emptyset)} \cdot \frac{|\psi_p \cap \psi_j||}{|\psi_p||}. \tag{70}$$

$\forall \phi_g \in \Phi$, Eq. (69) and Eq. (70) can be expressed as:

$$\mathrm{BetGQP}(\phi_g) = \sum_{\psi_p \subseteq \Phi} \frac{|\mathbb{Q}_{\mathbb{M}}(\psi_p)|^2}{1 - |\mathbb{Q}_{\mathbb{M}}(\emptyset)|^2} \cdot \frac{|\psi_p \cap \phi_g||}{|\psi_p||}, \tag{71}$$

$$\mathrm{BetGQP}(\phi_g) = \sum_{\psi_p \subseteq \Phi} \frac{\mathrm{M}(\psi_p)}{1 - \mathrm{M}(\emptyset)} \cdot \frac{|\psi_p \cap \phi_g||}{|\psi_p||}. \tag{72}$$

3.2. A quantum representation of GQET in Hilbert space

3.2.1. Quantum state representation of a GQBPA

Definition 42 (Basis event in GQET). Let $\Phi = \{\phi_1, ..., \phi_g, ..., \phi_n\}$ be a QFOD consisting of a set of mutually exclusive and collectively nonempty events. Let $\emptyset$ be an unknown event or unknown events, except for the events of Φ. A set of basis events in GQET is defined as:

$$\mathbb{B} = \{\emptyset, \phi_1, \ldots, \phi_g, \ldots, \phi_n\}, \tag{73}$$

where $e_w \in \mathbb{B}$ $(0 \leq w \leq n)$ denotes a basis event.

When $w = 0$, e_o represents an unknown event of $\emptyset$; when $w = g$ $(1 \leq g \leq n)$, e_g represents a certain event from the QFOD Φ.

Definition 43 (Standard basis vector representation of a basis event in GQET). Let $|e_w\rangle$ be a standard basis vector corresponding to the basis event e_w of GQET, where $|e_w\rangle \in \mathcal{B} = \{|\emptyset\rangle, |\phi_1\rangle, \ldots, |\phi_g\rangle, \ldots, |\phi_n\rangle\}$ $(0 \leq w \leq n)$. The standard basis vector $|e_w\rangle$ in Hilbert space is defined as:

$$|e_w^j\rangle = [\eta_0, \eta_1, \ldots, \eta_l, \ldots, \eta_n]^T, \quad \eta_l = \begin{cases} 1, & l = w, \\ 0, & l \neq w, \end{cases} \tag{74}$$

where any two standard basis vectors corresponding to the basis events: $|e_w^j\rangle$ and $|e_s^j\rangle$ $(0 \leq s, w \leq n; s \neq w)$ are orthonormal, namely, $\langle e_w^j|e_w^j\rangle = 1$, and $\langle e_w^j|e_s^j\rangle = 0$.

Definition 44 (State vector representation of a quantum proposition in GQET). Let $|\psi_j\rangle$ be a standard basis vector corresponding to the quantum proposition ψ_j $(\psi_j \in 2^\Phi)$ of GQBPA function $\mathbb{Q}_\mathbb{M}$ with basis events $e_w^j \in \psi_j$ $(0 \leq w \leq n)$. The state vector $|\psi_j\rangle$ in Hilbert space is defined as:

$$|\psi_j\rangle = \sum_{e_w^j \in \psi_j} \lambda_w^j |e_w^j\rangle, \tag{75}$$

where $\lambda_w^j = a_w^j + b_w^j i$ is a complex number satisfying $\sum_w |\lambda_w^j|^2 = 1$; $|\psi_j\rangle$ is a unit vector.

From Eq. (75), $|\psi_j\rangle$ is a pure quantum state. It can be seen that in the framework of GQET, a composite quantum proposition is represented in Hilbert space by a unit vector formed as a linear superposition of the basis vectors corresponding to its constituent elementary quantum propositions. In this context, a corresponding density operator of a quantum proposition is represented as follows.

Definition 45 (Density operator representation of a quantum proposition in GQET). Let $|\psi_j\rangle$ be a state vector corresponding to its quantum proposition ψ_j $(\psi_j \in 2^\Phi)$ on QFOD. Its density operator is defined as:

$$\rho_j = |\psi_j\rangle\langle\psi_j|. \tag{76}$$

On this basis, a GQBPA function is expressed as an ensemble of pure states as follows.

Definition 46 (Linear operator representation of a GQBPA function). Let $\mathbb{Q}_{\mathbb{M}}$ be a GQBPA function with $\mathbb{Q}_{\mathbb{M}}(\psi_j)$ on QFOD Φ, and $\{\mathbb{Q}_{\mathbb{M}}(\psi_j), |\psi_j\rangle\}$ be an ensemble of pure quantum states. The linear operator corresponding to a GQBPA function is defined as:

$$\tilde{\rho}_{\mathbb{Q}_{\mathbb{M}}} = \sum_j \mathbb{Q}_{\mathbb{M}}(\psi_j)|\psi_j\rangle\langle\psi_j| = \sum_j \mathbb{Q}_{\mathbb{M}}(\psi_j)\rho_j, \tag{77}$$

where a GQBPA function $\mathbb{Q}_{\mathbb{M}}$ is expressed as a quantum system of mixed quantum states.

Definition 47 (Normalized density operator representation of a GQBPA function). The normalized density operator corresponding to a GQBPA function is defined as:

$$\widehat{\rho}_{\mathbb{Q}_{\mathbb{M}}} = \sum_j |\mathbb{Q}_{\mathbb{M}}(\psi_j)|^2\rho_j = \sum_j |\mathbb{Q}_{\mathbb{M}}(\psi_j)|^2|\psi_j\rangle\langle\psi_j|, \tag{78}$$

where $|\cdot|^2$ is the modulus squared function.

From Eq. (78), it is obvious that

$$\widehat{\rho}_{\mathbb{Q}_{\mathbb{M}}} > 0,$$

and

$$\mathrm{tr}(\widehat{\rho}_{\mathbb{Q}_{\mathbb{M}}}) = \sum_j |\mathbb{Q}_{\mathbb{M}}(\psi_j)|^2\,\mathrm{tr}(|\psi_j\rangle\langle\psi_j|) = \sum_j |\mathbb{Q}_{\mathbb{M}}(\psi_j)|^2 = 1.$$

Therefore, $\widehat{\rho}_{\mathbb{Q}_{\mathbb{M}}}$ is a positive and normalized density operator.

Let $\widehat{\mathbb{E}}_\nu$ be the eigenvalues of $\widehat{\rho}_{\mathbb{Q}_{\mathbb{M}}}$. It follows that:

$$\widehat{\mathbb{E}}_\nu \geq 0, \quad \sum_\nu \widehat{\mathbb{E}}_\nu = 1.$$

Since $|\mathbb{Q}_{\mathbb{M}}|^2 = \varphi^2 = \mathrm{M}$, the density operator of a GQBPD can be expressed as follows.

Definition 48 (Density operator representation of a GQBPD). The density operator corresponding to a GQBPD is defined as:

$$\rho_{\mathrm{M}} = \sum_j \mathrm{M}(\psi_j)|\psi_j\rangle\langle\psi_j| = \sum_j \mathrm{M}(\psi_j)\rho_j.$$

3.2.2. Measurement operator for basis event in GQET

Definition 49 (Measurement operator for basis event in GQET). A measurement operator to measure the quantum probability of a basis event $e_w \in \mathbb{B} = \{\emptyset, \phi_1, \dots, \phi_g, \dots, \phi_n\}$ in GQET is defined as:

$$\mathbb{M}_{e_w} = |e_w\rangle\langle e_w|, \quad 0 \le w \le n, \tag{79}$$

satisfying

$$\sum_{w=0}^{n} \mathbb{M}_{e_w}^{\dagger}\mathbb{M}_{e_w} = I, \tag{80}$$

where $\mathbb{M}_{e_w}^{\dagger}$ is the Hermitian conjugate or adjoint of the $\mathbb{M}_{e_w}$ matrix, e.g., $\mathbb{M}_{e_w}^{\dagger} = \left(\mathbb{M}_{e_w}^{\mathrm{T}}\right)^*$; and I denotes the identity matrix.

