Special Theory of Relativity in 4-Dimensional Euclidean Space

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Abstract

The construction of space-time in a 4-dimensional Euclidean space is considered, where a scalar field is defined that has no distinguished directions. When constructing space-time, the hypothesis is used that the causality principle is applied separately and independently for different inertial reference frames. It is shown that when the relative velocity of two inertial reference frames tends to zero, the equations of direct transformations tend to the form that the equations of special relativity have. As a consequence, the sameness of the maximum velocity in all inertial reference frames is obtained and the principle of locality is derived.

Introduction

Hypothesis [1] suggests that the causality principle applies separately and independently to each different inertial reference frame (IFR). Two IFRs are considered to be different if they have non-zero relative velocity. The hypothesis also suggests that the smaller the difference in the velocities of two IFRs, the smaller the difference in the application of the causality principle to both IFRs simultaneously and to each IFR separately. If the hypothesis is true, there should be two types of transformations of space-time and fields when moving between IFRs. The first type of transformations are transformations from the observer's point of view. According to these transformations, events are the same in all IFRs. The second type of transformations of space-time and fields, direct transformations, are transformations of space-time and fields based on fields observed in different IFRs by observers stationary relative to the corresponding IFRs. The hypothesis makes it impossible for observers to obtain information about events in inertial reference frames moving relative to them and to directly compare them.

We need the possibility that an event can exist only in a part of an IFR. Since in any space-time any event exists in all IFRs, then space-time cannot be fundamental. The hypothesis leads to the conclusion that each IFR corresponds to its own space-time.

According to the hypothesis, the principle of causality should apply separately and independently for each different IFR. For each inertial reference frame K, the following equation should be satisfied:

$$\varphi(t + dt, K) = A\varphi(t, K) \tag{1}$$

Here φ is the state of the system or its wave function when using the quantum description, t is time, A is some operator.

As the relative velocity of two IFRs tends to zero, the difference in events must also tend to zero. This means that direct transformations must turn into transformations from the observer's point of view as the relative velocity of two IFRs tends to zero.

When considering the hypothesis, an example of constructing space-time on a plane was given. Let us expand this example. Let us consider constructing space-time in 4-dimensional Euclidean space.

Model

Consider a 4-dimensional Euclidean space. We assume that at each point of this space there is a scalar field. We assume that the field equation has no distinguished directions. The values of this field belong to the set of real numbers. The value of the field at each point is determined by the values of the field at neighboring points or, in other words, is described by differential equations in partial derivatives. This can be written as follows:

$$f(x) = g\big(x, S, f(S)\big) \qquad (2)$$

where x is some point in the fundamental space, $f(x)$ is the value of the field at the point x, S is a closed surface surrounding the point x, $f(S)$ is the value of the field on the surface S, g is some function.

Let us look for how to transform the space (x_1, x_2, x_3, x_4) into a set S consisting of space-times $((x, y, z, t), K)$,, where (x, y, z) is space, t is time, K is the inertial frame of reference to which the space-time (x, y, z, t) corresponds, and where equation 1 is satisfied.

Let us take some 3-dimensional hyperplane in space. On this hyperplane, we expand the field f by some complete system of orthonormal functions so that the field at each point is equal to the sum of functions with some coefficients. We seek such an expansion that when the hyperplane is shifted by a distance l perpendicularly, the following equation is satisfied for any l:

$$\varphi(l) = A(l)\varphi(0) \qquad (3)$$

Here φ is a set consisting of the values of the coefficients of the expansion of the field into functions at each point of the hyperplane, A is some operator.

We have a candidate for space and time. The hyperplane here acts as a candidate for space, l acts as a candidate for time. It is obvious that equations 1 and 3 are similar to each other.

Now, we require that equation 3 be satisfied for an arbitrary hyperplane, with shifts of this hyperplane by an arbitrary distance l perpendicularly. In this case, we require that the operator $A(l)$ be the same for all hyperplanes. That is, that it depends only on l and φ. It is clear that this is not possible for any field. Therefore, we consider only a field that allows this to be obtained.

Now let's look for how to add transitions between IFRs to such a model. Let's rotate the previous space-time (x, y, z, t) by an angle a in the space (x_1, x_2, x_3, x_4), and go to (x', y', z', t'). We assume that the time axis should always be perpendicular to the hyperplane of space. Equation 3 is still satisfied after the rotation, there is a parameter of changes. Obviously, the distance between any two points belonging to (x, y, z) and (x', y', z'), respectively, changes uniformly and proportionally to the time interval t or t', and the rate of its change depends on the angle a. Therefore, we can say that a candidate for an inertial reference system has been found. Accordingly, space-times (x, y, z, t) and (x', y', z', t') correspond to different inertial reference systems if their axes have a non-zero angle relative to each other.

