Deformed Lie products and involution

First part: Discussion in a three-dimensional space

©Thierry PERIAT - no affiliation

9 June 2025

This document is the first part of an exploration examining when a deformed Lie product can be an involution. The approach starts softly in a real three-dimensional space, introducing basic notions like (i) the already well-known link between involution and neutral element, (ii) the importance of some rules concerning the indexes when a discussion is developed in a three-dimensional space, (iii) a specific semantic for the diverse representations of the deforming matrices (effective, normalized, associated six-pack). It gives then the formalism of the matrices representing the repetition of the action of any deformed cross product. It starts a systematization of the discussion and finally criterion precising when a deformed cross product is an involution. It turns out that a classical cross product cannot be an involution if the discussion is not involving vectors with components in the set of complex numbers or of quaternions. ©Thierry PERIAT: Deformed Lie products and involution - first part: discussion in a three-dimensional space.

Contents

1 Deformed Lie products and involution in a three-dimensional space. 2

1.1 Basics 2

1.2 Involution and neutral element 4

1.3 A normalized formalism for the deforming matrices 7

1.4 An alternative representation for any matrix in $\mathrm{M}(3,\mathbb{R})$ 8

1.5 Characterizing the deforming matrix ${}^{(3)}[\mathrm{B}({}^{(3)}[\mathrm{A}], {}^{(3)}\mathbf{a})]$ 9

1.6 Starting a systematization of the discussion 15

1.7 Deformed cross product and its non-trivial decomposition in case of involution 20

1.8 The classical cross product and the involution 22

1.9 Conclusion 24

2 Bibliography 24

2.1 My contributions 24

2.2 International works 24

1 Deformed Lie products and involution in a three-dimensional space.

1.1 Basics

Definition 1.1. ***Deformed cross product***

The discussion involves vectors with three real components and Lie products (see definition in [a]) which are deformed by anti-symmetric cubes with knots in $\mathbb{R}$, the set of real numbers. The space is denoted $\mathrm{V}_3 = \{\mathrm{E}(3, \mathbb{R}), \mathrm{A} \in \boxplus^-(3, \mathbb{R})\}$.

Because of this anti-symmetry, the cubes contain no more $3^3 = 27$ knots but only at most $3 \times 3 = 9$ different non-necessarily vanishing positive real numbers. They can be regrouped inside an element [A] in $\mathrm{M}(3, \mathbb{R})$ which is a square matrix.

$$^{(3)}[A] = \begin{bmatrix} A^1_{12} & A^2_{12} & A^3_{12} \\ A^1_{23} & A^2_{23} & A^3_{23} \\ A^1_{13} & A^2_{13} & A^3_{13} \end{bmatrix} \in M(3, \mathbb{R})$$

Let suppose that V_3 can be referred to a canonical basis Ω ($\mathbf{e}_1$, $\mathbf{e}_2$, $\mathbf{e}_3$); then, per definition, the deformation is acting in the following manner:

$$\forall\, ^{(3)}\mathbf{q}_1,\, ^{(3)}\mathbf{q}_2 \in V_3 : [^{(3)}\mathbf{q}_1, ^{(3)}\mathbf{q}_2]_{[A]} = \sum_{d=1}^{3} \sum_{a<b=2}^{3} A^d_{ab} \cdot (q^a_1 \cdot q^b_2 - q^a_2 \cdot q^b_1) \cdot \mathbf{e}_d$$

Proposition 1.1. *Any deformed Lie product acting on pairs of elements in V_3 is a deformation of the classical cross product involving these elements.*

Proof. Effectively, any deformed Lie product acting on a pair ($\mathbf{q}_1$, $\mathbf{q}_2$) writes:

$$[\mathbf{q}_1, \mathbf{q}_2]_{[A]}$$

$$=$$

$$\sum_{a<b=2}^{3} A^1_{ab} \cdot (q^a_1 \cdot q^b_2 - q^a_2 \cdot q^b_1) \cdot \mathbf{e}_1 + A^2_{ab} \cdot (q^a_1 \cdot q^b_2 - q^a_2 \cdot q^b_1) \cdot \mathbf{e}_2 + A^3_{ab} \cdot (q^a_1 \cdot q^b_2 - q^a_2 \cdot q^b_1) \cdot \mathbf{e}_3$$

It can be represented in $\mathrm{V^*}_3$, the dual space of V_3:

$$|[\mathbf{q}_1, \mathbf{q}_2]_{[A]} >$$

$$=$$

$$\left| \begin{matrix} \sum_{a<b=2}^{3} A^1_{ab} \cdot (q^a_1 \cdot q^b_2 - q^a_2 \cdot q^b_1) \\ \sum_{a<b=2}^{3} A^2_{ab} \cdot (q^a_1 \cdot q^b_2 - q^a_2 \cdot q^b_1) \\ \sum_{a<b=2}^{3} A^3_{ab} \cdot (q^a_1 \cdot q^b_2 - q^a_2 \cdot q^b_1) \end{matrix} \right\rangle$$

$$=$$

$$\left| \begin{matrix} A^1_{12} \cdot (q^1_1 \cdot q^2_2 - q^2_2 \cdot q^1_1) + A^1_{23} \cdot (q^2_1 \cdot q^3_2 - q^3_2 \cdot q^2_1) + A^1_{13} \cdot (q^1_1 \cdot q^3_2 - q^3_2 \cdot q^1_1) \\ A^2_{12} \cdot (q^1_1 \cdot q^2_2 - q^2_2 \cdot q^1_1) + A^2_{23} \cdot (q^2_1 \cdot q^3_2 - q^3_2 \cdot q^2_1) + A^2_{13} \cdot (q^1_1 \cdot q^3_2 - q^3_2 \cdot q^1_1) \\ A^3_{12} \cdot (q^1_1 \cdot q^2_2 - q^2_2 \cdot q^1_1) + A^3_{23} \cdot (q^2_1 \cdot q^3_2 - q^3_2 \cdot q^2_1) + A^3_{13} \cdot (q^1_1 \cdot q^3_2 - q^3_2 \cdot q^1_1) \end{matrix} \right\rangle$$

Simultaneously, one knows that:

$$|\wedge(\mathbf{q}_1,\mathbf{q}_2)>= \left| \begin{array}{c} q_1^2 \cdot q_2^3 - q_2^2 \cdot q_1^3 \\ -(q_1^1 \cdot q_2^3 - q_2^1 \cdot q_1^3) \\ q_1^1 \cdot q_2^2 - q_2^1 \cdot q_1^2 \end{array} \right\rangle$$

and that:

$$[J]^t \,.\, [A] = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & -1 \\ 1 & 0 & 0 \end{bmatrix} . \begin{bmatrix} A_{12}^1 & A_{12}^2 & A_{12}^3 \\ A_{23}^1 & A_{23}^2 & A_{23}^3 \\ A_{13}^1 & A_{13}^2 & A_{13}^3 \end{bmatrix} = \begin{bmatrix} A_{23}^1 & A_{23}^2 & A_{23}^3 \\ -A_{13}^1 & -A_{13}^2 & -A_{13}^3 \\ A_{12}^1 & A_{12}^2 & A_{12}^3 \end{bmatrix}$$

or also:

$$[A]^t \,.\, [J] = \begin{bmatrix} A_{23}^1 & -A_{13}^1 & A_{23}^1 \\ A_{23}^2 & -A_{13}^2 & A_{23}^2 \\ A_{23}^3 & -A_{13}^3 & A_{23}^3 \end{bmatrix}$$

It is then easy to state that:

$$|[\mathbf{q}_1,\mathbf{q}_2]_{[A]}>= \{[A]^t \,.\, [J]\} \,.\, |\wedge(\mathbf{q}_1,\mathbf{q}_2)>$$

With:

$$[J] = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & -1 & 0 \end{bmatrix} \quad [J]^t = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & -1 \\ 1 & 0 & 0 \end{bmatrix}$$

The last term on the right hand side (r.h.s.) is nothing but the classical cross product (i.e.: the three-dimensional version of the wedge product). This first and basic statement explains why that new product has been coined with the label "deformed cross product". □

Proposition 1.2. *"A cross product is a Lie product deformed by the matrix [J]".*

Proof. Effectively, writing [A] = [J] in the definition of any deformed cross product immediately yields:

$$[\mathbf{q}_1,\mathbf{q}_2]_{[J]} = \wedge(\mathbf{q}_1,\mathbf{q}_2)$$

□

Since, historically, the cross product can be considered as the "point zero" for the definition of a Lie bracket/cross product (i.e.: as a non-deformed Lie product acting in a three-dimensional space), the spontaneous terminology attributing the deformation to the ${}^{(3)}$[A] matrix may be a little bit confusing. This is suggesting that one should either change it for convenience and harmonization with the intuitive way of doing; or consider the presence of [J] as a sign left by the nature and containing a subliminal message related to some underlying symmetry; for example: the Abel cyclic group C6.

$$[J]^2 = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & -1 & 0 \end{bmatrix} . \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & -1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & -1 & 0 \\ 0 & 0 & 1 \\ -1 & 0 & 0 \end{bmatrix}$$

$$[J]^3 = -Id_3$$

$$[J]^4 = ([J]^2)^2 \begin{bmatrix} 0 & -1 & 0 \\ 0 & 0 & 1 \\ -1 & 0 & 0 \end{bmatrix} . \begin{bmatrix} 0 & -1 & 0 \\ 0 & 0 & 1 \\ -1 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & -1 \\ -1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} = -[J]$$

$$[J]^5 = -[J]^2$$

$$[J]^6 = Id_3$$

1.2 Involution and neutral element

Definition 1.2. *The function $[\boldsymbol{a}, \ldots]_{[A]}$.*

Let consider a given element $\mathbf{a}$ in V_3; the function $[\mathbf{a}, \ldots]_{[A]}$ is an element in $\mathrm{End}(\mathrm{V}_3)$ such that:

$$\forall\, \mathbf{x} \in V_3 \xrightarrow{[\mathbf{a}, \ldots]_{[A]}} [\mathbf{a},\, \mathbf{x}]_{[A]} \in V_3$$

