Simple Formulas for the Horizontal Span of a Catenary Curve

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Abstract

This study derives formulas for the horizontal span of a catenary curve formed by a string fixed at two points. Given the heights h_1 and h_2 at the endpoints and the curve length l, we calculate the horizontal span s. We consider two cases based on the position of the minimum point, proving corresponding formulas. Theorems 1 and 2 provide complex logarithmic expressions, while Theorems 3 and 4 offer concise forms using hyperbolic functions. These results enhance the understanding of catenary curves and support applications in engineering and physics, where precise calculations of such curves are essential.

1 Introduction

A catenary curve is the curve formed by a string fixed at two arbitrary points, commonly observed in nature. The catenary curve is expressed as

$$y = a\cosh\frac{x}{a},$$

where its shape is determined by a single parameter a ($a > 0$).

The function y attains its minimum value a at $x = 0$, and the curvature decreases as a increases. This is illustrated in Figure 1.

This paper proves the following four theorems.

Theorem 1. For a catenary curve formed by a string fixed at two points, let the heights from the minimum point at the endpoints be h_1 and h_2, and the curve length be l. When the minimum point lies between the two endpoints, the horizontal span s is given by

$$s = \frac{l^2 - (h_1 + h_2)^2}{h_1 + h_2 + 2l\sqrt{\frac{h_1 h_2}{l^2 - (h_1 - h_2)^2}}} \log \frac{\sqrt{l^2 - (h_1 - h_2)^2} + 2\sqrt{h_1 h_2}}{l - h_1 - h_2}.$$

Theorem 2. For a catenary curve formed by a string fixed at two points, let the heights from the minimum point at the endpoints be h_1 and h_2, and the curve length be l. When the minimum point does not lie between the two endpoints, the horizontal span s is given by

$$s = \frac{l^2 - (h_1 + h_2)^2}{h_1 + h_2 - 2l\sqrt{\frac{h_1 h_2}{l^2 - (h_1 - h_2)^2}}} \log \frac{\sqrt{l^2 - (h_1 - h_2)^2} - 2\sqrt{h_1 h_2}}{l - h_1 - h_2}.$$

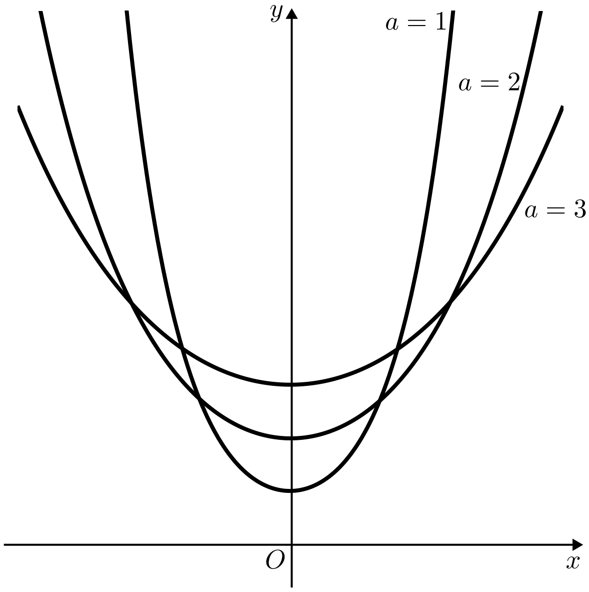


Figure 1: Shape of the catenary curve and parameter a

Theorem 3. For a catenary curve formed by a string fixed at two points, let the heights from the minimum point at the endpoints be h_1 and h_2, and the curve length be l. When the minimum point lies between the two endpoints, the horizontal span s is given by

$$s = \frac{X}{\sinh X}\sqrt{l^2 - (h_1 - h_2)^2},$$

where

$$X = \log \frac{\sqrt{l^2 - (h_1 - h_2)^2} + 2\sqrt{h_1 h_2}}{l - h_1 - h_2}.$$

Theorem 4. For a catenary curve formed by a string fixed at two points, let the heights from the minimum point at the endpoints be h_1 and h_2, and the curve length be l. When the minimum point does not lie between the two endpoints, the horizontal span s is given by

$$s = \frac{X}{\sinh X}\sqrt{l^2 - (h_1 - h_2)^2},$$

where

$$X = \log \frac{l + h_1 + h_2}{\sqrt{l^2 - (h_1 - h_2)^2} + 2\sqrt{h_1 h_2}}.$$

2 Problem Formulation

For simplicity, we consider the curve

$$y = a \cosh \frac{x}{a} - a, \tag{1}$$

where y takes its minimum value 0 at $x = 0$. Since any catenary curve can be reduced to this form via translation, this formulation does not lose generality.