3.2.3. Basis event measurement function in GQET

Definition 50 (Basis event measurement function in GQET). Let ρ_{M} be a density operator corresponding to a GQBPD M on QFOD Φ with basis event $e_w \in \mathbb{B} = \{\emptyset, \phi_1, \dots, \phi_g, \dots, \phi_n\}$, and $\left\{\mathbb{M}_{e_w} = |e_w\rangle\langle e_w|, 0 \le w \le n\right\}$ be a set of measurement operators. The basis event measurement function in GQET is defined as:

$$\mathrm{P}(\mathbb{Q}_{\mathbb{M}}(e_w)) = \mathrm{Tr}\left(\mathbb{M}_{e_w}^{\dagger}\mathbb{M}_{e_w}\rho_{\mathrm{M}}\right), \quad 0 \le w \le n, \tag{81}$$

where $\mathbb{M}_{e_w}^{\dagger}$ is the Hermitian conjugate or adjoint of the $\mathbb{M}_{e_w}$ matrix, e.g., $\mathbb{M}_{e_w}^{\dagger} = \left(\mathbb{M}_{e_w}^{\mathrm{T}}\right)^*$; and Tr is a function of the trace of a matrix.

3.2.4. GQBP measurement function

Definition 51 (GQBP measurement function). Let ρ_{M} be a density operator corresponding to a GQBPD M on QFOD Φ with basis event $e_w \in \mathbb{B} = \{\emptyset, \phi_1, \ldots, \phi_g, \ldots, \phi_n\}$, and $\left\{\mathbb{M}_{e_w} = |e_w\rangle\langle e_w|, 0 \leq w \leq n\right\}$ be a set of measurement operators. The GQBP measurement function is defined as:

$$\mathrm{M}(\psi_j) = \sum_{e_w \in \psi_j} \mathrm{P}(\mathbb{Q}_{\mathbb{M}}(e_w)), \quad \psi_j \in 2^{\Phi}, \tag{82}$$

satisfying

$$\mathrm{P}(\mathbb{Q}_{\mathbb{M}}(e_w)) = \mathrm{Tr}\left(\mathbb{M}_{e_w}^{\dagger} \mathbb{M}_{e_w} \rho_{\mathrm{M}}\right),$$

where $\mathbb{M}_{e_w}^{\dagger}$ is the Hermitian conjugate or adjoint of the $\mathbb{M}_{e_w}$ matrix, e.g., $\mathbb{M}_{e_w}^{\dagger} = \left(\mathbb{M}_{e_w}^{\mathrm{T}}\right)^*$; and Tr is a function of the trace of a matrix.

3.2.5. Belief and plausibility measurement functions in GQET

Definition 52 (Belief and plausibility measurement functions in GQET). Let ρ_{M} be a density operator corresponding to a GQBPD M on QFOD Φ with basis event $e_w \in \mathbb{B} = \{\emptyset, \phi_1, \ldots, \phi_g, \ldots, \phi_n\}$, and $\left\{\mathbb{M}_{e_w} = |e_w\rangle\langle e_w|, 0 \leq w \leq n\right\}$ be a set of measurement operators. The belief and plausibility measurement functions are defined as:

$$\begin{cases} \mathrm{GQBel}(\psi_j) = \min\left\{\sum\limits_{e_w \in \psi_j} \mathrm{P}(\mathbb{Q}_{\mathbb{M}}(e_w))\right\}, & \psi_j \in 2^{\Phi}, \\ \mathrm{GQPl}(\psi_j) = \max\left\{\sum\limits_{e_w \in \psi_j} \mathrm{P}(\mathbb{Q}_{\mathbb{M}}(e_w))\right\}, & \psi_j \in 2^{\Phi}, \end{cases} \tag{83}$$

satisfying

$$\mathrm{P}(\mathbb{Q}_{\mathbb{M}}(e_w)) = \mathrm{Tr}\left(\mathbb{M}_{e_w}^{\dagger} \mathbb{M}_{e_w} \rho_{\mathrm{M}}\right),$$

where $\mathbb{M}_{e_w}^{\dagger}$ is the Hermitian conjugate or adjoint of the $\mathbb{M}_{e_w}$ matrix, e.g., $\mathbb{M}_{e_w}^{\dagger} = \left(\mathbb{M}_{e_w}^{\mathrm{T}}\right)^*$; and Tr is a function of the trace of a matrix.

According to Eqs. (82) and (83), we have:

$$\mathrm{GQBel}(\psi_j) \leq \mathrm{M}(\psi_j) \leq \mathrm{GQPl}(\psi_j). \tag{84}$$

The belief and plausibility measurement functions represent the lower and upper probabilities for ψ_j.

3.2.6. Generalized progressive quantum evidential combination rule in Hilbert space

Definition 53 (Generalized progressive quantum evidential combination rule in Hilbert space). The generalized progressive quantum evidential combination rule in Hilbert space is defined as:

$$\begin{aligned}\rho_{\mathbb{Q}_{\mathbb{M}_1}} \oplus \cdots \oplus \rho_{\mathbb{Q}_{\mathbb{M}_h}} \cdots \oplus \rho_{\mathbb{Q}_{\mathbb{M}_k}} &= \frac{1}{\gamma}\sum_j \left(\sum_{\cap\psi_p=\psi_j}\prod_{1\leq h\leq k}\mathbb{Q}_{\mathbb{M}_h}(\psi_p)\right)|\psi_j\rangle\langle\psi_j| \\ &= \frac{1}{\gamma}\sum_j \left(\sum_{\cap\psi_p=\psi_j}\prod_{1\leq h\leq k}\mathbb{Q}_{\mathbb{M}_h}(\psi_p)\right)\rho_j,\end{aligned} \tag{85}$$

where γ is the normalization factor:

$$\gamma = \sum_{\psi_v\subseteq\Phi}\sum_{\cap\psi_p=\psi_v}\prod_{1\leq h\leq k}\mathbb{Q}_{\mathbb{M}_h}(\psi_p) + \prod_{1\leq h\leq k}\mathbb{Q}_{\mathbb{M}_h}(\emptyset). \tag{86}$$

4. Quantum evidence theory for a closed world

The previous section introduces the GQET in an open world. When $|\mathbb{Q}_{\mathbb{M}}(\emptyset)| = 0$, the quantum evidence theory (QET) is defined in a closed world. The main concepts and knowledge of QET are proposed in this section.

4.1. QET: Quantum evidence theory

4.1.1. Basic concepts of the quantum basic probability amplitude function

Definition 54 (Quantum basic probability amplitude function). A quantum basic probability amplitude (QBPA) function $\mathbb{Q}_{\mathbb{M}}$ in QFOD Φ, also referred to as a generalized quantum mass function (QMF), is defined as a mapping:

$$\mathbb{Q}_{\mathbb{M}} : 2^{\Phi} \to \mathbb{C}, \tag{87}$$

satisfying

$$\begin{aligned} &\mathbb{Q}_{\mathbb{M}}(\emptyset) = 0, \\ &\mathbb{Q}_{\mathbb{M}}(\psi_j) = \varphi(\psi_j) e^{i\theta(\psi_j)},\ \psi_j \subseteq \Phi, \\ &\sum_{\psi_j \subseteq \Phi} |\mathbb{Q}_{\mathbb{M}}(\psi_j)|^2 = 1, \end{aligned} \tag{88}$$

in which $i = \sqrt{-1}$; $\varphi(\psi_j) \in [0, 1]$ represents the modulus of $\mathbb{Q}_{\mathbb{M}}(\psi_j)$; $\theta(\psi_j)$ denotes a phase term of $\mathbb{Q}_{\mathbb{M}}(\psi_j)$; $\mathbb{Q}_{\mathbb{M}}(\psi_j)$ denote a quantum basic probability amplitude for ψ_j; and $|\mathbb{Q}_{\mathbb{M}}(\psi_j)|^2$ denotes the modulus squared of $\mathbb{Q}_{\mathbb{M}}(\psi_j)$.

The $\mathbb{Q}_{\mathbb{M}}(\psi_j)$, which is the generalization of basic probability assignment in DSET, is called a quantum basic probability amplitude, and can be represented as Algebraic form:

$$\mathbb{Q}_{\mathbb{M}}(\psi_j) = x_j + y_j i, \quad x_j^2 + y_j^2 \in [0, 1],$$

or alternatively, using Polar form:

$$\mathbb{Q}_{\mathbb{M}}(\psi_j) = \varphi(\psi_j)\big(\cos\theta(\psi_j) + i\sin\theta(\psi_j)\big).$$

Its amplitude is expressed as:

$$|\mathbb{Q}_{\mathbb{M}}(\psi_j)| = \varphi(\psi_j) = \sqrt{x_j^2 + y_j^2}.$$

Definition 55 (Quantum focal element in QET). Let $\mathbb{Q}_{\mathbb{M}}$ be a QBPA function. $\forall \psi_j \subseteq \Phi$, if $|\mathbb{Q}_{\mathbb{M}}(\psi_j)|$ or $\varphi(\psi_j) > 0$, ψ_j is called a focal element in QET.