Proposition 1.3. *If $f = [\boldsymbol{a}, \ldots][A]$ acts on a given element $\boldsymbol{x}$ in V_3 like an involution would do it, then there automatically exists an element $[B([A],\, \boldsymbol{a})]$ in $M(3,\, \mathbb{R})$ such that $\boldsymbol{a}$ behaves like a neutral element on the left side of $\boldsymbol{x}$ for the deformed cross product $[\ldots,\, \ldots][B([A],\, \boldsymbol{a})]$; concretely:*

$$[\mathbf{a},\, \underbrace{[\mathbf{a},\, \mathbf{x}]_{[A]}}_{=\mathbf{X}}]_{[A]} = \mathbf{x} \Rightarrow \exists\, [B([A],\, \mathbf{a})] \in M(3,\, \mathbb{R}) : [\mathbf{a}, \mathbf{x}]_{[B([A],\, \mathbf{a})]} = \mathbf{x}$$

Proof. Let consider the definition of any deformed cross product and let then calculate:

$$\mathbf{a} \in E(3,\, \mathbb{R}) : [\mathbf{a},\, \underbrace{\mathbf{X}}_{= \sum_{\gamma=1}^{3} X^\gamma\, .\, \mathbf{e}_\gamma}]_{[A]}$$

When:

$$\mathbf{X} = [\mathbf{a},\, \mathbf{x}]_{[A]}$$

Because:

$$\forall\, \gamma = 1,\, 2,\, 3 : X^\gamma = \sum_{\alpha<\beta=2}^{3} A^\gamma_{\alpha\beta} \,.\, (a^\alpha \,.\, x^\beta - a^\beta \,.\, x^\alpha)$$

This is:

$$\{[\mathbf{a},\, \mathbf{X}]_{[A]}\}^\eta$$

$$=$$

$$\sum_{\delta<\gamma=2}^{3} A^\eta_{\delta\gamma} \,.\, \{a^\delta \,.\, X^\gamma - a^\gamma \,.\, X^\delta\}$$

$$=$$

$$\sum_{\delta<\gamma=2}^{3} A^\eta_{\delta\gamma} \,.\, \{a^\delta \,.\, \sum_{\alpha<\beta=2}^{3} A^\gamma_{\alpha\beta} \,.\, (a^\alpha \,.\, x^\beta - x^\alpha \,.\, a^\beta) - a^\gamma \,.\, \sum_{\alpha<\beta=2}^{3} A^\delta_{\alpha\beta} \,.\, (a^\alpha \,.\, x^\beta - x^\alpha \,.\, a^\beta)\}$$

$$=$$

$$\sum_{\delta<\gamma=2}^{3} A^{\eta}_{\delta\gamma} \, . \, \{ \sum_{\alpha<\beta=2}^{3} \underbrace{(a^{\delta} \, . \, A^{\gamma}_{\alpha\beta} \, - \, a^{\gamma} \, . \, A^{\delta}_{\alpha\beta})}_{= C^{\delta\gamma}_{\alpha\beta}} \, . \, (a^{\alpha} \, . \, x^{\beta} \, - \, x^{\alpha} \, . \, a^{\beta}) \}$$

$$=$$

$$\sum_{\alpha<\beta=2}^{3} \underbrace{(\sum_{\delta<\gamma=2}^{3} A^{\eta}_{\delta\gamma} \, . \, C^{\delta\gamma}_{\alpha\beta})}_{= B^{\eta}_{\alpha\beta}} \, . \, (a^{\alpha} \, . \, x^{\beta} \, - \, x^{\alpha} \, . \, a^{\beta})$$

$$=$$

$$\sum_{\alpha<\beta=2}^{3} B^{\eta}_{\alpha\beta} \, . \, (a^{\alpha} \, . \, x^{\beta} \, - \, x^{\alpha} \, . \, a^{\beta})$$

With:

$$\forall \, \alpha, \, \beta, \, \eta \, = \, 1, \, 2, \, 3$$

$$\forall \, (\alpha, \, \beta) \, | \, \alpha \, < \, \beta :$$

$$B^{\eta}_{\alpha\beta} \, = \, \sum_{\delta<\gamma=2}^{3} A^{\eta}_{\delta\gamma} \, . \, C^{\delta\gamma}_{\alpha\beta}$$

$$C^{\delta\gamma}_{\alpha\beta} \, = \, A^{\delta}_{\alpha\beta} \, . \, a^{\gamma} \, - \, A^{\gamma}_{\alpha\beta} \, . \, a^{\delta}$$

Remark 1.1. *Concerning the indexes in a three-dimensional space.*

Due to the fact that for any given pair $(\alpha, \, \beta)$ of non-repeated indexes:

- taken in $\mathrm{Ind}_3 = \{1, \, 2, \, 3\}$ and
- such that $\alpha < \beta$

... can be replaced by the index missing in the pair at hand, let it for example denote with the Greek letter ϵ:

$$(\alpha, \, \beta) \, = \, (1, \, 2) \, \equiv \, \epsilon \, = \, 3$$

$$(\alpha, \, \beta) \, = \, (1, \, 3) \, \equiv \, \epsilon \, = \, 2$$

$$(\alpha, \, \beta) \, = \, (2, \, 3) \, \equiv \, \epsilon \, = \, 1$$

This remark is also true for the pairs $(\delta, \, \gamma)$ which can be replaced by an index denoted, e.g., μ.

Hence, one must decode the cube B in starting with:

$$\forall \, \alpha, \, \beta, \, \eta \, = \, 1, \, 2, \, 3$$

$$\forall \, (\alpha, \, \beta) \, | \, \alpha \, < \, \beta \, and \, \forall \, (\delta, \, \gamma) \, | \, \delta \, < \, \gamma$$

$$B^{\eta}_{\alpha\beta}$$

$$=$$

...

$$\sum_{\delta<\gamma=2}^{3} A^{\eta}_{\delta\gamma} \,.\, C^{\delta\gamma}_{\alpha\beta}$$

$$=$$

$$A^{\eta}_{12} \,.\, C^{12}_{\alpha\beta} \,+\, A^{\eta}_{23} \,.\, C^{23}_{\alpha\beta} \,+\, A^{\eta}_{13} \,.\, C^{13}_{\alpha\beta}$$

$$=$$

$$A^{\eta}_{3} \,.\, C^{3}_{\alpha\beta} \,+\, A^{\eta}_{1} \,.\, C^{1}_{\alpha\beta} \,+\, A^{\eta}_{2} \,.\, C^{2}_{\alpha\beta}$$

In a second step, one must write:

$$B^{\eta}_{12} \,=\, A^{\eta}_{3} \,.\, C^{3}_{12} \,+\, A^{\eta}_{1} \,.\, C^{1}_{12} \,+\, A^{\eta}_{2} \,.\, C^{2}_{12}$$

$$B^{\eta}_{13} \,=\, A^{\eta}_{3} \,.\, C^{3}_{13} \,+\, A^{\eta}_{1} \,.\, C^{1}_{13} \,+\, A^{\eta}_{2} \,.\, C^{2}_{13}$$

$$B^{\eta}_{23} \,=\, A^{\eta}_{3} \,.\, C^{3}_{23} \,+\, A^{\eta}_{1} \,.\, C^{1}_{23} \,+\, A^{\eta}_{2} \,.\, C^{2}_{23}$$

There relations are equivalent to:

$$B^{\eta}_{3} \,=\, A^{\eta}_{3} \,.\, C^{3}_{3} \,+\, A^{\eta}_{1} \,.\, C^{1}_{3} \,+\, A^{\eta}_{2} \,.\, C^{2}_{3}$$

$$B^{\eta}_{2} \,=\, A^{\eta}_{3} \,.\, C^{3}_{2} \,+\, A^{\eta}_{1} \,.\, C^{1}_{2} \,+\, A^{\eta}_{2} \,.\, C^{2}_{2}$$

$$B^{\eta}_{1} \,=\, A^{\eta}_{3} \,.\, C^{3}_{1} \,+\, A^{\eta}_{1} \,.\, C^{1}_{1} \,+\, A^{\eta}_{2} \,.\, C^{2}_{1}$$

At this stage, it becomes absolutely evident that the cube B and the hypercube C are in fact respectively equivalent to an element [B] and to an element [C] in $\mathrm{M}(3, \mathbb{R})$. It is obvious that the entries of [C] depend on the information contained in the pair $([\mathrm{A}], \mathbf{a})$ - nota bene: more details concerning this point will be given later in this document.

Therefore, as claimed:

1. What gave the initial illusion to be a cube B is only a (3-3) matrix [B] in $\mathrm{M}(3, \mathbb{R})$ depending on the information contained in the pair $([\mathrm{A}], \mathbf{a})$.

2. When f is an involution, one can write:

$$[\mathbf{a},\, [\mathbf{a},\, \mathbf{x}]_{[A]}]_{[A]} \,=\, [{}^{(3)}\mathbf{a},\, {}^{(3)}\mathbf{x}]_{[B({}^{(3)}[A],\, {}^{(3)}\mathbf{a})]} \,=\, {}^{(3)}\mathbf{x}$$

   The formalism of this relation proves that the vector ${}^{(3)}\mathbf{a}$ acts like a neutral element on the left side of ${}^{(3)}\mathbf{x}$ through the function $[\mathbf{a}, \ldots]_{[B]}$. One can also say that a repetition of the involutive action of $[{}^{(3)}\mathbf{a}, \ldots]_{[A]}$ on a given vector ${}^{(3)}\mathbf{x}$ modifies the deforming matrix but not this vector.

$\square$

1.3 A normalized formalism for the deforming matrices

Any anti-symmetric and real cube A can be condensed as an element [A] in $M(3, \mathbb{R})$

$$^{(3)}[A] = \begin{bmatrix} A^1_{12} & A^2_{12} & A^3_{12} \\ A^1_{23} & A^2_{23} & A^3_{23} \\ A^1_{13} & A^2_{13} & A^3_{13} \end{bmatrix}$$

Due to the remark 1.1 concerning the indexes, this element can also be rewritten as:

$$^{(3)}[A] = \begin{bmatrix} A^1_3 & A^2_3 & A^3_3 \\ A^1_1 & A^2_1 & A^3_1 \\ A^1_2 & A^2_2 & A^3_2 \end{bmatrix}$$

This writing is not corresponding to the usual (line, row) convention because a *normalized writing* of this matrix would be:

$$\begin{bmatrix} A^1_1 & A^1_2 & A^1_3 \\ A^2_1 & A^2_2 & A^2_3 \\ A^3_1 & A^3_2 & A^3_3 \end{bmatrix} = [A^\diamond]$$

The normalized formalism of the deforming matrix can always be recovered in introducing the matrix:

$$[W] = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} ; \; |K| = 1; \; [W] \,.\, [W]^t = Id_3$$

Because it is easy to verify that:

$$[A] = \{[A^\diamond] \,.\, [W]\}^t = [W]^t \,.\, [A^\diamond]^t$$

$$[A]^t = [A^\diamond] \,.\, [W]$$

$$[A^\diamond] = [A]^t \,.\, [W]^t = \{[W] \,.\, [A]\}^t$$

The transposed matrix $[W]^t$ is one of the three matrices representing the tetrahedron group in a three dimensional space [01; annex J, pp. 653-655]. It is also an element generating the cyclic group Z3 [02; p.18].