If we have found space-time, then the operator A describes the laws of physics from the observer's point of view. Therefore, if the operator is the same for all hyperplanes, then this means that the equations describing the laws of physics from the observer's point of view are the same in all IFRs. This means that Einstein's principle of relativity is fulfilled.

Before moving on, let us consider the following question. According to the hypothesis, each IFR, in general, has its own space-time. Let there be two IFRs, K and K', having non-zero relative velocity. In one of them, let it be K, there is some point (x, y, z, t) in the space-time of this IFR. We need to find which point (x', y', z', t') in the space-time corresponding to K' corresponds to this point.

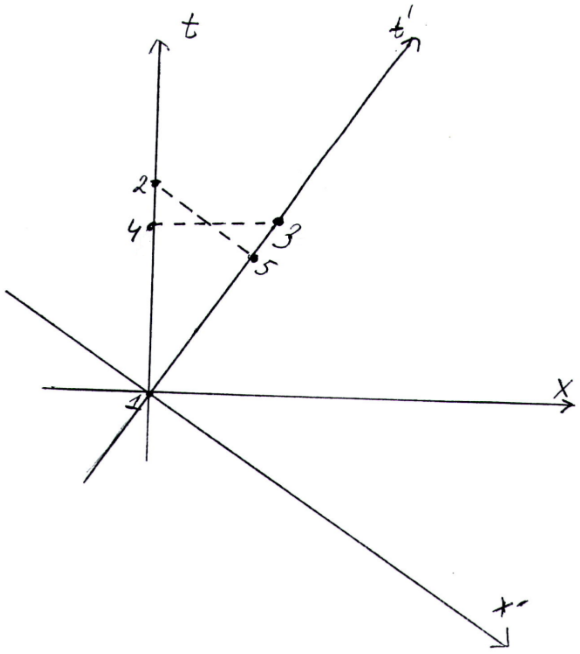
The first method that is visible for finding the correspondence of these points is geometric. The space of each IFR, at any moment in time, corresponds to a 3-dimensional hyperplane. IFRs that have a non-zero relative velocity have slopes of hyperplanes relative to each other. We look for where they intersect, and on this basis, we compare the points.

This method shows a problem with events. Let's assume that some event occurred in the first IFR. Let's assume that the second IFR also contains this event, but it is slightly shifted relative to the intersection point. Then it turns out that in one IFR at some point some event occurred, and in the other IFR this event occurred not at the point that corresponds to the first, but somewhere nearby. Can this be called a correct comparison of space-time points? It does not seem so. We can try to make a comparison based on events. If an event occurred in some space-time of some IFR, then we can assume that the corresponding point in the space-time of another IFR is the point where this event also occurred. However, this method is obviously ineffective, because an event can exist in the space-time of one IFR and be absent in the space-time of another IFR. Based on this, we can conclude that in the general case it is impossible to compare a point from the space-time of one IFR with a point in the space-time of another IFR.

According to the second postulate of the hypothesis, the smaller the relative difference in the velocities of two IFRs, the smaller the difference in events between the IFRs, and as the relative velocity tends to zero, this difference tends to zero. Based on this, the smaller the relative velocity, the more accurately the geometric comparison of points on the hypersurface will display the correspondence of events. That is, we can say that as the relative velocity of two IFRs tends to zero, direct transformations of space-time and fields tend to a simple geometric comparison of points on hyperplanes. Let us find this geometric comparison.

Let us call v_t the distance in the original 4-dimensional space corresponding to a unit of time. According to the above, this value is the same in all reference systems.

Let there be two IFRs moving relative to each other with velocity v along the x -axis, and their initial coordinate points coincide. Then this means that in the original 4-dimensional space, the hyperplanes of these IFRs are inclined to each other at some angle α. Since we consider that the time axis is always perpendicular to the hyperplane, we can say that this angle is the angle between the time axes of the two hyperplanes.



The figure above shows the x and t axes for the first frame of reference and the x' and t' axes for the second frame of reference. The y and z axes coincide. The second frame of reference, moving with a relative velocity v, is inclined at an angle α relative to the first. It should be emphasized that the t axis is an ordinary spatial axis in Euclidean space. The length l along this axis is related to the observed time by the following relation:

$$t = l/v_t$$

Instantaneous events are those events that occur on one hyperplane perpendicular to the t -axis.