Definition 56 (Bayesian QBPA function). When the quantum focal element of QBPA function $\mathbb{Q}_{\mathbb{M}}$ are singletons, such that $\forall \psi_j \subseteq \Phi$, $|\psi_j| > 1 \Rightarrow |\mathbb{Q}_{\mathbb{M}}(\psi_j)|^2 = 0$, $\mathbb{Q}_{\mathbb{M}}$ is called a Bayesian QBPA function.

Definition 57 (Vacuous QBPA function). In QET, when $|\mathbb{Q}_{\mathbb{M}}(\Phi)|^2 = 1$, $\mathbb{Q}_{\mathbb{M}}$ is called a vacuous QBPA function.

As $|\mathbb{Q}_{\mathbb{M}}(\emptyset)|^2 = 0$ indicates a closed world, a QBPA function is effective for uncertainty reasoning from the view of the quantum framework in a closed world.

4.1.2. Basic concepts of the quantum basic probability distribution

Definition 58 (Quantum basic probability distribution). The quantum basic probability distribution (QBPD) of $\mathbb{Q}_{\mathbb{M}}$, is defined as:

$$\mathrm{M} : 2^{\Phi} \to [0, 1], \tag{89}$$

and satisfies:

$$\begin{aligned}
&\mathrm{M}(\emptyset) = 0,\\
&\mathrm{M}(\psi_j) = |\mathbb{Q}_{\mathbb{M}}(\psi_j)|^2, \quad \psi_j \subseteq \Phi,\\
&\sum_{\psi_j \subseteq \Phi} \mathrm{M}(\psi_j) = 1,
\end{aligned}$$

where $|\mathbb{Q}_{\mathbb{M}}(\psi_j)|^2 = \mathbb{Q}_{\mathbb{M}}(\psi_j)\widehat{\mathbb{Q}}_{\mathbb{M}}(\psi_j) = \varphi^2(\psi_j) = x_j^2 + y_j^2$, in which $\widehat{\mathbb{Q}}_{\mathbb{M}}(\psi_j)$ is the complex conjugate of $\mathbb{Q}_{\mathbb{M}}(\psi_j)$, e.g., $\widehat{\mathbb{Q}}_{\mathbb{M}}(\psi_j) = x_j - y_j i$.

Definition 59 (Bayesian QBPD). When the quantum basic probabilities are only assigned to singleton states, M is called a Bayesian QBPD, and $\mathrm{M}(\phi_g)$ is called a Bayesian QBP. Mathematically, a Bayesian QBPD is just a quantum probability distribution.

Definition 60 (Vacuous QBPD). In QET, when $\mathrm{M}(\Phi) = 1$, M is called a vacuous QBPD.

Definition 61 (Quantum basic probability). In QET, $\mathrm{M}(\psi_j)$ $(\psi_j \subseteq \Phi)$ is called quantum basic probability (QBP), which represents the degree of belief or support to ψ_j.

The QET inherits the merits of DSET and has the following attractive characteristics:

- The QBPA function $\mathbb{Q}_{\mathbb{M}}$ in QET can be expressed by not only complex numbers but also positive real numbers, while the BPA m can only be expressed by positive real numbers in DSET.
- In contrast to DSET, $\forall \psi_j \subseteq \Phi$, the value of $|\mathbb{Q}_{\mathbb{M}}(\psi_j)|^2$ or $\varphi^2(\psi_j)$ represents the degree of belief or support to ψ_j.
- When $\mathrm{M} = m$, the QBPD M of QET is in accordance with the classical BPA m in DSET.

Comparison of QET with GQET, the following interpretations and properties can be obtained:

- It is unnecessary for $|\mathbb{Q}_{\mathbb{M}}(\emptyset)| = 0$ in GQET, such that $|\mathbb{Q}_{\mathbb{M}}(\emptyset)| \geq 0$, while $|\mathbb{Q}_{\mathbb{M}}(\emptyset)|$ must be equal to 0 in QET.

- $\emptyset$ can be a focal element as $|\mathbb{Q}_{\mathbb{M}}(\emptyset)| > 0$ in GQET, but $\emptyset$ cannot be a focal element in QET.

- When $|\mathbb{Q}_{\mathbb{M}}(\emptyset)| > 0$, it is utilized to model an open world in GQET, indicating that $\emptyset$ is a focal element or the union of focal elements not within the QFOD, rather than the empty set of QBPA function in QET.

- When $|\mathbb{Q}_{\mathbb{M}}(\emptyset)| = 0$, the GQBPA function $\mathbb{Q}_{\mathbb{M}}$ in GQET degrades into the QBPA function in QET.

4.1.3. Quantum belief and plausibility functions

Definition 62 (Quantum interference belief function in QET). Let $\mathbb{Q}_{\mathbb{M}}$ be a QBPA function with proposition $\psi_j \subseteq \Phi$. A quantum interference belief function QIBel for ψ_j in QET, mapping from 2^{Φ} to $[0, 1]$, is defined by:

$$\mathrm{QIBel}(\psi_j) = \max_{\psi_p} \left| \sum_{\psi_p \subseteq \psi_j} \mathbb{Q}_{\mathbb{M}}(\psi_p) \right|^2, \quad \psi_j \subseteq \Phi. \tag{90}$$

According to Eq. (90), for $\psi_j \subseteq \Phi$, we obtain:

$$\Lambda_A = \arg\max_{\psi_p} \left| \sum_{\psi_p \subseteq \psi_j} \mathbb{Q}_{\mathbb{M}}(\psi_p) \right|^2,$$

where Λ_A includes essential subsets, i.e., $\{\psi_p \subseteq \psi_j\}$, to achieve the maximal modulus squared of Eq. (90).

Consider $\psi_s, \psi_t \subseteq \Lambda_A = \{\psi_p \subseteq \psi_j\}$, and $\psi_s \neq \psi_t$. According to Feynman's rule, and $|\mathbb{Q}_{\mathbb{M}}|^2 = \varphi^2 = \mathrm{M}$, $\mathrm{QIBel}(\psi_j)$ is calculated as:

$$\begin{aligned}\mathrm{QIBel}(\psi_j) &= \sum_{\psi_s \subseteq \Lambda_A} |\mathbb{Q}_{\mathbb{M}}(\psi_s)|^2 + 2 \sum_{\psi_s \subseteq \Lambda_A} \sum_{\substack{\psi_t \subseteq \Lambda_A \\ \psi_s \neq \psi_t}} |\mathbb{Q}_{\mathbb{M}}(\psi_s)||\mathbb{Q}_{\mathbb{M}}(\psi_t)| \cos(\theta_s - \theta_t) \\ &= \sum_{\psi_s \subseteq \Lambda_A} \mathrm{M}(\psi_s) + 2 \sum_{\psi_s \subseteq \Lambda_A} \sum_{\substack{\psi_t \subseteq \Lambda_A \\ \psi_s \neq \psi_t}} \sqrt{\mathrm{M}(\psi_s)} \sqrt{\mathrm{M}(\psi_t)} \cos(\theta_s - \theta_t),\end{aligned} \tag{91}$$

where $-1 \leq \cos(\theta_s - \theta_t) \leq 1$.

Definition 63 (Interference effect in the QIBel function within QET). Let $\mathbb{Q}_{\mathbb{M}}$ be a QBPA function with proposition $\psi_j \subseteq \Phi$. Let $\Lambda_A = \{\psi_p \subseteq \psi_j \mid \psi_j \subseteq \Phi\}$ denote essential subsets to achieve the maximal modulus squared of QIBel, and $\psi_s, \psi_t \subseteq \Lambda_A$ with $\psi_s \neq \psi_t$. The interference effect in the QIBel function is defined by:

$$\begin{aligned}\mathrm{Int}_{\mathrm{QIBel}}(\psi_j) &= 2 \sum_{\psi_s \subseteq \Lambda_A} \sum_{\substack{\psi_t \subseteq \Lambda_A \\ \psi_s \neq \psi_t}} |\mathbb{Q}_{\mathbb{M}}(\psi_s)||\mathbb{Q}_{\mathbb{M}}(\psi_t)| \cos(\theta_s - \theta_t) \\ &= 2 \sum_{\psi_s \subseteq \Lambda_A} \sum_{\substack{\psi_t \subseteq \Lambda_A \\ \psi_s \neq \psi_t}} \sqrt{M(\psi_s)} \sqrt{M(\psi_t)} \cos(\theta_s - \theta_t),\end{aligned} \tag{92}$$

where $-1 \leq \cos(\theta_s - \theta_t) \leq 1$.

