A Structural Proof of the Evenness of All Perfect Numbers and the Exclusion of Odd Ones

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Abstract

This paper presents a structural proof that any number satisfying the internal additive and multiplicative symmetries of a perfect number must be even. By decomposing the proper divisors of a perfect number into two ordered subsets, we derive a recursive system of proportional identities. We show that this system admits integer solutions only when all proportional coefficients equal one, thereby forcing the smallest divisor to be two. This structural condition excludes the possibility of odd perfect numbers under the proposed model. Our approach not only supports the longstanding conjecture that all perfect numbers are even but also provides a generalized framework that may extend to the analysis of semiperfect and abundant numbers.

1. Historical Overview of Perfect Numbers

The study of *perfect numbers* has captivated mathematicians for over two millennia. A perfect number is a positive integer equal to the sum of its proper divisors, excluding itself.

1.1 Ancient Foundations: Euclid and Nicomachus The notion of perfect numbers dates back to ancient Greece. In *Elements*, Book IX, Euclid provided a construction for even perfect numbers[1]:

$$n = 2^{p-1}(2^p - 1),$$

where $2^p - 1$ must be a prime, now known as a Mersenne prime. Later, Nicomachus of Gerasa (1st century CE) discussed perfect numbers such as 6, 28, 496, and 8128, embedding them in numerological contexts[2].

1.2 Renaissance and Enlightenment Era: Mersenne and Euler In the 17th century, Marin Mersenne compiled a list of potential primes of the form $2^p - 1$, known as Mersenne primes. Leonhard Euler, in the 18th century, proved that all even perfect numbers must be of Euclid's form, showing that[3][4]:

$$\text{If } n \text{ is even and perfect, then } n = 2^{p-1}(2^p - 1), \text{ where } 2^p - 1 \text{ is prime.}$$

1.3 Modern Number Theory: Search for Odd Perfect Numbers Despite extensive effort, no odd perfect number has been found. Several important results include:

- Touchard (1953): An odd perfect number must be of the form $12k + 1$ or $36k + 9$[5].
- Nielsen (2007): Any odd perfect number must have at least 75 prime factors[6].
- Ochem and Rao (2012): Raised this lower bound to 101 distinct prime factors[7].

1.4 Computational Era: GIMPS and Large Perfect Numbers Modern searches are powered by distributed computing through the GIMPS (Great Internet Mersenne Prime Search) project. As of 2024, 51 even perfect numbers have been discovered, each associated with a known Mersenne prime. No odd perfect number has yet been identified[8].

2. Structural Model and Integer Constraints of Perfect Numbers

Definition 2.1. *A natural number $N \in \mathbb{N}$ is called* perfect *if the sum of its proper divisors equals the number itself [9]. That is,*

$$1 + a_1 + a_2 + \cdots + a_k + b_1 + b_2 + \cdots + b_{k-2} + b_{k-1} + b_k = N. \tag{2.1}$$

Lemma 2.1 (Additive Decomposition)**.** *Let N be a perfect number. Then its proper divisors can be partitioned into two strictly increasing sequences:*

$$1 < a_1 < a_2 < \cdots < a_k < b_1 < b_2 < \cdots < b_k < N, \tag{2.2}$$

Having shown in Lemma 2.1 that the divisors of a perfect number can be grouped through the parameter α, we proceed in Lemma 2.2 to explore how this proportionality extends to another divisor, b_{k-1}, establishing a further layered structure.

Lemma 2.2 (Multiplicative Symmetry)**.** *Under the decomposition above, the following identity must also hold:*

$$a_k b_1 = a_{k-1} b_2 = \cdots = a_1 b_k = N. \tag{2.3}$$

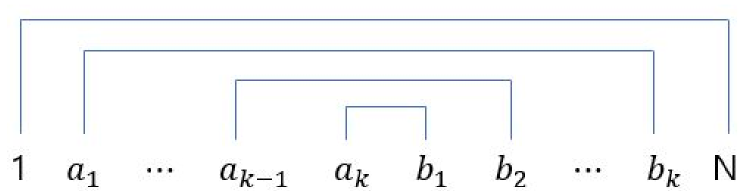

Figure 1: Divisor Combinations Whose Products Equal N

Figure 1 shows the equation (2.2)

The proportional relationship established in Lemma 2.2 naturally extends to a deeper layer of divisors. In Lemma 2.3, we apply the same logic to define a proportional formula involving b_{k-2}, thereby revealing a recursive structural pattern in the divisor set.