Figure 2 illustrates the concepts of the heights h_1 and h_2 at the endpoints, the curve length l, and the horizontal span s for a catenary curve formed by a string fixed at two points. The height at the right endpoint is h_1, and at the left endpoint is h_2, with $l, s \in \mathbb{R}^+$, $h_1, h_2 \in \mathbb{R}_0^+$, $h_1 + h_2 > 0$, and $l > |h_1 - h_2|$.

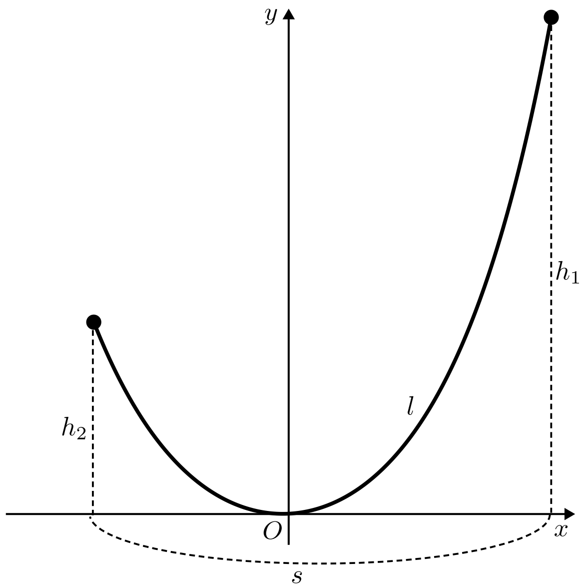
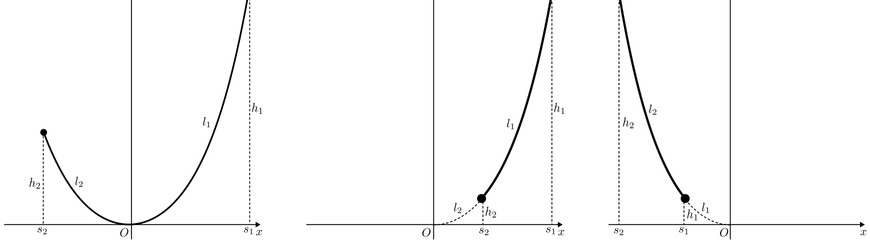


Figure 2: Conceptual diagram of heights h_1, h_2, curve length l, and horizontal span s in a catenary curve

In this paper, we denote the x-coordinate of the right endpoint as s_1, the left endpoint as s_2, the curve length from 0 to s_1 as l_1, and from s_2 to 0 as l_2, used in the proofs. Here, $s_1 > s_2$, $s = s_1 - s_2$, and $l = l_1 + l_2$. For generality, we consider cases where s_1 and s_2 are positive or negative. This is depicted in Figure 3. The case where the minimum point lies between the endpoints corresponds to s_1 and s_2 having opposite signs, i.e., $s_1 s_2 \leq 0$. Conversely, when the minimum point does not lie between the endpoints, s_1 and s_2 have the same sign, i.e., $s_1 s_2 \geq 0$.



(a) $s_1 > 0, s_2 < 0$ **(b)** $s_1 > 0, s_2 > 0$ **(c)** $s_1 < 0, s_2 < 0$

Figure 3: Conceptual diagram of s_1, s_2, l_1, l_2

The expressions for h_1, h_2, l_1, l_2 are derived as follows. Since h_1 is the value of y at $x = s_1$, from Equation (1),