When $\mathrm{Int}_{\mathrm{QIBel}} = 0$, Eq. (91) becomes:

$$\mathrm{QIBel}(\psi_j) = \sum_{\psi_s \subseteq \Lambda_A} |\mathbb{Q}_{\mathbb{M}}(\psi_s)|^2 = \sum_{\psi_s \subseteq \Lambda_A} \mathrm{M}(\psi_s). \tag{93}$$

Therefore, when $\mathrm{Int}_{\mathrm{QIBel}} = 0$ and $\mathrm{M} = m$, Eq. (90) becomes:

$$\mathrm{QIBel}(\psi_j) = \sum_{\psi_p \subseteq \psi_j} m(\psi_p), \ \psi_j \subseteq \Phi.$$

which is consistent with the classical Bel in DSET.

Definition 64 (Quantum interference plausibility function in QET). Let $\mathbb{Q}_{\mathbb{M}}$ be a QBPA function with proposition $\psi_j \subseteq \Phi$. A quantum interference plausibility function QIPl in QET, mapping from 2^{Φ} to $[0,1]$, is defined by:

$$\mathrm{QIPl}(\psi_j) = \max_{\psi_p} \left| \sum_{\psi_p \cap \psi_j \neq \emptyset} \mathbb{Q}_{\mathbb{M}}(\psi_p) \right|^2, \quad \psi_j \subseteq \Phi. \tag{94}$$

According to Eq. (94), for $\psi_j \subseteq \Phi$, we have:

$$\Lambda_B = \arg\max_{\psi_p} \left| \sum_{\psi_p \cap \psi_j \neq \emptyset} \mathbb{Q}_{\mathbb{M}}(\psi_p) \right|^2, \tag{95}$$

where Λ_B includes essential subsets to achieve the maximal modulus squared of Eq. (94).

Consider $\psi_u, \psi_v \subseteq \Lambda_B = \{\psi_p \cap \psi_j \neq \emptyset\}$, and $\psi_u \neq \psi_v$. According to Feynman's rule, and $|\mathbb{Q}_{\mathbb{M}}|^2 = \varphi^2 = \mathrm{M}$, $\mathrm{QIPl}(\psi_j)$ is given by:

$$\begin{aligned} \mathrm{QIPl}(\psi_j) &= \sum_{\psi_u \subseteq \Lambda_B} |\mathbb{Q}_{\mathbb{M}}(\psi_u)|^2 + 2 \sum_{\psi_u \subseteq \Lambda_B} \sum_{\substack{\psi_v \subseteq \Lambda_B \\ \psi_u \neq \psi_v}} |\mathbb{Q}_{\mathbb{M}}(\psi_u)||\mathbb{Q}_{\mathbb{M}}(\psi_v)| \cos(\theta_u - \theta_v) \\ &= \sum_{\psi_u \subseteq \Lambda_B} \mathrm{M}(\psi_u) + 2 \sum_{\psi_u \subseteq \Lambda_B} \sum_{\substack{\psi_v \subseteq \Lambda_B \\ \psi_u \neq \psi_t}} \sqrt{\mathrm{M}(\psi_u)}\sqrt{\mathrm{M}(\psi_v)} \cos(\theta_u - \theta_v), \end{aligned} \tag{96}$$

where $-1 \leq \cos(\theta_u - \theta_v) \leq 1$.

Definition 65 (Interference effect in QIPl function within QET). Let $\mathbb{Q}_{\mathbb{M}}$ be a QBPA function with proposition $\psi_j \subseteq \Phi$. Let $\Lambda_B = \{\psi_p \subseteq \psi_j \mid \psi_j \subseteq \Phi\}$ denote essential subsets to achieve the maximal modulus squared of QIPl, and $\psi_u, \psi_v \subseteq \Lambda_B$ with $\psi_u \neq \psi_v$. The interference effect in the QIPl function is defined by:

$$\begin{aligned}\mathrm{Int}_{\mathrm{QIPl}}(\psi_j) &= 2\sum_{\psi_u\subseteq\Lambda_B}\sum_{\substack{\psi_v\subseteq\Lambda_B\\ \psi_u\neq\psi_v}} |\mathbb{Q}_{\mathbb{M}}(\psi_u)||\mathbb{Q}_{\mathbb{M}}(\psi_v)|\cos(\theta_u-\theta_v)\\ &= 2\sum_{\psi_u\subseteq\Lambda_B}\sum_{\substack{\psi_v\subseteq\Lambda_B\\ \psi_u\neq\psi_v}} \sqrt{M(\psi_u)}\sqrt{M(\psi_v)}\cos(\theta_u-\theta_v),\end{aligned} \tag{97}$$

where $-1 \leq \cos(\theta_u-\theta_v) \leq 1$.

When $\mathrm{Int}_{\mathrm{QIPl}} = 0$, Eq. (96) becomes:

$$\mathrm{QIPl}(\psi_j) = \sum_{\psi_u\subseteq\Lambda_B} |\mathbb{Q}_{\mathbb{M}}(\psi_u)|^2 = \sum_{\psi_u\subseteq\Lambda_B} \mathrm{M}(\psi_u). \tag{98}$$

Therefore, when $\mathrm{Int}_{\mathrm{QIPl}} = 0$ and $\mathrm{M} = m$, Eq. (98) becomes:

$$\mathrm{QIPl}(\psi_j) = \sum_{\psi_p\cap\psi_j\neq\emptyset} m(\psi_p),\ \ \psi_j \subseteq \Phi, \tag{99}$$

which is consistent with the classical Pl in DSET.

The functions of QIBel and QIPl in QET have the following properties:

- Similar to those in DSET, $\mathrm{QIBel}(\psi_j)$ and $\mathrm{QIPl}(\psi_j)$ in QET are the lower and upper probabilities for ψ_j, respectively.

- When the QBPD M of QET is in accordance with the BPA m of DSET, and there are no interferences involving in the functions of QIBel and QIPl, QIBel and QIPl in QET degrade into the classical Bel and Pl in DSET, respectively.

Comparison of QIBel and QIPl functions in QET with GQIBel and GQIPl functions in GQET, the following properties can be obtained:

- It is unnecessary for $\text{GQIBel}(\emptyset) = 0$ and $\text{GQIPl}(\emptyset) = 0$ in GQET, such that $\text{GQIBel}(\emptyset) \geq 0$ and $\text{GQIPl}(\emptyset) \geq 0$, while $\text{QIBel}(\emptyset)$ and $\text{QIPl}(\emptyset)$ must be equal to 0 in QET.

- When $|\mathbb{Q}_{\mathbb{M}}(\emptyset)| = 0$, GQIBel and GQIPl in GQET degrade into the QIBel and QIPl in QET, respectively.

In addition, considering the application scenarios in the absence of interference, the alternative quantum belief and plausibility functions in QET are defined as follows.

Definition 66 (Quantum belief function in QET). Let $\mathbb{Q}_{\mathbb{M}}$ be a QBPA function with proposition $\psi_j \subseteq \Phi$. A quantum belief function QBel in QET, mapping from 2^{Φ} to $[0, 1]$, is defined by:

$$\text{QBel}(\psi_j) = \sum_{\psi_p \subseteq \psi_j} \left| \mathbb{Q}_{\mathbb{M}}(\psi_p) \right|^2, \ \psi_j \subseteq \Phi. \tag{100}$$

According to Eq. (31), Eq. (100) can also be represented as:

$$\text{QBel}(\psi_j) = \sum_{\psi_p \subseteq \psi_j} \text{M}(\psi_p), \ \psi_j \subseteq \Phi, \tag{101}$$

which is consistent with the QIBel in Eq. (93) in the case without interference.

Therefore, when $\text{M} = m$, Eq. (101) becomes:

$$\text{QBel}(\psi_j) = \sum_{\psi_p \subseteq \psi_j} m(\psi_p), \ \psi_j \subseteq \Phi. \tag{102}$$

which is consistent with the classical Bel in DSET.

Definition 67 (Quantum plausibility function in QET). Let $\mathbb{Q}_{\mathbb{M}}$ be a QBPA function with proposition $\psi_j \subseteq \Phi$. A generalized quantum plausibility function QPl in QET, mapping from 2^{Φ} to $[0,1]$, is defined by:

$$\mathrm{QPl}(\psi_j) = \sum_{\psi_p \cap \psi_j \neq \emptyset} \left| \mathbb{Q}_{\mathbb{M}}(\psi_p) \right|^2, \ \psi_j \subseteq \Phi. \tag{103}$$

According to Eq. (31), Eq. (103) can also be represented as:

$$\mathrm{QPl}(\psi_j) = \sum_{\psi_p \cap \psi_j \neq \emptyset} \mathrm{M}(\psi_p), \ \psi_j \subseteq \Phi, \tag{104}$$

which is consistent with the QIPl in Eq. (98) in the case without interference.