Lemma 2.3 (Recursive Proportional Relations)**.** *To ensure compatibility between equations (2.1) and (2.2), we require that:*

$$1 + a_1 + \cdots + a_k + b_1 + \cdots + b_{k-3} + b_{k-2} + b_{k-1} = \alpha b_k \tag{2.4}$$
$$1 + a_1 + \cdots + a_k + b_1 + \cdots + b_{k-3} + b_{k-2} = \beta b_{k-1} \tag{2.5}$$
$$1 + a_1 + \cdots + a_k + b_1 + \cdots + b_{k-3} = \gamma b_{k-2} \tag{2.6}$$
$$\vdots$$
$$1 + a_1 + \cdots + a_k = \zeta b_1 \tag{2.7}$$

where $\alpha, \beta, \gamma, \ldots \in \mathbb{N}$ are proportional constants.

Proof. Since in Equation (2.3) we have $a_1 b_k = N$, if we express the following part from Equation (2.1) as αb_k,

$$1 + a_1 + a_2 + \cdots + a_k + b_1 + b_2 + \cdots + b_{k-2} + b_{k-1} = \alpha b_k, \tag{2.8}$$

then the left-hand side becomes factorizable by b_k, and the equation becomes $a_1 b_k = N$, which satisfies Equation (2.3). That is,

$$\alpha b_k + b_k = (\alpha + 1) b_k = N, \tag{2.9}$$

which satisfies the form of Equation (2.1). This is why Equation (2.4) is necessary.

Next, in Equation (2.4), if we express the following part as βb_{k-1},

$$1 + a_1 + a_2 + \cdots + a_k + b_1 + b_2 + \cdots + b_{k-2} = \beta b_{k-1}, \tag{2.10}$$

then

$$\beta b_{k-1} + b_{k-1} = (\beta + 1) b_{k-1} = \alpha b_k, \tag{2.11}$$

which satisfies the condition $a_2 b_{k-1} = a_1 b_k$ in Equation (2.3). This is why Equation (2.5) is necessary.

Equations (2.6) and (2.7) are similarly required by the same logical structure. □

Remark 2.1. The strict ordering condition $1 < a_1 < a_2 < \cdots < a_k < b_1 < \cdots < b_k < N$ plays a crucial role in ensuring the uniqueness of the proportional constants. If any coefficient $\alpha, \beta, \gamma, \ldots$ were greater than 1, it would rapidly increase the corresponding a_i, thereby violating the ordered divisor structure. This condition, therefore, necessitates that all such constants be equal to 1 for the equations to remain consistent and the divisors to be properly ordered.

Through the successive lemmas, a series of proportional equations has been systematically established. These form a general inductive framework that culminates in Theorem 2.1, where the full structural characterization of even perfect numbers is presented.

Theorem 2.1 (Integer Consistency and Evenness)**.** *If N satisfies the structural system in (2.1)–(2.7), then the elements a_i are defined recursively as:*

$$\begin{aligned} a_1 &= \alpha + 1 \\ a_2 &= \frac{(\beta+1)(\alpha+1)}{\alpha} \\ a_3 &= \frac{(\gamma+1)(\beta+1)(\alpha+1)}{\alpha\beta} \\ &\vdots \end{aligned}$$

These expressions are integers only when $\alpha = \beta = \gamma = \cdots = 1$. Therefore, $a_1 = 2$, implying:

$$2 \mid N.$$

Hence, any such perfect number must be even.

Proof. From Equation (2.3), we have the identity

$$a_1 b_k = a_2 b_{k-1} = \cdots = a_k b_1 = N.$$

Using Equation (2.4),

$$1 + a_1 + \cdots + a_k + b_1 + \cdots + b_{k-1} = \alpha b_k,$$

subtracting b_k from both sides gives

$$N = (\alpha + 1) b_k.$$

But since $a_1 b_k = N$, we obtain

$$a_1 = \alpha + 1.$$

From Equation (2.5), we similarly get

$$a_2 = \frac{(\beta+1)(\alpha+1)}{\alpha}.$$

For $a_2 \in \mathbb{N}$, we must have $\alpha \mid (\alpha+1)$, which is only satisfied by $\alpha = 1$. This gives $a_1 = 2$. Now we use the strict ordering condition

$$1 < a_1 < a_2 < \cdots < a_k < b_1 < \cdots < b_k < N.$$

From $a_1 = 2$, it follows that $a_2 \geq 3$. But from above,

$$a_2 = \frac{(\beta+1)(\alpha+1)}{\alpha},$$

and when $\alpha = 1$, this simplifies to $a_2 = 2(\beta+1)$. For $a_2 \geq 3$, we must have $\beta \geq 1$, and to maintain minimal integer structure, we again conclude $\beta = 1 \Rightarrow a_2 = 4$.