$$h_1 = a \cosh \frac{s_1}{a} - a. \tag{2}$$

Similarly, h_2 is the value of y at $x = s_2$, so

$$h_2 = a \cosh \frac{s_2}{a} - a. \tag{3}$$

For l_1, using the arc length formula,

$$\begin{aligned}
l_1 &= \int_0^{s_1} \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx \\
&= \int_0^{s_1} \sqrt{1 + \left(\frac{d}{dx}\left(a \cosh \frac{x}{a} - a\right)\right)^2}\, dx \\
&= \int_0^{s_1} \sqrt{1 + \sinh^2 \frac{x}{a}}\, dx \\
&= \int_0^{s_1} \sqrt{\cosh^2 \frac{x}{a}}\, dx \\
&= \int_0^{s_1} \cosh \frac{x}{a}\, dx \\
&= \left[a \sinh \frac{x}{a}\right]_0^{s_1} \\
&= a \sinh \frac{s_1}{a}. \qquad (4)
\end{aligned}$$

Similarly, for l_2,

$$\begin{aligned}
l_2 &= \int_{s_2}^{0} \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx \\
&= \left[a \sinh \frac{x}{a}\right]_{s_2}^{0} \\
&= -a \sinh \frac{s_2}{a}. \qquad (5)
\end{aligned}$$

3 Related Work

Chatterjee et al. [1] proposed a method to derive the curve parameter a using known h_1, h_2, l when $s_1 s_2 \leq 0$. Specifically,

$$\begin{aligned}
a &= \frac{(h_1 + h_2)(l^2 - (h_1 - h_2)^2) - 2l\sqrt{h_1 h_2 (l^2 - (h_1 - h_2)^2)}}{2(h_1 - h_2)^2}, \\
l_1 &= \sqrt{h_1^2 + 2ah_1}, \\
l_2 &= \sqrt{h_2^2 + 2ah_2}, \\
s_1 &= a \log \frac{l_1 + h_1 + a}{a}, \\
s_2 &= a \log \frac{l_2 + h_2 + a}{a},
\end{aligned}$$

and derived s from the sum of s_1 and s_2. Note that the definitions of s_1 and s_2 in [1] differ slightly, representing the horizontal spans on the h_1 and h_2 sides, respectively.

By deriving a using the method in [1], s can be indirectly computed from known h_1, h_2, l. However, [1] does not express s in a single formula and does not consider the case $s_1 s_2 > 0$.

4 Proof of Theorem 1

4.1 General Approach

First, compute s_1 and s_2, and obtain s from their difference. Then, eliminate the unknown parameters l_1, l_2, a.

4.2 Derivation of s

4.2.1 Derivation of s_1

From Equation (4),

$$\begin{aligned} \frac{l_1}{a} &= \sinh \frac{s_1}{a}, \\ \frac{s_1}{a} &= \operatorname{arsinh} \frac{l_1}{a}, \\ s_1 &= a \operatorname{arsinh} \frac{l_1}{a}, \\ &= a \log \left(\frac{l_1}{a} + \sqrt{\frac{l_1^2}{a^2} + 1} \right). \end{aligned} \tag{6}$$

For a, from Equations (2) and (4),

$$\begin{aligned} h_1^2 &= a^2 \cosh^2 \frac{s_1}{a} - 2a^2 \cosh \frac{s_1}{a} + a^2, \\ l_1^2 &= a^2 \sinh^2 \frac{s_1}{a}, \end{aligned}$$

and solving for a,

$$\begin{aligned} h_1^2 - l_1^2 &= a^2 \left(\cosh^2 \frac{s_1}{a} - \sinh^2 \frac{s_1}{a} \right) - 2a^2 \cosh \frac{s_1}{a} + a^2, \\ &= 2a^2 - 2a^2 \cosh \frac{s_1}{a}, \\ &= -2a \left(a \cosh \frac{s_1}{a} - a \right), \\ &= -2ah_1, \\ a &= \frac{l_1^2 - h_1^2}{2h_1}. \end{aligned} \tag{7}$$

Substituting Equation (7) into the logarithmic term of Equation (6),

$$
\begin{aligned}
s_1 &= a \log\left(\frac{2h_1 l_1}{l_1^2 - h_1^2} + \sqrt{\frac{4h_1^2 l_1^2 + (l_1^2 - h_1^2)^2}{(l_1^2 - h_1^2)^2}}\right), \\
&= a \log\left(\frac{2h_1 l_1}{l_1^2 - h_1^2} + \sqrt{\frac{(l_1^2 + h_1^2)^2}{(l_1^2 - h_1^2)^2}}\right), \\
&= a \log \frac{l_1^2 + 2h_1 l_1 + h_1^2}{l_1^2 - h_1^2}, \\
&= a \log \frac{(l_1 + h_1)^2}{(l_1 + h_1)(l_1 - h_1)}, \\
&= a \log \frac{l_1 + h_1}{l_1 - h_1}. \qquad (8)
\end{aligned}
$$