Therefore, when $\mathrm{M} = m$, Eq. (104) becomes:

$$\mathrm{QPl}(\psi_j) = \sum_{\psi_p \cap \psi_j \neq \emptyset} m(\psi_p), \ \psi_j \subseteq \Phi. \tag{105}$$

which is consistent with the classical Pl in DSET.

The functions of QBel and QPl in QET have the following properties:

- When considering scenarios in the absence of interference, $\mathrm{QBel}(\psi_j)$ and $\mathrm{QPl}(\psi_j)$ in QET are the lower and upper probabilities for ψ_j, respectively.

- In QET, when ther are no interferences involving in the function of QIBel and QIPl, QIBel and QIPl degrade into the QBel and QPl, respectively.

- When the QBPD M of QET is in accordance with the BPA m of DSET, QBel and QPl in QET degrade into the classical Bel and Pl in DSET, respectively.

Comparison of QIBel and QIPl functions in QET with GQIBel and GQIPl functions in GQET, the following properties can be obtained:

- It is unnecessary for $\mathrm{GQBel}(\emptyset) = 0$ and $\mathrm{GQPl}(\emptyset) = 0$ in GQET, such that $\mathrm{GQBel}(\emptyset) \geq 0$ and $\mathrm{GQPl}(\emptyset) \geq 0$, while $\mathrm{QBel}(\emptyset)$ and $\mathrm{QPl}(\emptyset)$ must be equal to 0 in QET.

- When $|\mathbb{Q}_{\mathbb{M}}(\emptyset)| = 0$, GQBel and GQPl in GQET degrade into the QBel and QPl in QET, respectively.

4.1.4. Quantum evidential combination rules

Definition 68 (Quantum evidential combination rule). Let $\{\mathbb{Q}_{\mathbb{M}_1}, \ldots, \mathbb{Q}_{\mathbb{M}_h}, \ldots, \mathbb{Q}_{\mathbb{M}_k}\}$ be a set of independent QBPAs with proposition ψ_j in QFOD Φ. The quantum evidential combination rule (QECR), denoted as $\mathbb{Q}_{\mathbb{M}_1} \oplus \cdots \oplus \mathbb{Q}_{\mathbb{M}_h} \oplus \cdots \oplus \mathbb{Q}_{\mathbb{M}_k}$, is defined as:

$$\mathbb{Q}_{\mathbb{M}_1} \oplus \cdots \oplus \mathbb{Q}_{\mathbb{M}_h} \oplus \cdots \oplus \mathbb{Q}_{\mathbb{M}_k}(\psi_j) = \frac{\left| \sum\limits_{\cap\psi_p=\psi_j} \prod\limits_{1\leq h\leq k} \mathbb{Q}_{\mathbb{M}_h}(\psi_p) \right|^2}{\sum\limits_{\psi_v\subseteq\Phi} \left| \sum\limits_{\cap\psi_p=\psi_v} \prod\limits_{1\leq h\leq k} \mathbb{Q}_{\mathbb{M}_h}(\psi_p) \right|^2}, \quad (106)$$

$$\mathbb{Q}_{\mathbb{M}_1} \oplus \cdots \oplus \mathbb{Q}_{\mathbb{M}_h} \oplus \cdots \oplus \mathbb{Q}_{\mathbb{M}_k}(\emptyset) = 0. \quad (107)$$

Using Feynman's rule, due to $|\mathbb{Q}_{\mathbb{M}_h}|^2 = \varphi_h^2 = \mathrm{M}_h$, Eq. (106) can be rewritten as:

$$\mathbb{Q}_{\mathbb{M}_1} \oplus \cdots \oplus \mathbb{Q}_{\mathbb{M}_h} \oplus \cdots \oplus \mathbb{Q}_{\mathbb{M}_k}(\psi_j) = \frac{\sum\limits_{\cap\psi_p=\psi_j} \prod\limits_{1\leq h\leq k} \mathrm{M}_h(\psi_p) + \mathrm{Int}_{\mathrm{QECR}}}{\sum\limits_{\psi_v\subseteq\Phi} \left(\sum\limits_{\cap\psi_p=\psi_v} \prod\limits_{1\leq h\leq k} \mathrm{M}_h(\psi_p) + \mathrm{Int}_{\mathrm{QECR}} \right)}, \quad (108)$$

where the interference term $\mathrm{Int}_{\mathrm{QECR}}$ is given by:

$$\mathrm{Int}_{\mathrm{QECR}} = 2 \sum_{\cap\psi_p=\psi_j} \sum_{\substack{\cap\psi_q=\psi_j \\ \psi_p\neq\psi_q}} \prod_{1\leq h\leq k} \sqrt{\mathrm{M}_h(\psi_p)} \prod_{1\leq h\leq k} \sqrt{\mathrm{M}_h(\psi_q)} \cos(\theta_p - \theta_q), \quad (109)$$

with $-1 \leq \cos(\theta_p - \theta_q) \leq 1$.

Definition 69 (Interference effect in QECR function). Let $\{\mathbb{Q}_{\mathbb{M}_1}, \ldots, \mathbb{Q}_{\mathbb{M}_h}, \ldots, \mathbb{Q}_{\mathbb{M}_k}\}$ be a set of independent QBPAs with propositions ψ_p and ψ_q ($\psi_p \neq \psi_q$) in QFOD Φ. The interference effect in QECR function is defined as:

$$\begin{aligned} \mathrm{Int}_{\mathrm{QECR}} &= 2 \sum_{\cap\psi_p=\psi_j} \sum_{\substack{\cap\psi_q=\psi_j \\ \psi_p\neq\psi_q}} \prod_{1\leq h\leq k} |\mathbb{Q}_{\mathbb{M}_h}(\psi_p)| \prod_{1\leq h\leq k} |\mathbb{Q}_{\mathbb{M}_h}(\psi_q)| \cos(\theta_p - \theta_q) \\ &= 2 \sum_{\cap\psi_p=\psi_j} \sum_{\substack{\cap\psi_q=\psi_j \\ \psi_p\neq\psi_q}} \prod_{1\leq h\leq k} \sqrt{M_h(\psi_p)} \prod_{1\leq h\leq k} \sqrt{M_h(\psi_q)} \cos(\theta_p - \theta_q), \end{aligned} \tag{110}$$

where $-1 \leq \cos(\theta_p - \theta_q) \leq 1$.

When $\mathrm{Int}_{\mathrm{QECR}} = 0$ and $\mathrm{M_h} = m_h$, Eq. (108) becomes:

$$\begin{aligned} \mathbb{Q}_{\mathbb{M}_1} \oplus \cdots \oplus \mathbb{Q}_{\mathbb{M}_h} \oplus \cdots \oplus \mathbb{Q}_{\mathbb{M}_k}(\psi_j) &= \frac{\sum\limits_{\cap\psi_p=\psi_j} \prod\limits_{1\leq h\leq k} m_h(\psi_p)}{\sum\limits_{\psi_v\subseteq\Phi} \left\{ \sum\limits_{\cap\psi_p=\psi_v} \prod\limits_{1\leq h\leq k} m_h(\psi_p) \right\}} \\ &= \frac{\sum\limits_{\cap\psi_p=\psi_j} \prod\limits_{1\leq h\leq k} m_h(\psi_p)}{1 - K_G}, \end{aligned} \tag{111}$$

in which

$$K_G = \sum_{\cap|\psi_p=\emptyset} \prod_{1\leq h\leq k} m_h(\psi_p). \tag{112}$$

From Eqs. (111) and (112), it is learned that in the case of $\mathrm{M}_h = m_h$ and $\mathrm{Int}_{\mathrm{QECR}} = 0$, QECR degrades into DRC of DSET.