Per convention:

- The matrix [A] is the *deforming matrix*,
- The matrix $[A]^t.[J]$ is the *effective deforming matrix* appearing in proposition 1.1 proving that a deformed Lie product acting on elements in V_3 is a deformation of the cross product involving these elements and
- The matrix $[A^\diamond]$ is the *normalized deforming matrix*; note that the effective deforming matrix is related to the normalized one through the relation:

$$[A]^t \,.\, [J] = [A^\diamond] \,.\, [W] \,.\, [J]$$

Note also by the way that:

$$[W] . [J] = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} . \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & -1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$[J] . [W] = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & -1 & 0 \end{bmatrix} . \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

And that:

$$[W] . [J] . {}_{[J]}\Phi(\mathbf{a}) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} . \begin{bmatrix} 0 & -a^3 & a^2 \\ a^3 & 0 & -a^1 \\ -a^2 & a^1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & -a^3 & a^2 \\ -a^3 & 0 & a^1 \\ -a^2 & a^1 & 0 \end{bmatrix}$$

$${}_{[J]}\Phi(\mathbf{a}) . [J]^t = \begin{bmatrix} 0 & -a^3 & a^2 \\ a^3 & 0 & -a^1 \\ -a^2 & a^1 & 0 \end{bmatrix} . \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & -1 \\ 1 & 0 & 0 \end{bmatrix} = \begin{bmatrix} a^2 & 0 & a^3 \\ -a^1 & a^3 & 0 \\ 0 & -a^2 & -a^1 \end{bmatrix}$$

These results will be useful later in this document.

1.4 An alternative representation for any matrix in $\mathbf{M(3, \mathbb{R})}$

Any element in M(3, $\mathbb{R}$), for example [A], can be understood as a set of six intricate elements in E(3, $\mathbb{R}$).

Proof. Each (3-3) matrix contains implicitly six vectors because, as a matter of facts:

$${}^{(3)}[A] = \begin{bmatrix} A^1_{12} & A^2_{12} & A^3_{12} \\ A^1_{23} & A^2_{23} & A^3_{23} \\ A^1_{13} & A^2_{13} & A^3_{13} \end{bmatrix}$$

can always be decoded as the juxtaposition of three columns:

$${}^{(3)}[A] = [|\mathbf{a}_1 >, |\mathbf{a}_2 >, |\mathbf{a}_3 >]$$

with:

$$|\mathbf{a}_\eta > \equiv \left| \begin{matrix} A^\eta_{12} \\ A^\eta_{23} \\ A^\eta_{13} \end{matrix} \right\rangle ; \ \eta = 1, 2, 3.$$

and as a superposition of three lines:

$$[A] = \begin{bmatrix} |\mathbf{a}^1 > \\ |\mathbf{a}^2 > \\ |\mathbf{a}^3 > \end{bmatrix}$$

Each of these six vectors has three components in $\mathbb{R}$. Each of them is simultaneously a component of another vector in that six-pack. □

Definition 1.3. *Six-pack*

A "six-pack" is a set of six elements in $E(3, \mathbb{R})$ such that their components allow the reconstruction of an element in $M(3, \mathbb{R})$. Per convention, the element in $M(3, \mathbb{R})$ is the mother of a six-pack.

Note that:

- any element in $M(3, \mathbb{R})$ is systematically equivalent to a six-pack; but, the converse is false.
- any six-pack contains two subsets of each three vectors: three rows and three lines. The subset containing the three lines can be crossed with the subset containing the three rows and the maneuver generates a set of nine classical cross products. Each of these products can be deformed by an element chosen in $M(3, \mathbb{R})$; eventually, they can be simultaneously deformed by only one element, for example [X]. Exceptionally, the deforming matrix [X] can coincide with the mother [A] of the six-pack.

1.5 Characterizing the deforming matrix $^{(3)}[B(^{(3)}[A], {}^{(3)}\mathbf{a})]$

Coming back to the main topic of this document, one now looks for more information concerning the formalism of the generic deforming matrix $[B(^{(3)}[A], {}^{(3)}\mathbf{a})]$ which is obtained when the action of $[\mathbf{a}, ...]_{[A]}$ is repeated. This subsection gives four important characteristics for this matrix:

- It is the simplest decomposition for some cross product which is deformed by a cube D in $\boxplus(3, \mathbb{R})$: $[\mathbf{a}, ...]_D$; proposition 1.4.
- It is a specific and degenerated representation in $M(3, \mathbb{R})$ of a repetition of the action of $f = [\mathbf{a}, ...]_{[A]}$; proposition 1.5.
- It is a weighted sum of Pythagorean tables which are built with the vectors implicitly contained in the deforming matrix $^{(3)}[A]$; proposition 1.6.
- Its determinant is null; proposition 1.7.

Proposition 1.4. *The (non-normalized) formalism of the deforming matrix $^{(3)}[B]$ is a trivial one.*

Proof. Let recall the result which has been obtained during the demonstration of proposition 1.3:

$$\eta = 1, 2, 3\,;\, \alpha < \beta\,:$$

$$B^{\eta}_{\alpha\beta} = A^{\eta}_{12} . (A^{1}_{\alpha\beta} . a^2 - A^{2}_{\alpha\beta} . a^1) + A^{\eta}_{23} . (A^{2}_{\alpha\beta} . a^3 - A^{3}_{\alpha\beta} . a^2) + A^{\eta}_{13} . (A^{1}_{\alpha\beta} . a^3 - A^{3}_{\alpha\beta} . a^1)$$

The relation can be reorganized as:

$$B^{\eta}_{\alpha\beta}$$

$$=$$

$$-(A^{\eta}_{12} . A^{2}_{\alpha\beta} + A^{\eta}_{13} . A^{3}_{\alpha\beta}) . a^1 + (A^{\eta}_{23} . A^{3}_{\alpha\beta} - A^{\eta}_{12} . A^{1}_{\alpha\beta}) . a^2 + (A^{\eta}_{23} . A^{2}_{\alpha\beta} + A^{\eta}_{13} . A^{1}_{\alpha\beta}) . a^3$$

They can then be condensed:

$$\eta \,=\, 1,2,3\,;\, \alpha \,<\, \beta \,:\, B^{\eta}_{(\alpha\beta)} \,=\, \sum_{\lambda\,=\,1}^{3} D^{\eta}_{\lambda(\alpha\beta)} \,.\, a^{\lambda}$$

With:

$$\eta \,=\, 1,2,3\,;\, \alpha \,<\, \beta \,:\, D^{\eta}_{1\alpha\beta} \,=\, -(A^{\eta}_{12} \,.\, A^{2}_{\alpha\beta} \,+\, A^{\eta}_{13} \,.\, A^{3}_{\alpha\beta})$$
$$\eta \,=\, 1,2,3\,;\, \alpha \,<\, \beta \,:\, D^{\eta}_{2\alpha\beta} \,=\, (A^{\eta}_{23} \,.\, A^{3}_{\alpha\beta} \,-\, A^{\eta}_{12} \,.\, A^{1}_{\alpha\beta})$$
$$\eta \,=\, 1,2,3\,;\, \alpha \,<\, \beta \,:\, D^{\eta}_{3\alpha\beta} \,=\, (A^{\eta}_{23} \,.\, A^{2}_{\alpha\beta} \,+\, A^{\eta}_{13} \,.\, A^{1}_{\alpha\beta})$$

The pairs $(\alpha,\, \beta)$ take only three values: (1, 2), (2, 3) and (1, 3). Due to the fact that the discussion is developed in a three-dimensional space, these pairs can be replaced by a unique subscript, for example ϵ, the value of which is the missing one in the set $\{1,\, 2,\, 3\}$:

$$\eta\,,\, \epsilon \,=\, 1,2,3 \,:\, D^{\eta}_{1\epsilon} \,=\, -(A^{\eta}_{3} \,.\, A^{2}_{\epsilon} \,+\, A^{\eta}_{2} \,.\, A^{3}_{\epsilon})$$
$$\eta\,,\, \epsilon \,=\, 1,2,3 \,:\, D^{\eta}_{2\epsilon} \,=\, (A^{\eta}_{1} \,.\, A^{3}_{\epsilon} \,-\, A^{\eta}_{3} \,.\, A^{1}_{\epsilon})$$
$$\eta\,,\, \epsilon \,=\, 1,2,3 \,:\, D^{\eta}_{3\epsilon} \,=\, (A^{\eta}_{1} \,.\, A^{2}_{\epsilon} \,+\, A^{\eta}_{2} \,.\, A^{1}_{\epsilon})$$