4.2.2 Derivation of s_2

Similarly,

$$
a = \frac{l_2^2 - h_2^2}{2h_2}, \qquad (9)
$$

$$
s_2 = -a \log \frac{l_2 + h_2}{l_2 - h_2}. \qquad (10)
$$

4.2.3 Derivation of s

Since $s = s_1 - s_2$, using Equations (8) and (10),

$$
\begin{aligned}
s &= a \log \frac{l_1 + h_1}{l_1 - h_1} + a \log \frac{l_2 + h_2}{l_2 - h_2}, \\
&= a \log \frac{(l_1 + h_1)(l_2 + h_2)}{(l_1 - h_1)(l_2 - h_2)}, \\
&= a \log \frac{\left(\frac{l_1}{h_1} + 1\right)\left(\frac{l_2}{h_2} + 1\right)}{\left(\frac{l_1}{h_1} - 1\right)\left(\frac{l_2}{h_2} - 1\right)}. \qquad (11)
\end{aligned}
$$

4.3 Elimination of l_1, l_2

Express l_1 in terms of l. From Equations (7) and (9),

$$
\frac{l_1^2 - h_1^2}{2h_1} = \frac{l_2^2 - h_2^2}{2h_2},
$$

and substituting $l_2 = l - l_1$,

$$
\begin{aligned}
\frac{l_1^2 - h_1^2}{2h_1} &= \frac{(l - l_1)^2 - h_2^2}{2h_2}, \\
&= \frac{l^2 - 2ll_1 + l_1^2 - h_2^2}{2h_2}.
\end{aligned}
$$

Multiplying both sides by $2h_1h_2$,

$$h_2l_1^2 - h_1^2h_2 = h_1l^2 - 2h_1ll_1 + h_1l_1^2 - h_1h_2^2,$$

and rearranging,

$$(h_1 - h_2)l_1^2 - 2h_1ll_1 + h_1l^2 + h_1^2h_2 - h_1h_2^2 = 0.$$

Solving for l_1,

$$\begin{aligned}
l_1 &= \frac{2h_1l \pm \sqrt{4h_1^2l^2 - 4(h_1 - h_2)(h_1l^2 + h_1^2h_2 - h_1h_2^2)}}{2(h_1 - h_2)}, \\
&= \frac{h_1l \pm \sqrt{h_1^2l^2 - (h_1 - h_2)(h_1l^2 + h_1^2h_2 - h_1h_2^2)}}{h_1 - h_2}, \\
&= \frac{h_1l \pm \sqrt{h_1h_2(l^2 - h_1^2 + 2h_1h_2 - h_2^2)}}{h_1 - h_2}, \\
&= \frac{h_1l \pm \sqrt{h_1h_2(l^2 - (h_1 - h_2)^2)}}{h_1 - h_2}.
\end{aligned}$$

Then, $l_2 = l - l_1$ gives

$$\begin{aligned}
l_2 &= l - \frac{h_1l \pm \sqrt{h_1h_2(l^2 - (h_1 - h_2)^2)}}{h_1 - h_2}, \\
&= \frac{(h_1 - h_2)l - h_1l \mp \sqrt{h_1h_2(l^2 - (h_1 - h_2)^2)}}{h_1 - h_2}, \\
&= \frac{-h_2l \mp \sqrt{h_1h_2(l^2 - (h_1 - h_2)^2)}}{h_1 - h_2}.
\end{aligned}$$