Definition 70 (Quantum evidential conflict coefficient). The quantum evidential conflict coefficient (QECC) among QBPAs $\{\mathbb{Q}_{\mathbb{M}_1}, \cdots, \mathbb{Q}_{\mathbb{M}_h},$

$\cdots, \mathbb{Q}_{\mathbb{M}_k}\}$, denoted as $\mathbb{K}_{\mathbb{Q}}$, is defined by:

$$\mathbb{K}_{\mathbb{Q}} = \left| \sum_{\cap\psi_p=\emptyset} \prod_{1\le h\le k} \mathbb{Q}_{\mathbb{M}_h}(\psi_p) \right|^2 . \tag{113}$$

According to Feynman's rule, since $|\mathbb{Q}_{\mathbb{M}_h}|^2 = \varphi_h^2 = \mathrm{M}_h$, Eq. (113) can also be expressed as:

$$\begin{aligned}
\mathbb{K}_{\mathbb{Q}} = & \sum_{\cap\psi_p=\emptyset} \prod_{1\le h\le k} |\mathbb{Q}_{\mathbb{M}_h}(\psi_p)|^2 \\
& + 2 \sum_{\cap\psi_p=\emptyset} \sum_{\cap\psi_q=\emptyset} \prod_{1\le h\le k} |\mathbb{Q}_{\mathbb{M}_h}(\psi_p)| \prod_{1\le h\le k} |\mathbb{Q}_{\mathbb{M}_h}(\psi_q)| \cos(\theta_p - \theta_q) \\
= & \sum_{\cap\psi_p=\emptyset} \prod_{1\le h\le k} \mathrm{M}_h(\psi_p) \\
& + 2 \sum_{\cap\psi_p=\emptyset} \sum_{\cap\psi_q=\emptyset} \prod_{1\le h\le k} \sqrt{\mathrm{M}_h(\psi_p)} \prod_{1\le h\le k} \sqrt{\mathrm{M}_h(\psi_q)} \cos(\theta_p - \theta_q),
\end{aligned} \tag{114}$$

in which $-1 \le \cos(\theta_p - \theta_q) \le 1$.

Definition 71 (Interference effect in QECC function). The interference effect involved in QECC function among QBPAs $\{\mathbb{Q}_{\mathbb{M}_1}, \cdots, \mathbb{Q}_{\mathbb{M}_h}, \cdots, \mathbb{Q}_{\mathbb{M}_k}\}$, denoted as $\mathrm{Int}_{\mathbb{K}_{\mathbb{Q}}}$, is defined by:

$$\begin{aligned}
\mathrm{Int}_{\mathbb{K}_{\mathbb{Q}}} &= 2 \sum_{\cap\psi_p=\emptyset} \sum_{\cap\psi_q=\emptyset} \prod_{1\le h\le k} |\mathbb{Q}_{\mathbb{M}_h}(\psi_p)| \prod_{1\le h\le k} |\mathbb{Q}_{\mathbb{M}_h}(\psi_q)| \cos(\theta_p - \theta_q) \\
&= 2 \sum_{\cap\psi_p=\emptyset} \sum_{\cap\psi_q=\emptyset} \prod_{1\le h\le k} \sqrt{\mathrm{M}_h(\psi_p)} \prod_{1\le h\le k} \sqrt{\mathrm{M}_h(\psi_q)} \cos(\theta_p - \theta_q),
\end{aligned} \tag{115}$$

in which $-1 \le \cos(\theta_p - \theta_q) \le 1$.

When $\mathrm{Int}_{\mathbb{K}_{\mathbb{Q}}} = 0$ and $\mathrm{M_h} = m_h$, Eq. (113) becomes:

$$\mathbb{K}_{\mathbb{Q}} = \sum_{\cap\psi_p=\emptyset} \prod_{1\le h\le k} m_h(\psi_p),$$

which is consistent with the conflict coefficient K in DSET.

QECR has the following characteristics:

- When $\mathbb{Q}_{\mathbb{M}_1} \oplus \cdots \oplus \mathbb{Q}_{\mathbb{M}_h} \oplus \cdots \oplus \mathbb{Q}_{\mathbb{M}_k}(\emptyset) = 0$, GQECR reduces to the QECR.

- When $\mathbb{Q}_{\mathbb{M}_1} \oplus \cdots \oplus \mathbb{Q}_{\mathbb{M}_h} \oplus \cdots \oplus \mathbb{Q}_{\mathbb{M}_k}(\emptyset) = 0$, GQECC $\mathbb{K}_{\mathbb{Q}}$ in GQET reduces to QECC in QET.

- Since $|\mathbb{Q}_{\mathbb{M}}(\emptyset)| = 0$ indicating a closed world and $|\mathbb{Q}_{\mathbb{M}}(\emptyset)| > 0$ indicating an open world, the GQECR can merge arbitrary multiple GQBPAs to facilitate uncertainty reasoning in an open world, while QECR can merge arbitrary multiple QBPAs to facilitate uncertainty reasoning in a closed world.

- When the QBPDs of QET are in accordance with the BPAs of DSET, and there are no interferences involving in QECR, QECR reduces to the DRC in DSET.

- When the QBPDs of QET are in accordance with the BPAs of DSET, and there are no interferences involving in QECR, QECC $\mathbb{K}_{\mathbb{Q}}$ reduces to the conflict coefficient K in DSET.

To support a wide range of applications, an alternative progressive quantum evidential combination rule is proposed to enable the continuous, progressive, and incremental fusion of QBPAs.

Definition 72 (Progressive quantum evidential combination rule). Let $\{\mathbb{Q}_{\mathbb{M}_1}, \ldots, \mathbb{Q}_{\mathbb{M}_h}, \ldots, \mathbb{Q}_{\mathbb{M}_k}\}$ be a set of independent QBPAs with proposition ψ_j in QFOD Φ. The progressive quantum evidential combination rule

(PQECR), denoted as $\mathbb{Q}_{\mathbb{M}_1} \oplus \cdots \oplus \mathbb{Q}_{\mathbb{M}_h} \oplus \cdots \oplus \mathbb{Q}_{\mathbb{M}_k}$, is defined as:

$$\mathbb{Q}_{\mathbb{M}_1} \oplus \cdots \oplus \mathbb{Q}_{\mathbb{M}_h} \oplus \cdots \oplus \mathbb{Q}_{\mathbb{M}_k}(\psi_j) = \frac{\sum\limits_{\cap\psi_p=\psi_j} \prod\limits_{1\leq h\leq k} \mathbb{Q}_{\mathbb{M}_h}(\psi_p)}{\sum\limits_{\psi_v\subseteq\Phi} \sum\limits_{\cap\psi_p=\psi_v} \prod\limits_{1\leq h\leq k} \mathbb{Q}_{\mathbb{M}_h}(\psi_p)}, \tag{116}$$

$$\mathbb{Q}_{\mathbb{M}_1} \oplus \cdots \oplus \mathbb{Q}_{\mathbb{M}_h} \oplus \cdots \oplus \mathbb{Q}_{\mathbb{M}_k}(\emptyset) = 0. \tag{117}$$

Definition 73 (Progressive quantum evidential conflict coefficient). The progressive quantum evidential conflict coefficient (PQECC) among QBPAs $\{\mathbb{Q}_{\mathbb{M}_1}, \cdots, \mathbb{Q}_{\mathbb{M}_h}, \cdots, \mathbb{Q}_{\mathbb{M}_k}\}$, denoted as $\mathbb{K}_{\mathbb{Q}}$, is defined by:

$$\mathbb{K}_{\mathbb{Q}} = \sum_{\cap\psi_p=\emptyset} \prod_{1\leq h\leq k} \mathbb{Q}_{\mathbb{M}_h}(\psi_p). \tag{118}$$

PQECR satisfies the following properties:

- **Commutativity:** Ensures that the fusion result of multisource uncertain information is independent of the order of information.

$$\mathbb{Q}_{\mathbb{M}_1} \oplus \mathbb{Q}_{\mathbb{M}_2} = \mathbb{Q}_{\mathbb{M}_2} \oplus \mathbb{Q}_{\mathbb{M}_1}.$$

- **Associativity:** Grants the system the capability of flexibly structured fusion of multisource uncertain information.

$$(\mathbb{Q}_{\mathbb{M}_1} \oplus \mathbb{Q}_{\mathbb{M}_2}) \oplus \mathbb{Q}_{\mathbb{M}_3} = \mathbb{Q}_{\mathbb{M}_1} \oplus (\mathbb{Q}_{\mathbb{M}_2} \oplus \mathbb{Q}_{\mathbb{M}_3}).$$

- **Identity:** Ensures that the fusion result remains unchanged when no new effective information is added.

$$\text{Let } \mathbb{Q}_{\mathbb{M}_0}(\Phi) = 1, \quad \mathbb{Q}_{\mathbb{M}_1} \oplus \mathbb{Q}_{\mathbb{M}_0} = \mathbb{Q}_{\mathbb{M}_1}.$$

- **Focalization:** Enables the fusion of consistent quantum evidence to reduce the quantum focal element, guiding the fusion result toward convergence.

PQECR has the following characteristics:

- When $\mathbb{Q}_{\mathbb{M}_1} \oplus \cdots \oplus \mathbb{Q}_{\mathbb{M}_h} \oplus \cdots \oplus \mathbb{Q}_{\mathbb{M}_k}(\emptyset) = 0$, GPQECR reduces to the PQECR.