Hence:

$$\eta\,,\, \epsilon \,=\, 1,2,3 \,:\, B^{\eta}_{\epsilon} \,=\, \sum_{\lambda\,=\,1}^{3} D^{\eta}_{\lambda\epsilon} \,.\, a^{\lambda}$$

or, more concisely and as it was claimed, the *non-normalized* writing of [B] has the same formalism than the simplest decomposition for some deformed cross product of the $[\mathbf{a}, \ldots]_D$ type:

$$[B([A], \mathbf{a})] \,=\, {}_{D}\Phi(\mathbf{a})$$

At this stage, one don't know if the cube D owns symmetries or not. The three matrices $_{\lambda}$[D] (for $\lambda = 1,\, 2,\, 3$) composing it are:

$$_{1}[D] \,=\, [D^{\eta}_{1\epsilon}] \,=\, -\begin{bmatrix} D^{1}_{11} & D^{1}_{12} & D^{1}_{13} \\ D^{2}_{11} & D^{2}_{12} & D^{2}_{13} \\ D^{3}_{11} & D^{3}_{12} & D^{3}_{13} \end{bmatrix} \,=\, \begin{bmatrix} A^{1}_{3} . A^{2}_{1} + A^{1}_{2} . A^{3}_{1} & A^{1}_{3} . A^{2}_{2} + A^{1}_{2} . A^{3}_{2} & A^{1}_{3} . A^{2}_{3} + A^{1}_{2} . A^{3}_{3} \\ A^{2}_{3} . A^{2}_{1} + A^{2}_{2} . A^{3}_{1} & A^{2}_{3} . A^{2}_{2} + A^{2}_{2} . A^{3}_{2} & A^{2}_{3} . A^{2}_{3} + A^{2}_{2} . A^{3}_{3} \\ A^{3}_{3} . A^{2}_{1} + A^{3}_{2} . A^{3}_{1} & A^{3}_{3} . A^{2}_{2} + A^{3}_{2} . A^{3}_{2} & A^{3}_{3} . A^{2}_{3} + A^{3}_{2} . A^{3}_{3} \end{bmatrix}$$

$$_{2}[D] \,=\, [D^{\eta}_{2\epsilon}] \,=\, \begin{bmatrix} D^{1}_{21} & D^{1}_{22} & D^{1}_{23} \\ D^{2}_{21} & D^{2}_{22} & D^{2}_{23} \\ D^{3}_{21} & D^{3}_{22} & D^{3}_{23} \end{bmatrix} \,=\, \begin{bmatrix} A^{1}_{1} . A^{3}_{1} - A^{1}_{3} . A^{1}_{1} & A^{1}_{1} . A^{3}_{2} - A^{1}_{3} . A^{1}_{2} & A^{1}_{1} . A^{3}_{3} - A^{1}_{3} . A^{1}_{3} \\ A^{2}_{1} . A^{3}_{1} - A^{2}_{3} . A^{1}_{1} & A^{2}_{1} . A^{3}_{2} - A^{2}_{3} . A^{1}_{2} & A^{2}_{1} . A^{3}_{3} - A^{2}_{3} . A^{1}_{3} \\ A^{3}_{1} . A^{3}_{1} - A^{3}_{3} . A^{1}_{1} & A^{3}_{1} . A^{3}_{2} - A^{3}_{3} . A^{1}_{2} & A^{3}_{1} . A^{3}_{3} - A^{3}_{3} . A^{1}_{3} \end{bmatrix}$$

$$_{3}[D] \,=\, [D^{\eta}_{3\epsilon}] \,=\, \begin{bmatrix} D^{1}_{31} & D^{1}_{32} & D^{1}_{33} \\ D^{2}_{31} & D^{2}_{32} & D^{2}_{33} \\ D^{3}_{31} & D^{3}_{32} & D^{3}_{33} \end{bmatrix} \,=\, \begin{bmatrix} A^{1}_{1} . A^{2}_{1} + A^{1}_{2} . A^{1}_{1} & A^{1}_{1} . A^{2}_{2} + A^{1}_{2} . A^{1}_{2} & A^{1}_{1} . A^{2}_{3} + A^{1}_{2} . A^{1}_{3} \\ A^{2}_{1} . A^{2}_{1} + A^{2}_{2} . A^{1}_{1} & A^{2}_{1} . A^{2}_{2} + A^{2}_{2} . A^{1}_{2} & A^{2}_{1} . A^{2}_{3} + A^{2}_{2} . A^{1}_{3} \\ A^{3}_{1} . A^{2}_{1} + A^{3}_{2} . A^{1}_{1} & A^{3}_{1} . A^{2}_{2} + A^{3}_{2} . A^{1}_{2} & A^{3}_{1} . A^{2}_{3} + A^{3}_{2} . A^{1}_{3} \end{bmatrix}$$

□

Proposition 1.5. *The matrix [B] is a specific and degenerated representation in $M(3,\mathbb{R})$ of a repetition of the action of $f = [\boldsymbol{a}, \ldots]_{[A]}$.*

Proof. First of all, let recall the description of the entries of the matrix [B] (remark 1.1) in re-ordering them to get a normalized formalism; one obtains:

- For the first row:

$$B_1^1 \,=\, A_3^1\,.\,C_1^3 \,+\, A_1^1\,.\,C_1^1 \,+\, A_2^1\,.\,C_1^2$$

$$B_1^2 \,=\, A_3^2\,.\,C_1^3 \,+\, A_1^2\,.\,C_1^1 \,+\, A_2^2\,.\,C_1^2$$

$$B_1^3 \,=\, A_3^3\,.\,C_1^3 \,+\, A_1^3\,.\,C_1^1 \,+\, A_2^3\,.\,C_1^2$$

- For the second row:

$$B_2^1 \,=\, A_3^1\,.\,C_2^3 \,+\, A_1^1\,.\,C_2^1 \,+\, A_2^1\,.\,C_2^2$$

$$B_2^2 \,=\, A_3^2\,.\,C_2^3 \,+\, A_1^2\,.\,C_2^1 \,+\, A_2^2\,.\,C_2^2$$

$$B_2^3 \,=\, A_3^3\,.\,C_2^3 \,+\, A_1^3\,.\,C_2^1 \,+\, A_2^3\,.\,C_2^2$$

- For the third row:

$$B_3^1 \,=\, A_3^1\,.\,C_3^3 \,+\, A_1^1\,.\,C_3^1 \,+\, A_2^1\,.\,C_3^2$$

$$B_3^2 \,=\, A_3^2\,.\,C_3^3 \,+\, A_1^2\,.\,C_3^1 \,+\, A_2^2\,.\,C_3^2$$

$$B_3^3 \,=\, A_3^3\,.\,C_3^3 \,+\, A_1^3\,.\,C_3^1 \,+\, A_2^3\,.\,C_3^2$$

This simple action gives the opportunity to get a product of normalized matrices:

$$B_\epsilon^\eta \,=\, \sum_{\mu=1}^{3} A_\mu^\eta\,.\,C_\epsilon^\mu \;\iff\; [B^\diamond] \,=\, [A^\diamond]\,.\,[C^\diamond]$$

Furthermore, with the conventions which have been introduced in remark 1.1:

$$\mu \,=\, 1 \,:\, C_\epsilon^{(2,3)} \,=\, C_\epsilon^1 \,=\, A_\epsilon^2\,.\,a^3 \,-\, A_\epsilon^3\,.\,a^2$$

$$\mu \,=\, 2 \,:\, C_\epsilon^{(1,3)} \,=\, C_\epsilon^2 \,=\, A_\epsilon^1\,.\,a^3 \,-\, A_\epsilon^3\,.\,a^1$$

$$\mu \,=\, 3 \,:\, C_\epsilon^{(1,2)} \,=\, C_\epsilon^3 \,=\, A_\epsilon^1\,.\,a^2 \,-\, A_\epsilon^2\,.\,a^1$$

One can write:

$$B^{\eta}_{\epsilon}$$

$$=$$

$$A^{\eta}_{1}\,.\,C^{1}_{\epsilon} \,+\, A^{\eta}_{2}\,.\,C^{2}_{\epsilon} \,+\, A^{\eta}_{3}\,.\,C^{3}_{\epsilon}$$

$$=$$

$$A^{\eta}_{1}\,.\,(A^{2}_{\epsilon}\,.\,a^{3} \,-\, A^{3}_{\epsilon}\,.\,a^{2}) \,+\, A^{\eta}_{2}\,.\,(A^{1}_{\epsilon}\,.\,a^{3} \,-\, A^{3}_{\epsilon}\,.\,a^{1}) \,+\, A^{\eta}_{3}\,.\,(A^{1}_{\epsilon}\,.\,a^{2} \,-\, A^{2}_{\epsilon}\,.\,a^{1})$$

This writing allows the construction of the normalized version of matrix [C]:

$$C^{1}_{1} \,=\, A^{2}_{1}\,.\,a^{3} \,-\, A^{3}_{1}\,.\,a^{2}\,;\, C^{1}_{2} \,=\, A^{2}_{2}\,.\,a^{3} \,-\, A^{3}_{2}\,.\,a^{2}\,;\, C^{1}_{3} \,=\, A^{2}_{3}\,.\,a^{3} \,-\, A^{3}_{3}\,.\,a^{2}$$

$$C^{2}_{1} \,=\, -(A^{3}_{1}\,.\,a^{1} - A^{1}_{1}\,.\,a^{3})\,;\, C^{2}_{2} \,=\, -(A^{3}_{2}\,.\,a^{1} - A^{1}_{2}\,.\,a^{3})\,;\, C^{2}_{3} \,=\, -(A^{3}_{3}\,.\,a^{1} - A^{1}_{3}\,.\,a^{3})$$

$$C^{3}_{1} \,=\, A^{1}_{1}\,.\,a^{2} \,-\, A^{2}_{1}\,.\,a^{1}\,;\, C^{3}_{2} \,=\, A^{1}_{2}\,.\,a^{2} \,-\, A^{2}_{2}\,.\,a^{1}\,;\, C^{3}_{3} \,=\, A^{1}_{3}\,.\,a^{2} \,-\, A^{2}_{3}\,.\,a^{1}$$

With an attentive observation, one recognizes that this normalized version is a product of two matrices. The first one can be identified with the help of results which have been obtained at the end of subsection 1.3 whilst the second simply is the normalized formulation of the deforming matrix [A]:

$${}^{(3)}[C^{\diamond}]$$

$$=$$

$$\begin{bmatrix} 0 & a^{3} & -a^{2} \\ a^{3} & 0 & -a^{1} \\ a^{2} & -a^{1} & 0 \end{bmatrix} \,.\, \begin{bmatrix} A^{1}_{1} & A^{1}_{2} & A^{1}_{3} \\ A^{2}_{1} & A^{2}_{2} & A^{2}_{3} \\ A^{3}_{1} & A^{3}_{2} & A^{3}_{3} \end{bmatrix}$$

$$=$$

$$-{}^{(3)}[W]\,.\,{}^{(3)}[J]\,.\,{}_{{}^{(3)}[J]}\Phi({}^{(3)}\mathbf{a})\,.\,{}^{(3)}[A^{\diamond}]$$

At the end, one gets the relation:

$${}^{(3)}[B^{\diamond}] \,=\, -{}^{(3)}[A^{\diamond}]\,.\,{}^{(3)}[W]\,.\,{}^{(3)}[J]\,.\,{}_{{}^{(3)}[J]}\Phi({}^{(3)}\mathbf{a})\,.\,{}^{(3)}[A^{\diamond}]$$

There is a non-normalized formulation for it:

$$[B]$$

$$= \{{}^{(3)}[B^{\diamond}]\,.\,[W]\}^{t}$$

$$= \{\{{}^{(3)}[A^{\diamond}]\,.\,{}^{(3)}[W]\}\,.\,\{{}^{(3)}[J]\,.\,{}_{{}^{(3)}[J]}\Phi(-{}^{(3)}\mathbf{a})\}\,.\,\{{}^{(3)}[A^{\diamond}]\,.\,[W]\}\}^{t}$$

$$= \{{}^{(3)}[A^{\diamond}]\,.\,[W]\}^{t}\,.\,\{{}_{{}^{(3)}[J]}\Phi({}^{(3)}\mathbf{a})\,.\,[J]^{t}\}\,.\,\{{}^{(3)}[A^{\diamond}]\,.\,[W]\}^{t}$$

$$= [A]\,.\,\{{}_{{}^{(3)}[J]}\Phi({}^{(3)}\mathbf{a})\,.\,[J]^{t}\}\,.\,[A]$$

The product:

$${}_{{}^{(3)}[J]}\Phi({}^{(3)}\mathbf{a})\,.\,[J]^{t} \,=\, \begin{bmatrix} a^{2} & 0 & a^{3} \\ -a^{1} & a^{3} & 0 \\ 0 & -a^{2} & -a^{1} \end{bmatrix}$$

... is an isomorphic representation of $\mathbf{a}$ in $\mathrm{M}(3, \mathbb{R})$.

