Vertex-Edge-Combinatorial Polytopes: A Class Defined by the Local Structure at Each Vertex

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Abstract

We introduce a class of geometric bodies, which we call *vertex-edge-combinatorial polytopes*, defined by a local structure in which each vertex is connected to a number of edges equal to the dimension of the body, and where any subset of those edges belongs to a face whose dimension equals the subset's cardinality. These polytopes satisfy an empirical formula for the number of vertices, from which a general combinatorial expression for the number of faces can be deduced. The class includes simplices, hypercubes, and the dodecahedron, and excludes the octahedron, the icosahedron, and any higher-dimensional polytopes derived from them. In some cases where the formula diverges, such as the hexagonal tiling, an infinite regular structure does exist, although this is not always the case.

1 Regular polygons and their natural extensions

Regular polygons have historically served as the foundation for building highly symmetrical three-dimensional structures: the Platonic solids. However, **not all regular polygons admit a closed three-dimensional extension** in which all faces are congruent copies of the base polygon:

- For $\alpha = 3$ (triangle), three regular solids exist: the **tetrahedron**, the **octahedron**, and the **icosahedron**.
- For $\alpha = 4$ (square), only one: the **hexahedron** or cube.
- For $\alpha = 5$ (pentagon), only one: the **dodecahedron**.
- For $\alpha = 6$ (hexagon), no closed solid exists: the polygon **tiles the Euclidean plane without gaps or overlaps**, forming an **infinite** regular lattice.
- For $\alpha \geq 7$, not even that: **no regular 3D solids** can be constructed from congruent faces of such polygons.

It is also possible to construct higher-dimensional polytopes from simple 3D solids, such as the simplices (based on the tetrahedron) and the hypercubes (based on the cube).

We present an **empirical formula** that describes the number of vertices of a significant proportion of these structures:

- In dimension 2, all regular polygons (trivially, since each vertex has 2 edges).
- In dimension 3, includes the tetrahedron, cube, and dodecahedron, and excludes the octahedron and icosahedron.
- In higher dimensions, the **simplices** and **hypercubes**, which fully satisfy both the formula and the local combinatorial conditions.

When the formula yields a **finite positive integer**, the result is a closed polytope.

When it yields an **infinite value**, this indicates that no closed polytope can be built under the same vertex-edge combinatorial rules. In some cases (such as $\alpha = 6$, $\beta = 3$), it may correspond to an actual infinite tessellation. The case $\alpha = 5$, $\beta = 4$, where the formula diverges, predicts the breakdown of combinatorial extensibility: while the 120-cell exists as a finite closed structure, it fails to satisfy the local inductive criteria defined here.

When it yields a **fractional or negative value**, this clearly signals the impossibility of a coherent polytope satisfying the same structural principles.

The exclusion of the octahedron and icosahedron is not arbitrary; as we will show in Section 3, it is rooted in their local vertex structure. This is due to the fact that their vertices are connected to more than β edges, violating the condition of local regularity.

2 The empirical formula for the number of vertices

Although we lack a formal derivation, this expression appears to encode the critical balance between connectivity and closure at each dimensional level.

Let $N_0(\alpha, \beta)$ denote the number of vertices of a β-dimensional body generated from a regular polygon with α vertices. The empirical formula observed for these bodies is recursive:

$$N_0(\alpha,\beta) = \begin{cases} 1 & \text{if } \beta = 0 \\ R(\alpha,\beta)\cdot N_0(\alpha,\beta-1) & \text{if } \beta \geq 1 \end{cases}$$

where the growth coefficient $R(\alpha, \beta)$ is given by:

$$R(\alpha,\beta) = \frac{(3\alpha-8)+(4-\alpha)\beta}{(2\alpha-6)+(4-\alpha)\beta}$$

This formula exhibits several notable properties:

- For $\alpha = 3$, we get $N_0(3, \beta) = \beta + 1$, corresponding to simplices.
- For $\alpha = 4$, we get $N_0(4, \beta) = 2^\beta$, corresponding to hypercubes.
- For $\alpha = 5$, it correctly reproduces the number of vertices of the dodecahedron ($\beta = 3$).

The behavior of $R(\alpha, \beta)$ varies with both parameters and determines when the formula ceases to yield coherent results. In particular, for certain combinations of α and β, the denominator vanishes or the expression becomes negative or fractional, which signals a breakdown in the construction process.