Since $h_1 > 0$, $h_2 > 0$, $l > |h_1 - h_2|$, we have $h_1l > 0$, $h_2l > 0$, $\sqrt{h_1h_2(l^2 - (h_1 - h_2)^2)} > 0$. If

$$\begin{aligned}
l_1 &= \frac{h_1l + \sqrt{h_1h_2(l^2 - (h_1 - h_2)^2)}}{h_1 - h_2}, \\
l_2 &= \frac{-h_2l - \sqrt{h_1h_2(l^2 - (h_1 - h_2)^2)}}{h_1 - h_2},
\end{aligned}$$

either l_1 or l_2 becomes negative, which is invalid. Thus,

$$l_1 = \frac{h_1l - \sqrt{h_1h_2(l^2 - (h_1 - h_2)^2)}}{h_1 - h_2}, \tag{12}$$

$$l_2 = \frac{-h_2l + \sqrt{h_1h_2(l^2 - (h_1 - h_2)^2)}}{h_1 - h_2}. \tag{13}$$

Dividing Equation (12) by h_1,

$$\frac{l_1}{h_1} = \frac{l - \sqrt{\frac{h_2}{h_1}(l^2 - (h_1 - h_2)^2)}}{h_1 - h_2},$$

yielding

$$\frac{l_1}{h_1}+1=\frac{l+h_1-h_2-\sqrt{\frac{h_2}{h_1}(l^2-(h_1-h_2)^2)}}{h_1-h_2}, \tag{14}$$

$$\frac{l_1}{h_1}-1=\frac{l-h_1+h_2-\sqrt{\frac{h_2}{h_1}(l^2-(h_1-h_2)^2)}}{h_1-h_2}. \tag{15}$$

Dividing Equation (13) by h_2,

$$\frac{l_2}{h_2}=\frac{-l+\sqrt{\frac{h_1}{h_2}(l^2-(h_1-h_2)^2)}}{h_1-h_2},$$

yielding

$$\frac{l_2}{h_2}+1=\frac{-l+h_1-h_2+\sqrt{\frac{h_1}{h_2}(l^2-(h_1-h_2)^2)}}{h_1-h_2}, \tag{16}$$

$$\frac{l_2}{h_2}-1=\frac{-l-h_1+h_2+\sqrt{\frac{h_1}{h_2}(l^2-(h_1-h_2)^2)}}{h_1-h_2}. \tag{17}$$

Substituting Equations (14), (15), (16), and (17) into Equation (11),

$$\begin{aligned} s &= a\log\frac{\left(l+h_1-h_2-\sqrt{\frac{h_2}{h_1}(l^2-(h_1-h_2)^2)}\right)\left(-l+h_1-h_2+\sqrt{\frac{h_1}{h_2}(l^2-(h_1-h_2)^2)}\right)}{\left(l-h_1+h_2-\sqrt{\frac{h_2}{h_1}(l^2-(h_1-h_2)^2)}\right)\left(-l-h_1+h_2+\sqrt{\frac{h_1}{h_2}(l^2-(h_1-h_2)^2)}\right)}, \\ &= a\log\frac{2(l^2-(h_1-h_2)^2)-(l+h_1-h_2)\sqrt{\frac{h_1}{h_2}(l^2-(h_1-h_2)^2)}-(l-h_1+h_2)\sqrt{\frac{h_2}{h_1}(l^2-(h_1-h_2)^2)}}{2(l^2-(h_1-h_2)^2)-(l-h_1+h_2)\sqrt{\frac{h_1}{h_2}(l^2-(h_1-h_2)^2)}-(l+h_1-h_2)\sqrt{\frac{h_2}{h_1}(l^2-(h_1-h_2)^2)}}. \end{aligned}$$

Multiplying the numerator and denominator of the logarithmic term by $-\sqrt{\frac{h_1h_2}{l^2-(h_1-h_2)^2}}$,

$$\begin{aligned} s &= a\log\frac{-2\sqrt{h_1h_2(l^2-(h_1-h_2)^2)}+h_1(l+h_1-h_2)+h_2(l-h_1+h_2)}{-2\sqrt{h_1h_2(l^2-(h_1-h_2)^2)}+h_1(l-h_1+h_2)+h_2(l+h_1-h_2)}, \\ &= a\log\frac{\left(\sqrt{h_1(l+h_1-h_2)}-\sqrt{h_2(l-h_1+h_2)}\right)^2}{\left(\sqrt{h_1(l-h_1+h_2)}-\sqrt{h_2(l+h_1-h_2)}\right)^2}, \\ &= 2a\log\frac{\sqrt{h_1(l+h_1-h_2)}-\sqrt{h_2(l-h_1+h_2)}}{\sqrt{h_1(l-h_1+h_2)}-\sqrt{h_2(l+h_1-h_2)}}. \end{aligned}$$