- When $\mathbb{Q}_{\mathbb{M}_1} \oplus \cdots \oplus \mathbb{Q}_{\mathbb{M}_h} \oplus \cdots \oplus \mathbb{Q}_{\mathbb{M}_k}(\emptyset) = 0$, GPQECC $\mathbb{K}_{\mathbb{Q}}$ in GQET reduces to PQECC in QET.

- Since $|\mathbb{Q}_{\mathbb{M}}(\emptyset)| = 0$ indicating a closed world and $|\mathbb{Q}_{\mathbb{M}}(\emptyset)| > 0$ indicating an open world, the GPQECR can merge arbitrary multiple GQBPAs to facilitate uncertainty reasoning in an open world, while PQECR can merge arbitrary multiple QBPAs to facilitate uncertainty reasoning in a closed world.

- When the QBPDs of QET are in accordance with the BPAs of DSET, PQECR reduces to the DRC in DSET.

- When the QBPDs of QET are in accordance with the BPAs of DSET, PQECC $\mathbb{K}_{\mathbb{Q}}$ reduces to the conflict coefficient K in DSET.

4.1.5. Quantum Pignistic transformations

When considering the interference during the process of decision-making, a quantum interference Pignistic transformation is defined to transform a QBPA function into a quantum interference Pignistic probability function.

Definition 74 (Quantum interference Pignistic transformation). Let $\mathbb{Q}_{\mathbb{M}}$ be a QBPA function on QFOD Φ and ψ_j be a quantum proposition with $\psi_j \subseteq \Phi$. Its quantum interference Pignistic probability function, denoted as BetQIP, is defined as:

$$\mathrm{BetQIP}(\psi_j) = \left| \sum_{\psi_p \subseteq \Phi} \mathbb{Q}_{\mathbb{M}}(\psi_p) \cdot \frac{|\psi_p \cap \psi_j|}{|\psi_p|} \right|^2, \tag{119}$$

where $|\psi|$ represents the cardinality of ψ. The transformation between $\mathbb{Q}_{\mathbb{M}}$ and BetQIP is called a quantum interference Pignistic transformation.

$\forall \phi_g \in \Phi$, Eq. (119) can be expressed as:

$$\mathrm{BetQIP}(\phi_g) = \left| \sum_{\phi_p \subseteq \Phi} \mathbb{Q}_{\mathbb{M}}(\psi_p) \cdot \frac{|\psi_p \cap \phi_g|}{|\psi_p|} \right|^2. \tag{120}$$

When considering scenarios in the absence of interference, a quantum Pignistic transformation is defined to transform a QBPA function into a quantum Pignistic probability function.

Definition 75 (Quantum Pignistic transformation). Let $\mathbb{Q}_{\mathbb{M}}$ be a QBPA function on QFOD Φ and ψ_j be a quantum proposition with $\psi_j \subseteq \Phi$. Its quantum Pignistic probability function, denoted as BetQP, is defined as:

$$\mathrm{BetQP}(\psi_j) = \sum_{\psi_p \subseteq \Phi} |\mathbb{Q}_{\mathbb{M}}(\psi_p)|^2 \cdot \frac{|\psi_p \cap \psi_j|}{|\psi_p|}, \tag{121}$$

where $|\psi|$ represents the cardinality of ψ. The transformation between $\mathbb{Q}_{\mathbb{M}}$ and BetQP is called a quantum Pignistic transformation.

Since $|\mathbb{Q}_{\mathbb{M}}|^2 = \varphi^2 = \mathrm{M}$, Eq. (121) can be expressed as:

$$\mathrm{BetQP}(\psi_j) = \sum_{\psi_p \subseteq \Phi} \mathrm{M}(\psi_p) \cdot \frac{|\psi_p \cap \psi_j|}{|\psi_p|}. \tag{122}$$

$\forall \phi_g \in \Phi$, Eq. (121) and Eq. (122) can be expressed as:

$$\text{BetGQP}(\phi_g) = \sum_{\psi_p \subseteq \Phi} |\mathbb{Q}_{\mathbb{M}}(\psi_p)|^2 \cdot \frac{|\psi_p \cap \phi_g||}{|\psi_p||}, \tag{123}$$

$$\text{BetGQP}(\phi_g) = \sum_{\psi_p \subseteq \Phi} \text{M}(\psi_p) \cdot \frac{|\psi_p \cap \phi_g||}{|\psi_p||}. \tag{124}$$

4.2. A quantum representation of QET in Hilbert space

4.2.1. Quantum state representation of a QBPA function

Definition 76 (Basis event in QET). Let Φ be a QFOD consisting of a set of mutually exclusive and collectively nonempty events $\{\phi_1, ..., \phi_g, ..., \phi_n\}$. A set of basis events in QET is defined as:

$$\mathbb{B} = \{\phi_1, \ldots, \phi_g, \ldots, \phi_n\}, \tag{125}$$

where $e_w \in \mathbb{B}$ $(1 \leq w \leq n)$ is called a basis event.

Definition 77 (Standard basis vector representation of a basis event in QET). Let $|e_w\rangle$ be a standard basis vector corresponding to the basis event e_w of QET, where $|e_w\rangle \in \mathcal{B} = \{|\phi_1\rangle, \ldots, |\phi_g\rangle, \ldots, |\phi_n\rangle\}$ $(1 \leq w \leq n)$. A standard basis vector $|e_w\rangle$ in Hilbert space is defined as:

$$|e_w^j\rangle = [\eta_1, \ldots, \eta_l, \ldots, \eta_n]^T, \quad \eta_l = \begin{cases} 1, & l = w, \\ 0, & l \neq w. \end{cases} \tag{126}$$

where any two standard basis vectors corresponding to the basis events: $|e_w^j\rangle$ and $|e_s^j\rangle$ $(1 \leq s, w \leq n; s \neq w)$ are orthonormal, namely, $\langle e_w^j | e_w^j\rangle = 1$, and $\langle e_w^j | e_s^j\rangle = 0$.

Definition 78 (State vector representation of a quantum proposition in QET). Let $|\psi_j\rangle$ be a standard basis vector corresponding to the quantum proposition ψ_j $(\psi_j \subseteq \Phi)$ of QBPA function $\mathbb{Q}_{\mathbb{M}}$ with basis event $e_w^j \in \psi_j$ $(1 \leq w \leq n)$. The state vector $|\psi_j\rangle$ in Hilbert space is defined as:

$$|\psi_j\rangle = \sum_{e_w^j \in \psi_j} \lambda_w^j |e_w^j\rangle, \tag{127}$$

where $\lambda_w^j = a_w^j + b_w^j i$ is a complex number satisfying $\sum_w |\lambda_w^j|^2 = 1$.

From Eq. (127), $|\psi_j\rangle$ is a pure quantum state. It can be seen that in the framework of QET, a composite quantum proposition is represented in Hilbert space by a unit vector formed as a linear superposition of the basis vectors corresponding to its constituent elementary quantum propositions. In this context, a corresponding density operator of a quantum proposition is represented as follows.

Definition 79 (Density operator representation of a quantum proposition in GET). Let $|\psi_j\rangle$ be a state vector corresponding to its quantum proposition ψ_j $\psi_j \subseteq \Phi$ on QFOD. Its density operator is defined as:

$$\rho_j = |\psi_j\rangle\langle\psi_j|. \tag{128}$$

On this basis, a QBPA function is expressed as an ensemble of pure states as follows.

Definition 80 (Linear operator representation of a QBPA function). Let $\mathbb{Q}_{\mathbb{M}}$ be a QBPA function with $\mathbb{Q}_{\mathbb{M}}(\psi_j)$ on QFOD $|\Phi\rangle$, and $\{\mathbb{Q}_{\mathbb{M}}(\psi_j), |\psi_j\rangle\}$ be an ensemble of pure quantum states. The linear operator corresponding to a QBPA function is defined as:

$$\tilde{\rho}_{\mathbb{Q}_{\mathbb{M}}} = \sum_j \mathbb{Q}_{\mathbb{M}}(\psi_j)|\psi_j\rangle\langle\psi_j| = \sum_j \mathbb{Q}_{\mathbb{M}}(\psi_j)\rho_j, \tag{129}$$

where a QBPA function $\mathbb{Q}_{\mathbb{M}}$ is expressed as a quantum system of mixed quantum states.