Definition 1.4. *The isomorphic function $\odot_3$.*

The isomorphic function $\odot$ is such that:

$$\forall\, {}^{(3)}\mathbf{a} \in V_3 \xrightarrow{\odot_3} \odot_3(\mathbf{a}) \,=\, {}_{{}^{(3)}[J]}\Phi({}^{(3)}\mathbf{a})\,.\,[J]^t$$

Definition 1.5. *The function $f^2 = f \text{ o } f$.*

The function $f^2 = \text{f o f}$ is such that:

$$\forall\, ({}^{(3)}[A],\, {}^{(3)}\mathbf{a}) \in M(3,\, \mathbb{R}) \times V_3 \xrightarrow{f^2} \in M(3,\, \mathbb{R})$$

$$f^2({}^{(3)}[A],\, {}^{(3)}\mathbf{a}) \,=\, [A]\,.\, {}_{{}^{(3)}[J]}\Phi({}^{(3)}\mathbf{a})\,.\,[J]^t\,.\,[A]$$

Remark 1.2. *The determinant of [B].*

It is known that:

- The determinant of the simplest decomposition of any classical cross product is null.

$$|{}_{[J]}\Phi({}^{(3)}\mathbf{a})| \,=\, -a^3\,.\,(-a^2\,.\,a^1) \,-\, a^2\,.\,(-a^3\,.\,-a^1) \,=\, 0$$

- The determinant of any product of matrices is equal to the product of their respective determinants.

Therefore, the determinant of [B] is obligatorily null too:

$$|{}^{(3)}B| \,=\, 0,\, \forall\, ({}^{(3)}[A],\, {}^{(3)}\mathbf{a})$$

□

And, as claimed:

Lemma 1.1. *The matrix [B] is a specific and degenerated representation in $M(3,\mathbb{R})$ of a repetition of the action of $f = [\mathbf{a}, \ldots]_{[A]}$.*

Proposition 1.6. *The matrix $[B^\diamond]$ is the sum of six kernels of class II (see [a] for the definition and the theory concerning the kernels in a three-dimensional space).*

Proof. Since (recall previous proposition):

$$B^\eta_\epsilon$$

$$=$$

$$A^\eta_1\,.\,C^1_\epsilon \,+\, A^\eta_2\,.\,C^2_\epsilon \,+\, A^\eta_3\,.\,C^3_\epsilon$$

$$=$$

$$A^\eta_1\,.\,(A^2_\epsilon\,.\,a^3 \,-\, A^3_\epsilon\,.\,a^2) \,+\, A^\eta_2\,.\,(A^1_\epsilon\,.\,a^3 \,-\, A^3_\epsilon\,.\,a^1) \,+\, A^\eta_3\,.\,(A^1_\epsilon\,.\,a^2 \,-\, A^2_\epsilon\,.\,a^1)$$

$$=$$

$$-a^1\,.\,(A^\eta_2\,.\,A^3_\epsilon \,+\, A^\eta_3\,.\,A^2_\epsilon) \,+\, a^2\,.\,(A^\eta_3\,.\,A^1_\epsilon \,+\, A^\eta_1\,.\,A^3_\epsilon) \,+\, a^3\,.\,(A^\eta_1\,.\,A^2_\epsilon \,-\, A^\eta_2\,.\,A^1_\epsilon)$$

One can now alternatively write the matrix $[B^\diamond]$ as a weighted sum:

$$[B^\diamond]$$

$$=$$

$$\begin{aligned} &-a^1 . \{T_2(\otimes)(\mathbf{a}_2^\diamond, \mathbf{a}^{\diamond 3}) + T_2(\otimes)(\mathbf{a}_3^\diamond, \mathbf{a}^{\diamond 2})\} \\ &+ a^2 . \{T_2(\otimes)(\mathbf{a}_3^\diamond, \mathbf{a}^{\diamond 1}) - T_2(\otimes)(\mathbf{a}_1^\diamond, \mathbf{a}^{\diamond 3})\} \\ &+ a^3 . \{T_2(\otimes)(\mathbf{a}_1^\diamond, \mathbf{a}^{\diamond 2}) + T_2(\otimes)(\mathbf{a}_2^\diamond, \mathbf{a}^{\diamond 1})\} \end{aligned}$$

This is a sum of six Pythagorean tables which is weighted by the components of vector **a**. These tables are built with the six-pack implicitly contained in the normalized deforming matrix $[A^\diamond]$.

It has been explained in [a] that each Pythagorean table might be a kernel of class II for the non-trivial decomposition of some deformed cross product when this decomposition is associated with a polynomial of which the Hessian is degenerated. □

Proposition 1.7. *The matrix $[B^\diamond]$ is associated with a degenerated linear system:*

$$|B^\diamond| = 0$$

Proof. This proposition is though as a basic exercise confirming the statement which has already been done at the end of proposition 1.5. Let consider any pair $(\mathbf{u}_1, \mathbf{u}_2)$ in $V^2{}_3 = V_3 \times V_3$ and a second pair $(\mathbf{w}_1, \mathbf{w}_2)$ in $V^2{}_3$ too. Let build the Pythagorean tables $T_2(\otimes)(\mathbf{u}_1, \mathbf{u}_2)$ and $T_2(\otimes)(\mathbf{w}_1, \mathbf{w}_2)$. The determinant of each of them is null. Now, let ask the question: "What is the determinant of the sum of these tables?" Basic knowledge in linear algebra gives the well known answer: zero.

One can verify this affirmation in making all calculations in details:

$$|T_2(\otimes)(\mathbf{u}_1, \mathbf{u}_2) + T_2(\otimes)(\mathbf{w}_1, \mathbf{w}_2)|$$

$$=$$

$$\begin{vmatrix} u_1^1 . u_2^1 + w_1^1 . w_2^1 & u_1^2 . u_2^1 + w_1^2 . w_2^1 & u_1^3 . u_2^1 + w_1^3 . w_2^1 \\ u_1^1 . u_2^2 + w_1^1 . w_2^2 & u_1^2 . u_2^2 + w_1^2 . w_2^2 & u_1^3 . u_2^2 + w_1^3 . w_2^2 \\ u_1^1 . u_2^3 + w_1^1 . w_2^3 & u_1^2 . u_2^3 + w_1^2 . w_2^3 & u_1^3 . u_2^3 + w_1^3 . w_2^3 \end{vmatrix}$$

$$=$$

$$(u_1^1 . u_2^1 + w_1^1 . w_2^1)$$

$$. \{(u_1^2 . u_2^2 + w_1^2 . w_2^2) . (u_1^3 . u_2^3 + w_1^3 . w_2^3) - (u_1^2 . u_2^3 + w_1^2 . w_2^3) . (u_1^3 . u_2^2 + w_1^3 . w_2^2)\}$$

$$-(u_1^2 . u_2^1 + w_1^2 . w_2^1)$$

$$. \{(u_1^1 . u_2^2 + w_1^1 . w_2^2) . (u_1^3 . u_2^3 + w_1^3 . w_2^3) - (u_1^1 . u_2^3 + w_1^1 . w_2^3) . (u_1^3 . u_2^2 + w_1^3 . w_2^2)\}$$

$$+(u_1^3 . u_2^1 + w_1^3 . w_2^1)$$

$$. \{(u_1^1 . u_2^2 + w_1^1 . w_2^2) . (u_1^2 . u_2^3 + w_1^2 . w_2^3) - (u_1^1 . u_2^3 + w_1^1 . w_2^3) . (u_1^2 . u_2^2 + w_1^2 . w_2^2)\}$$

$$=$$

$$(u_1^1 . u_2^1 + w_1^1 . w_2^1) . (u_1^2 . w_1^3 - u_1^3 . w_1^2) . (u_2^2 . w_2^3 - u_2^3 . w_2^2)$$
$$- (u_1^2 . u_2^1 + w_1^2 . w_2^1) . (u_1^1 . w_1^3 - u_1^3 . w_1^1) . (u_2^2 . w_2^3 - u_2^3 . w_2^2)$$
$$+ (u_1^3 . u_2^1 + w_1^3 . w_2^1) . (u_1^1 . w_1^2 - u_1^2 . w_1^1) . (u_2^2 . w_2^3 - u_2^3 . w_2^2)$$

$$=$$

$$(u_2^2 . w_2^3 - u_2^3 . w_2^2)$$
$$. \{(u_1^1 . u_2^1 + w_1^1 . w_2^1) . (u_1^2 . w_1^3 - u_1^3 . w_1^2)$$
$$- (u_1^2 . u_2^1 + w_1^2 . w_2^1) . (u_1^1 . w_1^3 - u_1^3 . w_1^1)$$
$$+ (u_1^3 . u_2^1 + w_1^3 . w_2^1) . (u_1^1 . w_1^2 - u_1^2 . w_1^1)\}$$

$$=$$

$$(u_2^2 . w_2^3 - u_2^3 . w_2^2) . 0$$

$$=$$

$$0$$

Since $[\mathrm{B}^\diamond]$ is a linear combination of sums involving pairs of Pythagorean tables, its determinant vanishes and this matrix can be associated with a degenerated linear system. This fact is confirming what has already been obtained in proposition 1.5.