Multiplying the numerator and denominator by $\sqrt{h_1(l-h_1+h_2)}+\sqrt{h_2(l+h_1-h_2)}$,

$$s=2a\log\frac{(h_1-h_2)\sqrt{l^2-(h_1-h_2)^2}+2\sqrt{h_1h_2}(h_1-h_2)}{(h_1-h_2)l-(h_1-h_2)(h_1+h_2)}.$$

Dividing the numerator and denominator by h_1-h_2,

$$s=2a\log\frac{\sqrt{l^2-(h_1-h_2)^2}+2\sqrt{h_1h_2}}{l-h_1-h_2}. \tag{18}$$

4.4 Elimination of a

From Equation (7),

$$\begin{aligned} 2a &= \frac{l_1^2 - h_1^2}{h_1}, \\ &= \frac{(l_1+h_1)(l_1-h_1)}{h_1}, \\ &= h_1\left(\frac{l_1}{h_1}+1\right)\left(\frac{l_1}{h_1}-1\right). \end{aligned}$$

Substituting Equations (14) and (15),

$$\begin{aligned} 2a &= h_1\left(\frac{l+h_1-h_2-\sqrt{\frac{h_2}{h_1}(l^2-(h_1-h_2)^2)}}{h_1-h_2}\right)\left(\frac{l-h_1+h_2-\sqrt{\frac{h_2}{h_1}(l^2-(h_1-h_2)^2)}}{h_1-h_2}\right), \\ &= h_1\left(\frac{\left(1+\frac{h_2}{h_1}\right)(l^2-(h_1-h_2)^2)-2l\sqrt{\frac{h_2}{h_1}(l^2-(h_1-h_2)^2)}}{(h_1-h_2)^2}\right), \\ &= \frac{(h_1+h_2)(l^2-(h_1-h_2)^2)-2l\sqrt{h_1h_2(l^2-(h_1-h_2)^2)}}{(h_1-h_2)^2}. \end{aligned}$$

Dividing the numerator and denominator by $\sqrt{l^2-(h_1-h_2)^2}$,

$$2a = \frac{(h_1+h_2)\sqrt{l^2-(h_1-h_2)^2}-2l\sqrt{h_1h_2}}{\frac{(h_1-h_2)^2}{\sqrt{l^2-(h_1-h_2)^2}}}.$$

Multiplying the numerator and denominator by $(h_1+h_2)\sqrt{l^2-(h_1-h_2)^2} + 2l\sqrt{h_1h_2}$,

$$\begin{aligned} 2a &= \frac{(h_1+h_2)^2(l^2-(h_1-h_2)^2)-4l^2h_1h_2}{(h_1+h_2)(h_1-h_2)^2+\frac{2l\sqrt{h_1h_2}(h_1-h_2)^2}{\sqrt{l^2-(h_1-h_2)^2}}}, \\ &= \frac{(h_1-h_2)^2l^2-(h_1+h_2)^2(h_1-h_2)^2}{(h_1+h_2)(h_1-h_2)^2+\frac{2l\sqrt{h_1h_2}(h_1-h_2)^2}{\sqrt{l^2-(h_1-h_2)^2}}}. \end{aligned}$$

Dividing by $(h_1-h_2)^2$,

$$2a = \frac{l^2-(h_1+h_2)^2}{h_1+h_2+2l\sqrt{\frac{h_1h_2}{l^2-(h_1-h_2)^2}}}. \tag{19}$$

Substituting Equation (19) into Equation (18),

$$s = \frac{l^2-(h_1+h_2)^2}{h_1+h_2+2l\sqrt{\frac{h_1h_2}{l^2-(h_1-h_2)^2}}}\log\frac{\sqrt{l^2-(h_1-h_2)^2}+2\sqrt{h_1h_2}}{l-h_1-h_2}.$$

Thus, the theorem is proved. For $h_1 = h_2 = h$,

$$s = \frac{l^2-4h^2}{4h}\log\frac{l+2h}{l-2h}.$$

If either h_1 or h_2 is 0, say $h_2 = 0$ and $h_1 = h$,

$$s = \frac{l^2 - h^2}{2h} \log \frac{l+h}{l-h}.$$

5 Proof of Theorem 2

5.1 General Approach

Consider the alternative solutions for Equations (12) and (13) that were deemed invalid.