Since v_t is the same in all inertial reference systems, then $v = v_t \operatorname{tg}(\alpha)$, where α is the angle between the t and t' axes.

Let t be the time that has passed in the first frame of reference from point 1, and t' be the time that has passed in the moving frame of reference during time t. The time interval t in the first frame of reference corresponds to the distance $v_t t$, this is the distance between points 1 and 4. The same time interval t in the second frame of reference corresponds to the same distance, this is the distance between points 1 and 5. Point 2 is the intersection of the line perpendicular to the t' axis and passing through point 5. Similarly, point 3 is the intersection of the line perpendicular to the t axis and passing through point 4. In order to determine what time interval in the first frame of reference corresponds to the time t' in the second, you need to find the length of the hypotenuse of the triangle from points 1, 5 and 2. From the figure it is clear that

$$t = \frac{t'}{\cos(\alpha)}$$

Now let us consider how these equations obtained above will behave as α tends to zero, which corresponds to the relative velocity tending to zero.

At small angles

$$\operatorname{tg}(\alpha) \approx \sin(\alpha)$$

From here

$$\sin(\alpha) \approx v/v_t$$

Then, from the known value of sine, we obtain:

$$\cos(\alpha) = \sqrt{1 - sin^2(\alpha)} = \sqrt{1 - \left(\frac{v}{v_t}\right)^2}$$

$$t = \frac{t'}{\sqrt{1 - \left(\frac{v}{v_t}\right)^2}}$$

From the same figure it is clear that

$$t' = \frac{t}{\cos{(\alpha)}} = \frac{t}{\sqrt{1 - \left(\frac{v}{v_t}\right)^2}}$$

Now consider coordinate transformations. Let the velocity v be directed along the x-axis. Then, when the coordinate system is rotated, y and z remain unchanged:

$$y = y'$$

$$z = z'$$

In the second frame of reference, after rotation, $x' = x_0/\cos{(\alpha)}$

Then

$$x' = (x - vt)/\cos(\alpha) = \frac{x - vt}{\sqrt{1 - \left(\frac{v}{v_t}\right)^2}}$$

$$t' = \frac{t - (v/{v_t}^2)x}{\sqrt{1 - \left(\frac{v}{v_t}\right)^2}}$$

These equations take on a known form if

$$v_t = \mathrm{c}$$

where c is the speed of light.

From the obtained equations it follows that there is a maximum possible speed equal to v_t. Since we are considering such a field equation in a 4-dimensional space that does not have distinguished directions, then v_t is the same for all IFRs. Let me remind you that we construct space-time so that the principle of causality is fulfilled. This speed is maximum, since if it is exceeded, the principle of causality will not be fulfilled.

It can also be stated that the principle of locality has been obtained. However, equation 2 has nothing similar to locality. This equation is defined not on space-time, but on a more fundamental 4-dimensional space. The principle of locality obtained in this way acts on the states described by equation 1. It can be assumed that equation 1 leads to the appearance of some effective fields. Then, the principle of locality is applicable only to these effective fields.

We have obtained that when the relative velocity of two IFR tends to zero, direct transformations of space-time and fields tend to the equations of the special theory of relativity[2][3]. The only thing that was used in the model under consideration to obtain such a result was the requirement of the absence of a distinguished direction in equation 1. It can be said that to some extent, Einstein's principle of relativity was used here, albeit indirectly. However, the second postulate of STR and the principle of locality were not used in this model. At the same time, they were obtained. Thus, this model reduces the number of entities.

Conclusion

We have considered the construction of a set consisting of space-times for each IFR in a 4-dimensional space. In doing so, we derived these space-times based on the hypothesis that the causality principle is applied separately and independently for each different IFR. This hypothesis implies two types of transformations of space-time and fields. The first type is transformations from the observer's point of view, preserving events. The second type of transformations is direct transformations. It follows from the hypothesis that when the relative velocity of two IFRs tends to zero, direct transformations should transform into transformations from the observer's point of view. We found what the equations of direct transformations tend to when the relative velocity of two IFRs tends to zero. They tend to equations that are identical to the Lorentz transformations.

Thus, it can be argued that the second postulate of STR and the principle of locality are derived in a 4-dimensional space. The number of entities decreases.

It follows from the hypothesis that all modern widely accepted theories are not fundamental. The obtained result shows how STR can be derived from more fundamental principles, using the example of 4-dimensional space. This result, indirectly, supports the hypothesis of the principle of causality and IFR.

References

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