$$|{}^{(3)}[B^\diamond({}^{(3)}[A],\, {}^{(3)}\mathbf{a})| = 0$$

$\square$

1.6 Starting a systematization of the discussion

Remark 1.3. *The reiteration of the action of $f = [\mathbf{a}, ...]_{[A]}$.*

Let now calculate:

$$[\mathbf{a},\, \underbrace{[\mathbf{a},\, [\mathbf{a},\, \mathbf{x}]_{[A]}]_{[A]}}_{=\mathbf{Y}}]_{[A]} = ...$$

This is also:

$$[\mathbf{a},\, \mathbf{Y}]_{[A]} = \sum_{\lambda < \mu} A^{\psi}_{\lambda\mu} . (a^{\lambda} . Y^{\mu} - a^{\mu} . Y^{\lambda}) . \mathbf{e}_{\psi}$$

With:

$$Y^{\lambda} = \sum_{\alpha<\beta} B^{\lambda}_{\alpha\beta} . (a^{\alpha} . x^{\beta} - x^{\alpha} . a^{\beta})$$

In the language of components, that is in the dual space $\mathrm{V}^*{}_3$ (summations have been omitted because the rules governing them here are clear):

$$\{[\mathbf{a},\, \mathbf{Y}]_{[A]}\}^{\psi}$$

$$=$$

$$A^{\psi}_{\lambda\mu} \,.\, (a^{\lambda} \,.\, B^{\mu}_{\alpha\beta} \,.\, (a^{\alpha} \,.\, x^{\beta} \;-\; x^{\alpha} \,.\, a^{\beta}) \;-\; a^{\mu} \,.\, B^{\lambda}_{\alpha\beta} \,.\, (a^{\alpha} \,.\, x^{\beta} \;-\; x^{\alpha} \,.\, a^{\beta}))$$

$$=$$

$$A^{\psi}_{\lambda\mu} \,.\, (a^{\lambda} \,.\, B^{\mu}_{\alpha\beta} \;-\; a^{\mu} \,.\, B^{\lambda}_{\alpha\beta}) \,.\, (a^{\alpha} \,.\, x^{\beta} \;-\; x^{\alpha} \,.\, a^{\beta})$$

One recognizes a pattern similar to the one which has been observed in calculating $[\mathbf{a}, \mathbf{X}]_{[A]}$. Hence, in starting with the pair ([A], $\mathbf{a}$) representing the function f $= [\mathbf{a}, \boldsymbol{...}]_{[A]}$ acting on $\boldsymbol{...}$, and calculating f o f, one gets the pair ([B], $\mathbf{a}$). In continuing with the latter representing the function f o f $= [\mathbf{a}, \boldsymbol{...}]_{[B]}$ acting on $\boldsymbol{...}$, and calculating f o (f o f), one gets a cube E such that:

$$E^{\psi}_{\alpha\beta} \;=\; A^{\psi}_{\lambda\mu} \,.\, (a^{\lambda} \,.\, B^{\mu}_{\alpha\beta} \;-\; a^{\mu} \,.\, B^{\lambda}_{\alpha\beta})$$

The cube E inherits from the anti-symmetry of cube B. Hence, as the cubes A and B, it can be condensed inside an element [E] in $\mathrm{M}(3, \mathbb{R})$. This maneuver can be repeated as meany times as desired and it would always bring the same type of formal result. Therefore, at this stage, one can start a systematization of the discussion.

$$\begin{tikzcd}
The\,function\,f\,and\,its\,reiteration \arrow[r] \arrow[d] & Representations\,in\,V_3^* \arrow[d] \\
{[\mathbf{a},\, ...]_{[J]} \;=\; \mathbf{a} \wedge ...} \arrow[r] \arrow[d] & \underbrace{\{[J]^t \,.\, [J]\}}_{=Id_3} \,.\, |\mathbf{a} \wedge ... > \arrow[d] \\
{[\mathbf{a},\, ...]_{[_0A]}} \arrow[r] \arrow[d] & \{[_0A]^t \,.\, [J]\} \,.\, |\mathbf{a} \wedge ... > \arrow[d] \\
{[\mathbf{a},\, [\mathbf{a},\, ...]_{[_0A]}]_{[_0A]} \;=\; [\mathbf{a},\, ...]_{[_1A]}} \arrow[r] \arrow[d] & \{[_1A]^t \,.\, [J]\} \,.\, |\mathbf{a} \wedge ... > \arrow[d] \\
{[\mathbf{a},\, [\mathbf{a},\, [\mathbf{a},\, ...]_{[_0A]}]_{[_0A]}]_{[_0A]} \;=\; [\mathbf{a},\, ...]_{[_2A]}} \arrow[r] \arrow[d] & \{[_2A]^t \,.\, [J]\} \,.\, |\mathbf{a} \wedge ... > \arrow[d] \\
etc. & etc.
\end{tikzcd}$$

With the correspondences:

$$[_0A] \;=\; [A]$$

$$[_1A] \;=\; [B] \;=\; [_0A] \,.\, [_0C],\; [_0C] \;=\; [a^{\lambda} \,.\, (_0A)^{\mu}_{\alpha\beta} \;-\; a^{\mu} \,.\, (_0A)^{\lambda}_{\alpha\beta}]$$

$$[_2A] \;=\; [E] \;=\; [_0A] \,.\, [_1C],\; [_1C] \;=\; [a^{\lambda} \,.\, (_1A)^{\mu}_{\alpha\beta} \;-\; a^{\mu} \,.\, (_1A)^{\lambda}_{\alpha\beta}]$$

$$etc.$$

$$[_pA] \;=\; [_0A] \,.\, [_{p-1}C],\; [_{p-1}C] \;=\; [a^{\lambda} \,.\, (_{p-1}A)^{\mu}_{\alpha\beta} \;-\; a^{\mu} \,.\, (_{p-1}A)^{\lambda}_{\alpha\beta}]$$

The *deforming matrix* changes with the number p of iterations.

Remark 1.4. *The effective deforming matrices.*

In continuing the calculations, one gets:

$$E^{\psi}_{\alpha\beta}$$
$$=$$
$$A^{\psi}_{12} \,.\, (B^{1}_{\alpha\beta} \,.\, a^{2} - B^{2}_{\alpha\beta} \,.\, a^{1}) + A^{\psi}_{23} \,.\, (B^{2}_{\alpha\beta} \,.\, a^{3} - B^{3}_{\alpha\beta} \,.\, a^{2}) + A^{\psi}_{13} \,.\, (B^{1}_{\alpha\beta} \,.\, a^{3} - B^{3}_{\alpha\beta} \,.\, a^{1})$$
$$=$$
$$(A^{\psi}_{12} \,.\, a^{2} + A^{\psi}_{13} \,.\, a^{3}) \,.\, B^{1}_{\alpha\beta} + (-A^{\psi}_{12} \,.\, a^{1} + A^{\psi}_{23} \,.\, a^{3}) \,.\, B^{2}_{\alpha\beta} - (A^{\psi}_{13} \,.\, a^{1} + A^{\psi}_{23} \,.\, a^{2}) \,.\, B^{3}_{\alpha\beta}$$
$$=$$
$$\sum_{\lambda} (\sum_{\epsilon} A^{\psi}_{\epsilon\lambda} \,.\, a^{\epsilon}) \,.\, B^{\lambda}_{\alpha\beta}$$

With:

$$B^{\lambda}_{\alpha\beta}$$
$$=$$
$$A^{\lambda}_{12} \,.\, (A^{1}_{\alpha\beta} \,.\, a^{2} - A^{2}_{\alpha\beta} \,.\, a^{1}) + A^{\lambda}_{23} \,.\, (A^{2}_{\alpha\beta} \,.\, a^{3} - A^{3}_{\alpha\beta} \,.\, a^{2}) + A^{\lambda}_{13} \,.\, (A^{1}_{\alpha\beta} \,.\, a^{3} - A^{3}_{\alpha\beta} \,.\, a^{1})$$
$$=$$
$$(A^{\lambda}_{12} \,.\, a^{2} + A^{\lambda}_{13} \,.\, a^{3}) \,.\, A^{1}_{\alpha\beta} + (-A^{\lambda}_{12} \,.\, a^{1} + A^{\lambda}_{23} \,.\, a^{3}) \,.\, A^{2}_{\alpha\beta} - (A^{\lambda}_{13} \,.\, a^{1} + A^{\lambda}_{23} \,.\, a^{2}) \,.\, A^{3}_{\alpha\beta}$$
$$=$$
$$\sum_{\mu} (\sum_{\pi} A^{\lambda}_{\pi\mu} \,.\, a^{\pi}) \,.\, A^{\mu}_{\alpha\beta}$$

Here too, one recognizes a pattern:

$$[B^{\diamond}] \;=\; {}_{A}\Phi(\mathbf{a}) \,.\, [A^{\diamond}] \;=\; {}_{0A}\Phi(\mathbf{a}) \,.\, [{}_{0}A^{\diamond}] \;=\; [{}_{1}A^{\diamond}]$$
$$[E^{\diamond}] \;=\; {}_{A}\Phi(\mathbf{a}) \,.\, [B^{\diamond}] \;=\; {}_{0A}\Phi(\mathbf{a}) \,.\, [{}_{1}A^{\diamond}] \;=\; {}_{0A}\Phi^{2}(\mathbf{a}) \,.\, [{}_{0}A^{\diamond}] \;=\; [{}_{2}A^{\diamond}]$$
$$Etc.$$
$$[{}_{p}A^{\diamond}] \;=\; {}_{0A}\Phi^{p}(\mathbf{a}) \,.\, [{}_{0}A^{\diamond}]$$

If one prefers, this relation can be written with non-normalized matrices:

$$\{[W] \,.\, [{}_{p}A]\}^{t} \;=\; {}_{0A}\Phi^{p}(\mathbf{a}) \,.\, \{[W] \,.\, [{}_{0}A]\}^{t} \iff [W] \,.\, [{}_{p}A] \;=\; [W] \,.\, [{}_{0}A] \,.\, \{{}_{0A}\Phi^{p}(\mathbf{a})\}^{t}$$

And one gets in general:

$$[{}_{p}A] \;=\; [{}_{0}A] \,.\, \{{}_{0A}\Phi^{p}(\mathbf{a})\}^{t}$$

Hence, the *effective deforming matrix* for the pth iteration is:

$$[{}_{p}A]^{t} \,.\, [J] \;=\; {}_{0A}\Phi^{p}(\mathbf{a}) \,.\, \{[{}_{0}A]^{t} \,.\, [J]\}$$