Definition 81 (Normalized density operator representation of a QBPA function). The normalized density operator corresponding to a QBPA function is defined as:

$$\widehat{\rho}_{\mathbb{Q}_{\mathbb{M}}} = \sum_{j} |\mathbb{Q}_{\mathbb{M}}(\psi_j)|^2 \rho_j = \sum_{j} |\mathbb{Q}_{\mathbb{M}}(\psi_j)|^2 |\psi_j\rangle\langle\psi_j|, \tag{130}$$

where $|\cdot|^2$ is the modulus squared function.

From Eq. (130), it is obvious that

$$\widehat{\rho}_{\mathbb{Q}_{\mathbb{M}}} > 0,$$

and

$$\operatorname{tr}(\widehat{\rho}_{\mathbb{Q}_{\mathbb{M}}}) = \sum_{j} |\mathbb{Q}_{\mathbb{M}}(\psi_j)|^2 \operatorname{tr}(|\psi_j\rangle\langle\psi_j|) = \sum_{j} |\mathbb{Q}_{\mathbb{M}}(\psi_j)|^2 = 1.$$

Therefore, $\widehat{\rho}_{\mathbb{Q}_{\mathbb{M}}}$ is a positive and normalized density operator.

Let $\widehat{\mathbb{E}}_\nu$ be the eigenvalues of $\widehat{\rho}_{\mathbb{Q}_{\mathbb{M}}}$. It follows that:

$$\widehat{\mathbb{E}}_\nu \geq 0, \quad \sum_{\nu} \widehat{\mathbb{E}}_\nu = 1.$$

Since $|\mathbb{Q}_{\mathbb{M}}|^2 = \varphi^2 = \mathrm{M}$, the density operator of a QBPD can be expressed as follows.

Definition 82 (Density operator representation of a QBPD). The density operator corresponding to a QBPD is defined as:

$$\rho_{\mathrm{M}} = \sum_{j} \mathrm{M}(|\psi_j\rangle) |\psi_j\rangle\langle\psi_j| = \sum_{j} \mathrm{M}(|\psi_j\rangle) \rho_j. \tag{131}$$

4.2.2. Measurement operator for basis event in QET

Definition 83 (Measurement operator for basis event in QET). A measurement operator to measure the quantum probability of a basis event $e_w \in \mathbb{B} = \{\phi_1, \ldots, \phi_g, \ldots, \phi_n\}$ in QET is defined as:

$$\mathbb{M}_{e_w} = |e_w\rangle\langle e_w|, \quad 1 \leq w \leq n, \tag{132}$$

satisfying

$$\sum_{w=1}^{n} \mathbb{M}_{e_w}^{\dagger} \mathbb{M}_{e_w} = I, \tag{133}$$

where $\mathbb{M}_{e_w}^{\dagger}$ is the Hermitian conjugate or adjoint of the $\mathbb{M}_{e_w}$ matrix, e.g., $\mathbb{M}_{e_w}^{\dagger} = \left(\mathbb{M}_{e_w}^{\mathrm{T}}\right)^*$; and I denotes the identity matrix.

4.2.3. Basis event measurement function in QET

Definition 84 (Basis event measurement function in QET). Let ρ_{M} be a density operator corresponding to a QBPD M on QFOD Φ with basis event $e_w \in \mathbb{B} = \{\phi_1, \ldots, \phi_g, \ldots, \phi_n\}$, and $\left\{\mathbb{M}_{e_w} = |e_w\rangle\langle e_w|, 1 \leq w \leq n\right\}$ be a set of measurement operators. The basis event measurement function in QET is defined as:

$$\mathrm{P}(\mathbb{Q}_{\mathbb{M}}(e_w)) = \mathrm{Tr}\left(\mathbb{M}_{e_w}^{\dagger} \mathbb{M}_{e_w} \rho_{\mathrm{M}}\right), \quad 1 \leq w \leq n, \tag{134}$$

where $\mathbb{M}_{e_w}^{\dagger}$ is the Hermitian conjugate or adjoint of the $\mathbb{M}_{e_w}$ matrix, e.g., $\mathbb{M}_{e_w}^{\dagger} = \left(\mathbb{M}_{e_w}^{\mathrm{T}}\right)^*$; and Tr is a function of the trace of a matrix.

4.2.4. QBP measurement function

Definition 85 (QBP measurement function). Let ρ_{M} be a density operator corresponding to a QBPD M on QFOD Φ with basis event $e_w \in \mathbb{B} =$

$\{\phi_1, \ldots, \phi_g, \ldots, \phi_n\}$, and $\{\mathbb{M}_{e_w} = |e_w\rangle\langle e_w|, 1 \leq w \leq n\}$ be a set of measurement operators. The QBP measurement function is defined as:

$$\mathrm{M}(\psi_j) = \sum_{e_w \in \psi_j} \mathrm{P}(\mathbb{Q}_{\mathbb{M}}(e_w)), \quad \psi_j \subseteq \Phi, \tag{135}$$

satisfying

$$\mathrm{P}(\mathbb{Q}_{\mathbb{M}}(e_w)) = \mathrm{Tr}\left(\mathbb{M}_{e_w}^{\dagger}\mathbb{M}_{e_w}\rho_{\mathrm{M}}\right),$$

where $\mathbb{M}_{e_w}^{\dagger}$ is the Hermitian conjugate or adjoint of the $\mathbb{M}_{e_w}$ matrix, e.g., $\mathbb{M}_{e_w}^{\dagger} = \left(\mathbb{M}_{e_w}^{\mathrm{T}}\right)^*$; and Tr is a function of the trace of a matrix.

4.2.5. Belief and plausibility measurement functions in QET

Definition 86 (Belief and plausibility measurement functions in QET). Let ρ_{M} be a density operator corresponding to a QBPD M on QFOD Φ with basis event $e_w \in \mathbb{B} = \{\phi_1, \ldots, \phi_g, \ldots, \phi_n\}$, and $\{\mathbb{M}_{e_w} = |e_w\rangle\langle e_w|, 1 \leq w \leq n\}$ be a set of measurement operators. The belief and plausibility measurement functions are defined as:

$$\begin{cases} \mathrm{QBel}(\psi_j) = \min\left\{\sum_{e_w \in \psi_j} \mathrm{P}(\mathbb{Q}_{\mathbb{M}}(e_w))\right\}, & \psi_j \subseteq \Phi, \\ \mathrm{QPl}(\psi_j) = \max\left\{\sum_{e_w \in \psi_j} \mathrm{P}(\mathbb{Q}_{\mathbb{M}}(e_w))\right\}, & \psi_j \subseteq \Phi, \end{cases} \tag{136}$$

satisfying

$$\mathrm{P}(\mathbb{Q}_{\mathbb{M}}(|e_w\rangle)) = \mathrm{Tr}\left(\mathbb{M}_{e_w}^{\dagger}\mathbb{M}_{e_w}\rho_{\mathrm{M}}\right),$$

where $\mathbb{M}_{e_w}^{\dagger}$ is the Hermitian conjugate or adjoint of the $\mathbb{M}_{e_w}$ matrix, e.g., $\mathbb{M}_{e_w}^{\dagger} = \left(\mathbb{M}_{e_w}^{\mathrm{T}}\right)^*$; and Tr is a function of the trace of a matrix.

According to Eqs. (135) and (136), we have:

$$\mathrm{QBel}(\psi_j) \leq \mathrm{M}(\psi_j) \leq \mathrm{QPl}(\psi_j). \tag{137}$$

The belief and plausibility measurement functions represent the lower and upper probabilities for ψ_j.

4.2.6. Progressive quantum evidential combination rule in Hilbert space

Definition 87 (Progressive quantum evidential combination rule in Hilbert space). The progressive quantum evidential combination rule in Hilbert space is defined as:

$$\rho_{\mathbb{Q}_{\mathbb{M}_1}} \oplus \cdots \oplus \rho_{\mathbb{Q}_{\mathbb{M}_h}} \cdots \oplus \rho_{\mathbb{Q}_{\mathbb{M}_k}} = \frac{1}{\gamma} \sum_j \left(\sum_{\cap \psi_p = \psi_j} \prod_{1 \leq h \leq k} \mathbb{Q}_{\mathbb{M}_h}(\psi_p) \right) |\psi_j\rangle\langle\psi_j| \quad (138)$$

$$= \frac{1}{\gamma} \sum_j \left(\sum_{\cap \psi_p = \psi_j} \prod_{1 \leq h \leq k} \mathbb{Q}_{\mathbb{M}_h}(\psi_p) \right) \rho_j, \quad (139)$$

where γ is the normalization factor:

$$\gamma = \sum_{\psi_v \subseteq \Phi} \sum_{\cap \psi_p = \psi_v} \prod_{1 \leq h \leq k} \mathbb{Q}_{\mathbb{M}_h}(\psi_p). \quad (140)$$

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