5.2 Elimination of l_1, l_2

Consider

$$l_1 = \frac{h_1 l + \sqrt{h_1 h_2 (l^2 - (h_1 - h_2)^2)}}{h_1 - h_2}, \tag{20}$$

$$l_2 = \frac{-h_2 l - \sqrt{h_1 h_2 (l^2 - (h_1 - h_2)^2)}}{h_1 - h_2}. \tag{21}$$

Dividing Equation (20) by h_1,

$$\frac{l_1}{h_1} = \frac{l + \sqrt{\frac{h_2}{h_1}(l^2 - (h_1 - h_2)^2)}}{h_1 - h_2},$$

yielding

$$\frac{l_1}{h_1} + 1 = \frac{l + h_1 - h_2 + \sqrt{\frac{h_2}{h_1}(l^2 - (h_1 - h_2)^2)}}{h_1 - h_2}, \tag{22}$$

$$\frac{l_1}{h_1} - 1 = \frac{l - h_1 + h_2 + \sqrt{\frac{h_2}{h_1}(l^2 - (h_1 - h_2)^2)}}{h_1 - h_2}. \tag{23}$$

Dividing Equation (21) by h_2,

$$\frac{l_2}{h_2} = \frac{-l - \sqrt{\frac{h_1}{h_2}(l^2 - (h_1 - h_2)^2)}}{h_1 - h_2},$$

yielding

$$\frac{l_2}{h_2} + 1 = \frac{-l + h_1 - h_2 - \sqrt{\frac{h_1}{h_2}(l^2 - (h_1 - h_2)^2)}}{h_1 - h_2}, \tag{24}$$

$$\frac{l_2}{h_2} - 1 = \frac{-l - h_1 + h_2 - \sqrt{\frac{h_1}{h_2}(l^2 - (h_1 - h_2)^2)}}{h_1 - h_2}. \tag{25}$$

Substituting Equations (22), (23), (24), and (25) into Equation (11),

$$s = a\log\frac{\left(l+h_1-h_2+\sqrt{\frac{h_2}{h_1}(l^2-(h_1-h_2)^2)}\right)\left(-l+h_1-h_2-\sqrt{\frac{h_1}{h_2}(l^2-(h_1-h_2)^2)}\right)}{\left(l-h_1+h_2+\sqrt{\frac{h_2}{h_1}(l^2-(h_1-h_2)^2)}\right)\left(-l-h_1+h_2-\sqrt{\frac{h_1}{h_2}(l^2-(h_1-h_2)^2)}\right)},$$

$$= a\log\frac{2(l^2-(h_1-h_2)^2)+(l+h_1-h_2)\sqrt{\frac{h_1}{h_2}(l^2-(h_1-h_2)^2)}+(l-h_1+h_2)\sqrt{\frac{h_2}{h_1}(l^2-(h_1-h_2)^2)}}{2(l^2-(h_1-h_2)^2)+(l-h_1+h_2)\sqrt{\frac{h_1}{h_2}(l^2-(h_1-h_2)^2)}+(l+h_1-h_2)\sqrt{\frac{h_2}{h_1}(l^2-(h_1-h_2)^2)}}.$$

Multiplying the numerator and denominator by $\sqrt{\frac{h_1h_2}{l^2-(h_1-h_2)^2}}$,

$$s = a\log\frac{2\sqrt{h_1h_2(l^2-(h_1-h_2)^2)}+h_1(l+h_1-h_2)+h_2(l-h_1+h_2)}{2\sqrt{h_1h_2(l^2-(h_1-h_2)^2)}+h_1(l-h_1+h_2)+h_2(l+h_1-h_2)},$$

$$= a\log\frac{\left(\sqrt{h_1(l+h_1-h_2)}+\sqrt{h_2(l-h_1+h_2)}\right)^2}{\left(\sqrt{h_1(l-h_1+h_2)}+\sqrt{h_2(l+h_1-h_2)}\right)^2},$$

$$= 2a\log\frac{\sqrt{h_1(l+h_1-h_2)}+\sqrt{h_2(l-h_1+h_2)}}{\sqrt{h_1(l-h_1+h_2)}+\sqrt{h_2(l+h_1-h_2)}}.$$