Example 1.1. *The case p = 1 and the involution of f.*

In that case:

$$\{[B]^t \,.\, [J]\} \;=\; [{}_1A]^t \,.\, [J] \;=\; {}_{0A}\Phi(\mathbf{a}) \,.\, \{[{}_0A]^t \,.\, [J]\}$$

It follows that:

- The matrix:
$${}_{0A}\Phi(\mathbf{a}) \,.\, \{[{}_0A]^t \,.\, [J]\} \,.\, {}_{0A}\Phi(\mathbf{a})$$
... is the simplest representation of f^2 = f o f (equivalently: of a unique iteration of the action of f);

- The function f is an involution when:
$${}_{0A}\Phi(\mathbf{a}) \,.\, \{[{}_0A]^t \,.\, [J]\} \,.\, {}_{0A}\Phi(\mathbf{a}) \;=\; Id_3$$
For example, when the cube ${}_0$A is equivalent to the matrix [A] = [J], the function f = $[\mathbf{a}, \ldots]_{[A]}$ is an involution when:
$${}_{[J]}\Phi^2(\mathbf{a}) \;=\; T_2(\otimes)(\mathbf{a},\,\mathbf{a}) - <\mathbf{a},\,\mathbf{a}>_{Id_3}\ .\,Id_3 \;=\; Id_3$$
Is it possible? When? This condition means that the matrix $T_2(\otimes)(\mathbf{a}, \mathbf{a})$ - $(1 + <\mathbf{a}, \mathbf{a}>_{Id3})\,.\,Id_3$ is degenerated; equivalently, that its determinant is null:
$$|T_2(\otimes)(\mathbf{a},\,\mathbf{a}) \,-\, (1+ <\mathbf{a},\,\mathbf{a}>_{Id_3})\,.\,Id_3| \;=\; 0$$
An involution can only be envisaged when the components of vector **a** are not free. With different words, the involution f exists only when there is a specific constraint on the components of **a**. Since the determinant of any Pythagorean table is null, it is easy to identify this constraint:
$$1+ <\mathbf{a},\,\mathbf{a}>_{Id_3} = \;0$$
This constraint cannot be realized with a vector **a** in E(3, $\mathbb{R}$), but it can when this vector has components in $\mathbb{C}$ or, only for the pedagogy, when this vector has components in $\mathbb{H}$, e.g.:
$$a^1 \;=\; \frac{I}{\sqrt{3}},\, a^2 \;=\; \frac{J}{\sqrt{3}},\, a^3 \;=\; \frac{K}{\sqrt{3}}$$
... with:
$$I^2 \;=\; J^2 \;=\; K^2 \;=\; -1$$
This constraint will be confirmed later in this document; please see remark 1.7 and lemma 1.2 below.

Definition 1.6. *The weighting and the weighted vectors.*

Let consider any element $\mathbf{a}$ in E(3, $\mathbb{R}$) acting as projectile inside a deformed cross product $[\mathbf{a}, ...]_{[A]}$. Per convention, this vector is also called a *weighting vector*. Any pair ([A], $\mathbf{a}$) generates a set of seven vectors: the weighting vector $\mathbf{a}$ plus the six-pack associated with the deforming matrix [A]. These seven vectors can always be organized to form a *weighted vector* $\mathbf{S}_{[X]}$([A], $\mathbf{a}$) which is the sum of all mixed cross products that can be built with the six-pack when these cross products are simultaneously deformed by some element [X] in M(3 $\mathbb{R}$); precisely:

$$\mathbf{S}_{[X]}([A],\, \mathbf{a})$$

$$=$$

$$\begin{gathered} -a^1 . ([\mathbf{a}_2,\, \mathbf{a}^3]_{[X]} \,+\, [\mathbf{a}_3,\, \mathbf{a}^2]_{[X]}) \\ +\, a^2 . ([\mathbf{a}_1,\, \mathbf{a}^3]_{[X]} \,-\, [\mathbf{a}_3,\, \mathbf{a}^1]_{[X]}) \\ +\, a^3 . ([\mathbf{a}_1,\, \mathbf{a}^2]_{[X]} \,+\, [\mathbf{a}_2,\, \mathbf{a}^1]_{[X]}) \end{gathered}$$

Example 1.2. *The classical Euclidean cross product.*

Let suppose that [A] = [J] because this eventuality may concern our own three-dimensional Euclidean space E(3, $\mathbb{R}$); in that case:

$$[A] \,=\, [J] \,=\, \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & -1 & 0 \end{bmatrix}$$

$$\downarrow$$

$$|\mathbf{a}_1 > \equiv \left| \begin{array}{c} A_1^1 \,=\, 0 \\ A_1^2 \,=\, 1 \\ A_1^3 \,=\, 0 \end{array} \right\rangle ;\, |\mathbf{a}_2 > \equiv \left| \begin{array}{c} A_2^1 \,=\, 0 \\ A_2^2 \,=\, 0 \\ A_2^3 \,=\, -1 \end{array} \right\rangle ;\, |\mathbf{a}_3 > \equiv \left| \begin{array}{c} A_3^1 \,=\, 1 \\ A_3^2 \,=\, 0 \\ A_3^3 \,=\, 0 \end{array} \right\rangle$$

and:

$$[J] \,=\, \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & -1 & 0 \end{bmatrix}$$

$$\downarrow$$

$$|\mathbf{a}^1 > \equiv \left| \begin{array}{c} A_1^1 \,=\, 0 \\ A_2^1 \,=\, 0 \\ A_3^1 \,=\, 1 \end{array} \right\rangle ;\, |\mathbf{a}^2 > \equiv \left| \begin{array}{c} A_1^2 \,=\, 1 \\ A_2^2 \,=\, 0 \\ A_3^2 \,=\, 0 \end{array} \right\rangle ;\, |\mathbf{a}^3 > \equiv \left| \begin{array}{c} A_1^3 \,=\, 0 \\ A_2^3 \,=\, -1 \\ A_3^3 \,=\, 0 \end{array} \right\rangle$$

This is implying:

$$\mathbf{a}_1 \,=\, -\mathbf{a}^3;\, \mathbf{a}_2 \,=\, -\mathbf{a}^1;\, \mathbf{a}_3 \,=\, \mathbf{a}^2$$

$$|\mathbf{a}_1 \wedge \mathbf{a}^2 > = |\mathbf{a}_1 \wedge \mathbf{a}_3 > = \left| \begin{array}{c} 0 \\ 0 \\ 1 \end{array} \right\rangle \,=\, -|\mathbf{a}_2 >$$

$$|\mathbf{a}_2 \wedge \mathbf{a}^3 > = -|\mathbf{a}_2 \wedge \mathbf{a}_1 > = \left| \begin{array}{c} -1 \\ 0 \\ 0 \end{array} \right\rangle \,=\, -|\mathbf{a}_3 >$$

$$|\mathbf{a}_3 \wedge \mathbf{a}^1> = -|\mathbf{a}_3 \wedge \mathbf{a}_2> = \left| \begin{array}{c} 0 \\ -1 \\ 0 \end{array} \right\rangle = -|\mathbf{a}_1>$$

and:

$$|\mathbf{a}_1 \wedge \mathbf{a}^3> = \left| \begin{array}{c} 0 \\ 0 \\ 0 \end{array} \right\rangle ; |\mathbf{a}_3 \wedge \mathbf{a}^2> = \left| \begin{array}{c} 0 \\ 0 \\ 0 \end{array} \right\rangle ; |\mathbf{a}_2 \wedge \mathbf{a}^1> = \left| \begin{array}{c} 0 \\ 0 \\ 0 \end{array} \right\rangle ; \qquad (1)$$

A first immediate consequence is the possibility to calculate the weighted vector in that special space:

$$\mathbf{S}_{[J]}([J], \mathbf{a})$$
$$=$$
$$-a^1 . [\mathbf{a}_2, \mathbf{a}^3]_{[J]} + a^2 . [\mathbf{a}_3, \mathbf{a}^1]_{[J]} + a^3 . [\mathbf{a}_1, \mathbf{a}^2]_{[J]}$$
$$=$$
$$-a^1 . (\mathbf{a}_2 \wedge \mathbf{a}^3) + a^2 . (\mathbf{a}_3 \wedge \mathbf{a}^1) + a^3 . (\mathbf{a}_1 \wedge \mathbf{a}^2)$$
$$=$$
$$a^1 . \mathbf{a}_3 - a^2 . \mathbf{a}_1 - a^3 . \mathbf{a}_2$$

Since there is a simple correspondence between the vectors composing [A] = [J] and the canonical basis Ω of E(3, $\mathbb{R}$), precisely:

$$\mathbf{e}_3 = -\mathbf{a}_2; \, \mathbf{e}_1 = \mathbf{a}_3; \, \mathbf{e}_2 = \mathbf{a}_1;$$

it becomes obvious that:

$$\forall \mathbf{a} : \mathbf{S}_{[J]}([J], \mathbf{a}) = a^1 . \mathbf{e}_1 - a^2 . \mathbf{e}_2 + a^3 . \mathbf{e}_3$$

One state that:

$$[W] . [J] . |\mathbf{S}_{[J]}([J], \mathbf{a})> = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} . \begin{bmatrix} a^1 \\ -a^2 \\ a^3 \end{bmatrix} = |\mathbf{a}>$$

1.7 Deformed cross product and its non-trivial decomposition in case of involution

Question: "What happens when f = $[\mathbf{a}, ...]_{[A]}$ (i) is an involution and (ii) owns at least one non-trivial decomposition?"