In addition, the formula satisfies $N_0(\alpha, 0) = 1$ for any value of α, since all constructions begin from a single point in dimension zero. For $\beta = 1$, it yields a segment with two vertices, regardless of α. For $\beta = 2$, it correctly returns the regular polygon with α vertices, recovering all planar cases.

α	$\beta = 2$	$\beta = 3$	$\beta = 4$
3	3	4	5
4	4	8	16
5	5	20	∞
6	6	∞	–
7	7	< 0	–

Note that for $\alpha = 5$, the function diverges at $\beta = 4$, aligning with the breakdown of local combinatorial construction.

3 Definition of vertex-edge-combinatorial polytopes

The bodies selected by the empirical formula for $N_0(\alpha, \beta)$ share a remarkable structural feature: **any combination of j edges emanating from a vertex belongs to a j-dimensional face**. This implies that the local organization of edges at each vertex determines the entire structure of the body.

Definition

A β-dimensional geometric body is called a **vertex-edge-combinatorial polytope** if it satisfies the following conditions:

1. Each vertex is connected to exactly β edges.
2. Any subset of $j \leq \beta$ of those edges defines a j-dimensional face, isomorphic to the body generated by the same empirical formula in dimension j.

This definition ensures that the edge combinatorics around each vertex fully determine the face structure of the body.

This naturally excludes bodies such as:

- The *octahedron*, where each vertex has 4 edges (more than its dimension).
- The *icosahedron*, which also features excess connectivity.

In contrast, the **tetrahedron**, **cube**, **hypercubes**, **simplices**, and **dodecahedron** fully satisfy these conditions.

4 Sharing relations between elements of different dimensions

Let $C_{j \to k}$ denote the number of k-dimensional faces that share a j-dimensional element. The following relation must hold:

$$C_{j \to k} = \frac{N_k(\alpha, \beta) \cdot N_j(\alpha, k)}{N_j(\alpha, \beta)} = \binom{\beta - j}{k - j}$$

5 On the deducibility of the empirical formula

Although consistent with the observed combinatorial structure, the empirical formula is not directly deduced from the number of faces of lower-dimensional bodies. Partial combinatorial interpretations may suggest how it arises, but a full derivation remains elusive.

6 Euler–Poincaré characteristic

In all known examples, the following identity holds:

$$\chi = \sum_{k=0}^{\beta} (-1)^k N_k(\alpha, \beta) = 1 + (-1)^\beta$$

This supports the conjecture that all such polytopes are topological spheres, as expected from a closed inductive construction, although a general proof for all α has not yet been found.

7 Interpretation of Divergence

Although the empirical formula yields infinite values for certain pairs (α, β), these should not be interpreted as necessarily indicating the existence of infinite tessellations.

For example:

- For $(\alpha = 6, \beta = 3)$, the formula diverges, and this corresponds to the well-known infinite tessellation of the Euclidean plane by regular hexagons.
- However, for $(\alpha = 5, \beta = 4)$, the formula also diverges, but no such infinite structure exists. The only known construction using regular dodecahedra in 4D is the finite 120-cell, which does not satisfy the local combinatorial conditions defined for vertex-edge-combinatorial polytopes.

Therefore, a divergent value of $N_0(\alpha, \beta)$ must be interpreted more generally as indicating the **impossibility of constructing a closed polytope that satisfies the recursive combinatorial structure**, not necessarily the existence of an infinite one.

This aligns with the interpretation of negative or fractional values, which more clearly signal incompatibility with a closed, locally consistent structure.

8 Final remarks

This article defines a coherent class of polytopes with elegant and predictive combinatorial properties. Outstanding challenges include:

- Deriving the empirical formula from internal principles.
- Proving the Euler–Poincaré identity for all α.
- Extending the formula to describe the number of vertices of bodies outside the defined class.

To the best of our knowledge, all known polytopes that satisfy the vertex-edge combinatorial structure defined here also satisfy the empirical formula for the number of vertices. Conversely, all polytopes for which the formula diverges or breaks down—whether by producing non-integer, negative, or divergent values—appear to violate the local structural conditions.

No counterexample has yet been identified: no polytope is known to fulfill the local rules while escaping the formula. This dual correspondence suggests that the empirical recurrence may not merely select the class of vertex-edge-combinatorial polytopes, but fully characterize it. The search for a potential exception remains open, and might provide deeper insight into the combinatorics of higher-dimensional geometry.

References

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