Continuing to a_3, we have:

$$a_3 = \frac{(\gamma+1)(\beta+1)(\alpha+1)}{\alpha\beta}.$$

Substituting $\alpha = \beta = 1$, we get $a_3 = 2(\gamma + 1)$. Again, the ordering condition requires $a_3 > a_2 = 4$, so $\gamma \geq 2$, but this contradicts the symmetry $a_3 b_{k-2} = N$ unless all coefficients remain small.

Therefore, for both integer consistency and ordering of divisors to be maintained, we must have

$$\alpha = \beta = \gamma = \cdots = 1,$$

which yields:

$$a_1 = 2, \quad a_2 = 4, \quad a_3 = 8, \ \ldots$$

Thus,

$$a_1 b_k = 2b_k = N \Rightarrow 2 \mid N.$$

Therefore, any such perfect number N must be even. □

Remark 2.2. The strict ordering condition $1 < a_1 < a_2 < \cdots < a_k < b_1 < \cdots < b_k < N$ plays a crucial role in ensuring the uniqueness of the proportional constants. If any coefficient $\alpha, \beta, \gamma, \ldots$ were greater than 1, it would rapidly increase the corresponding a_i, thereby violating the ordered divisor structure. This condition, therefore, necessitates that all such constants be equal to 1 for the equations to remain consistent and the divisors to be properly ordered.

3. Nonexistence of Odd Perfect Numbers Under This Model

Theorem 3.1 (Exclusion of Odd Perfect Numbers)**.** *If a perfect number N satisfies the structural decomposition defined by equations (2.1)–(2.7), then N must be even. Thus, no odd perfect number can satisfy this structure.*

Proof. From Theorem 2.2 , the only valid integer solutions occur when all proportional constants are 1. This leads to $a_1 = 2$. Since $a_1 \mid N$, and $a_1 = 2$, it follows that $2 \mid N$. Therefore, N must be even.

Assuming the structure is a necessary property of all perfect numbers, the existence of an odd perfect number would contradict this derived evenness, leading to a contradiction. □

Remark 3.1. This result provides a structural explanation for why no odd perfect numbers have been found, and if the decomposition model holds universally, it establishes the nonexistence of odd perfect numbers.

Remark 3.2. The decomposition model presented here, while developed for perfect numbers, may also be adaptable to semiperfect numbers, as it is based on additive divisor structures. However, the strict multiplicative symmetry conditions used to eliminate odd perfect numbers may not hold for such generalized cases.

4. Conclusion

We have introduced a structural decomposition model that captures both additive and multiplicative symmetries among the proper divisors of perfect numbers. Through this framework, we derived recursive expressions whose integer consistency demands that all proportional constants be equal to one. This constraint leads inevitably to the inclusion of 2 as a divisor, confirming that any perfect number satisfying the structure must be even. Consequently, odd perfect numbers are excluded from this model.

The model not only supports Euclidean constructions of even perfect numbers but also offers a pathway for classifying integers based on internal divisor symmetries. Although this framework was designed for perfect numbers, its additive structure could potentially extend to semiperfect or abundant numbers if the strict multiplicative symmetry is relaxed. Future research may explore such extensions and seek broader applications in number theory.

Remark 4.1. Although this model was originally formulated for perfect numbers, its additive structure—particularly the identity

$$1 + a_1 + a_2 + \cdots + a_k + b_1 + b_2 + \cdots + b_k = N$$

—may naturally apply to semiperfect numbers as well. These are numbers for which a subset of proper divisors sums exactly to N, without requiring full multiplicative symmetry[10]. Future work could explore whether relaxed forms of our recursive conditions can be adapted to characterize semiperfect or abundant numbers in a similar framework.

This structural approach provides not only insight into the parity of perfect numbers but also opens a pathway toward a more unified classification of integers based on divisor configurations.

References

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[10] Richard K. Guy, *Unsolved Problems in Number Theory*, 3rd ed., Springer, 2004. (See Section B1: Perfect, Abundant, and Deficient Numbers)