Remark 1.5. *First try*

When f is an involution (recall):

$$[{}^{(3)}\mathbf{a}, [{}^{(3)}\mathbf{a}, {}^{(3)}\mathbf{x}]_{{}^{(3)}[A]}]_{{}^{(3)}[A]} = [{}^{(3)}\mathbf{a}, {}^{(3)}\mathbf{x}]_{{}^{(3)}[B({}^{(3)}[A], {}^{(3)}\mathbf{a})]} = {}^{(3)}\mathbf{x}$$

Or, equivalently (recall the basic result of proposition 1.1):

$$\{ {}^{(3)}[B({}^{(3)}[A], {}^{(3)}\mathbf{a})]^t . [J] \} . |{}^{(3)}\mathbf{a} \wedge {}^{(3)}\mathbf{x}> = |{}^{(3)}\mathbf{x}>$$

When there exists at least one non-trivial decomposition denoted ([K], **z**)[1]:

$$\{{}^{(3)}[B({}^{(3)}[A],\, {}^{(3)}\mathbf{a})]^t\,.\,[J]\}\,.\,\{[K]\,.\,|{}^{(3)}\mathbf{x}>\,+|{}^{(3)}\mathbf{z}>\}\;=\;|{}^{(3)}\mathbf{x}>$$

This situation forces to work with:

$$\{\,{}^{(3)}[B({}^{(3)}[A],\, {}^{(3)}\mathbf{a})]^t\,.\,[J]\,.\,[K]\;-\;Id_3\}\,.\,|{}^{(3)}\mathbf{x}>=\,-|{}^{(3)}\mathbf{z}>$$

As consequence of proposition 1.5 and remark 1.2, one must recall that:

$${}^{(3)}[B({}^{(3)}[A],\, {}^{(3)}\mathbf{a})]^t\,.\,[J]\,.\,[K]\;\neq\;Id_3$$

... because, whatever the decomposition of $|\mathbf{a}\wedge\mathbf{x}>$ is (i.e.: simple or not) the determinant of [B] is null whilst the determinant of the identity matrix is of course equal to $1_{\mathbb{R}}$.

The decomposition $([\text{K}],\mathbf{z})$ depends on the nature of a polynomial $\Lambda(\mathbf{a})$ describing the behavior of the components of **a**. For more technical details, please see [a] and [b]. Any way, as usual, (i) when:

$$|{}^{(3)}[B({}^{(3)}[A],\, {}^{(3)}\mathbf{a})]^t\,.\,[J]\,.\,[K]\;-\;Id_3|\;\neq\;0$$

... and (ii) if the decomposition $([\text{K}],\mathbf{z})$ is known, then one can always find a vector **x** on which a given function $[\mathbf{a},\ldots]_{[A]}$ acts like an involution... even if the calculations are tedious.

Remark 1.6. *Second try*

One could make a logical objection concerning the approach exposed in previous remark. More precisely: "Why should the cross product $\mathbf{a}\wedge\mathbf{x}$ have a non-trivial decomposition and not $\text{f}(\mathbf{x}) = [\mathbf{a},\mathbf{x}]_{[A]}$? And if the latter has effectively a non-trivial decomposition, how does it impact $\text{f}^2(\mathbf{x}) = [\mathbf{a},[\mathbf{a},\mathbf{x}]_{[A]}]_{[A]}$?"

In accordance with the results which have been obtained in [a] and [b], one can envisage the existence of a non-trivial decomposition for the image of $\text{f}(\mathbf{x})$ in $\text{V}^*{}_3$:

$$|f(\mathbf{x})>=\,|[\mathbf{a},\,\mathbf{x}]_{[A]}>=\,\{[A]^t\,.\,[J]\}\,.\,\{[K]\,.\,|\mathbf{x}>\,+|\mathbf{z}>\}$$

Therefore, one can also envisage to calculate the image of $\text{f}^2(\mathbf{x})$ in $\text{V}^*{}_3$:

$$|f^2(\mathbf{x})>$$

$$=$$

$$|[\mathbf{a},\,[\mathbf{a},\,\mathbf{x}]_{[A]}]_{[A]}>$$

$$=$$

$$\{[A]^t\,.\,[J]\}\,.\,\{[K]\,.\,|[\mathbf{a},\,\mathbf{x}]_{[A]}>\,+|\mathbf{z}>\}$$

[1] Where [K] is the kernel -also called main part- and where **z** is the residual part of the decomposition.

$$=$$

$$\{[A]^t\,.\,[J]\}\,.\,\{[K]\,.\,\{[A]^t\,.\,[J]\}\,.\,\{[K]\,.\,|\mathbf{x}> \,+|\mathbf{z}>\}\,+\,|\mathbf{z}>\}$$

$$=$$

$$\{[A]^t\,.\,[J]\}\,.\,[K]\,.\,\{[A]^t\,.\,[J]\}\,.\,[K]\,.\,|\mathbf{x}> \,+\{[A]^t\,.\,[J]\}\,.\,\{Id_3+[K]\,.\,\{[A]^t\,.\,[J]\}\}\,.\,|\mathbf{z}>$$

As consequence, when f^2 is an involution, one can write:

$$\{\{[A]^t\,.\,[J]\}\,.\,[K]\,.\,\{[A]^t\,.\,[J]\}\,.\,[K]\,-\,Id_3\}\,.\,|\mathbf{x}>$$

$$=$$

$$-\{[A]^t\,.\,[J]\}\,.\,\{Id_3\,+\,[K]\,.\,\{[A]^t\,.\,[J]\}\}\,.\,|\mathbf{z}>$$

This relation represents a very classical linear system containing three equations depending on the three components of $\mathbf{x}$. One can *treat it* as usual when the deforming matrix [A] and the polynomial $\Lambda(\mathbf{a})$ are known through the local physical circumstances.

1.8 The classical cross product and the involution

The classical cross product, as already mentioned in proposition 1.2, is a deformed Lie product acting in a three-dimensional space which has been deformed by the matrix [A] = [J]. Hence - *if the approach which is promoted in this document is plausible*- to know if and when this cross product acts like an involution, one should consider the system:

$$\{\{[J]^t\,.\,[J]\}\,.\,[K]\,.\,\{[J]^t\,.\,[J]\}\,.\,[K]\,-\,Id_3\}\,.\,|\mathbf{x}>$$

$$=$$

$$-\{[J]^t\,.\,[J]\}\,.\,\{Id_3\,+\,[K]\,.\,\{[J]^t\,.\,[J]\}\}\,.\,|\mathbf{z}>$$

And this system is equivalent to:

$$\{[K]^2\,-\,Id_3\}\,.\,|\mathbf{x}>=\,-[K]\,.\,|\mathbf{z}>$$

Where [K] is the kernel of a given non-trivial decomposition whilst $\mathbf{z}$ is its residual part.

Remark 1.7. *Confrontation with the classical approach within a three-dimensional Euclidean context.*

Before going further and eventually before making unnecessary calculations, let consider what happens in calculating a repeated cross product ($\mathbf{a}\,\wedge$ ...) within a three-dimensional Euclidean context; as usual:

$$\mathbf{a}\,\wedge\,(\mathbf{a}\,\wedge\,\mathbf{x})\,=<\mathbf{a},\,\mathbf{x}>_{Id_3}\,.\mathbf{a}\,-\,\underbrace{<\mathbf{a},\,\mathbf{a}>_{Id_3}}_{=||\mathbf{a}||^2}.\,\mathbf{x}$$

...

Proposition 1.8. *The classical cross product is an involution when it is plausible to write:*

$$\mathbf{a} \wedge (\mathbf{a} \wedge \mathbf{x}) = <\mathbf{a}, \mathbf{x}>_{Id_3} .\mathbf{a} - ||\mathbf{a}||^2 .\mathbf{x} = \mathbf{x}$$

This is impossible as long as the discussion is developed with elements in E(3, $\mathbb{R}$).

Proof. There are two arguments justifying this affirmation:

- The condition representing the involution contains the information that $\mathbf{x}$ must be proportional to $\mathbf{a}$; in that case, the cross product $(\mathbf{a} \wedge \mathbf{x})$ vanishes and asking if it is an involution becomes totally meaningless.

- Envisaging the sub-case for which $\mathbf{x}$ is orthogonal to $\mathbf{a}$ would avoid the dependence between both vectors and the first obstruction, but the realization of the involution would impose a negative Euclidean norm: this is not allowed within a discussion with elements in $\mathrm{E}(3, \mathbb{R})$.

□

In opposition:

Lemma 1.2. *The involution of a classical cross product can be envisaged in developing the mathematical discussion in a three-dimensional space where the vectors have components in $\mathbb{C}$. The existence of an involution imposes four necessary conditions:*

- *The definition of what is called the scalar product between two vectors, here between $\boldsymbol{a}$ and $\boldsymbol{x}$, is similar to the definition which is used with elements in E(3, $\mathbb{R}$), but the result is now in $\mathbb{C}$. Therefore this scalar product can no more be involved in the definition of a classical Euclidean norm and the misleading notation ||...|| will be up to now discarded. The existence of an involution will be written:*

$$\mathbf{a} \wedge (\mathbf{a} \wedge \mathbf{x}) = <\mathbf{a}, \mathbf{x}>_{Id_3} .\mathbf{a} - <\mathbf{a}, \mathbf{a}>_{Id_3} .\mathbf{x} = \mathbf{x}$$

  With:

$$<\mathbf{a}, \mathbf{x}>_{Id_3} = <\mathbf{a}, \mathbf{x}>_{Id_3} = a^1 . x^1 + a^2 . x^2 + a^3 . x^3 \in \mathbb{C}$$

- *The vector $\boldsymbol{a}$ must not vanish and it must be different from the vector $\boldsymbol{x}$ on which it is acting:*

$$\forall z \in \mathbb{C} - \{0_{\mathbb{C}}\} : \mathbf{a} \neq z . \mathbf{x}$$

- *The vector $\boldsymbol{a}$ must act on a vector $\boldsymbol{x}$ which is "orthogonal" to it:*

$$\mathbf{a} \perp \mathbf{x} \iff <\mathbf{a}, \mathbf{x}>_{Id_3} = 0_{\mathbb{C}}$$

- *The vector $\boldsymbol{a}$ must be chosen to respect the relation:*

$$<\mathbf{a}, \mathbf{a}>_{Id_3} + 1 = 0_{\mathbb{C}}$$

1.9 Conclusion

There is much more to say about the involution in three-dimensional spaces, especially if vectors have representations in $\mathbb{C}^3$ or in $\mathbb{H}^3$. But the topic deserves an entire chapter and it will be developed later elsewhere. This document was only though as a pedagogical introduction and warm up to make students want to study the subject in more depth.

References

2 Bibliography

2.1 My contributions

[a] The (E) Question in a Three-Dimensional Space - Analyzing a Subset of Linear Systems with the Intrinsic Method; viXra:2503.0200, 32 pages.

[b] The so-called Extrinsic Method; viXra:2406.0127, 12 pages.

2.2 International works

[01] Quantenmechanik (Eine Einführung mit Anwendungen auf Atome, Moleküle und Festkörper); ©1999 B.G. Teubner, Stuttgart - Leipzig (Teubner-Studienbuecher ; Physik), ISBN 3-519-03246-5, 669 pages.

[02] Introduction to group theory for physicists; State University of New-York at Stony Brook; January 12, 2011.