Multiplying by $\sqrt{h_1(l-h_1+h_2)}-\sqrt{h_2(l+h_1-h_2)}$,

$$s = 2a\log\frac{(h_1-h_2)\sqrt{l^2-(h_1-h_2)^2}-2\sqrt{h_1h_2}(h_1-h_2)}{(h_1-h_2)l-(h_1-h_2)(h_1+h_2)}.$$

Dividing by h_1-h_2,

$$s = 2a\log\frac{\sqrt{l^2-(h_1-h_2)^2}-2\sqrt{h_1h_2}}{l-h_1-h_2}. \tag{26}$$

5.3 Elimination of a

From Equation (7),

$$2a = \frac{l_1^2-h_1^2}{h_1},$$

$$= \frac{(l_1+h_1)(l_1-h_1)}{h_1},$$

$$= h_1\left(\frac{l_1}{h_1}+1\right)\left(\frac{l_1}{h_1}-1\right).$$

Substituting Equations (22) and (23),

$$2a = h_1\left(\frac{l+h_1-h_2+\sqrt{\frac{h_2}{h_1}(l^2-(h_1-h_2)^2)}}{h_1-h_2}\right)\left(\frac{l-h_1+h_2+\sqrt{\frac{h_2}{h_1}(l^2-(h_1-h_2)^2)}}{h_1-h_2}\right),$$

$$= h_1\left(\frac{\left(1+\frac{h_2}{h_1}\right)(l^2-(h_1-h_2)^2)+2l\sqrt{\frac{h_2}{h_1}(l^2-(h_1-h_2)^2)}}{(h_1-h_2)^2}\right),$$

$$= \frac{(h_1+h_2)(l^2-(h_1-h_2)^2)+2l\sqrt{h_1h_2(l^2-(h_1-h_2)^2)}}{(h_1-h_2)^2}.$$

Dividing by $\sqrt{l^2-(h_1-h_2)^2}$,

$$2a = \frac{(h_1+h_2)\sqrt{l^2-(h_1-h_2)^2}+2l\sqrt{h_1h_2}}{\frac{(h_1-h_2)^2}{\sqrt{l^2-(h_1-h_2)^2}}}.$$

Multiplying by $(h_1+h_2)\sqrt{l^2-(h_1-h_2)^2}-2l\sqrt{h_1h_2}$,

$$\begin{aligned}2a &= \frac{(h_1+h_2)^2(l^2-(h_1-h_2)^2)-4l^2h_1h_2}{(h_1+h_2)(h_1-h_2)^2-\frac{2l\sqrt{h_1h_2}(h_1-h_2)^2}{\sqrt{l^2-(h_1-h_2)^2}}}, \\ &= \frac{(h_1-h_2)^2l^2-(h_1+h_2)^2(h_1-h_2)^2}{(h_1+h_2)(h_1-h_2)^2-\frac{2l\sqrt{h_1h_2}(h_1-h_2)^2}{\sqrt{l^2-(h_1-h_2)^2}}}.\end{aligned}$$

Dividing by $(h_1-h_2)^2$,

$$2a = \frac{l^2-(h_1+h_2)^2}{h_1+h_2-2l\sqrt{\frac{h_1h_2}{l^2-(h_1-h_2)^2}}}. \tag{27}$$

Substituting Equation (27) into Equation (26),

$$s = \frac{l^2-(h_1+h_2)^2}{h_1+h_2-2l\sqrt{\frac{h_1h_2}{l^2-(h_1-h_2)^2}}}\log\frac{\sqrt{l^2-(h_1-h_2)^2}-2\sqrt{h_1h_2}}{l-h_1-h_2}.$$

For $l=h_1+h_2$, the expression becomes indeterminate, and

$$\begin{aligned}s &= \lim_{l\to h_1+h_2}\frac{l^2-(h_1+h_2)^2}{h_1+h_2-2l\sqrt{\frac{h_1h_2}{l^2-(h_1-h_2)^2}}}\log\frac{\sqrt{l^2-(h_1-h_2)^2}-2\sqrt{h_1h_2}}{l-h_1-h_2}, \\ &= \frac{8h_1h_2(h_1+h_2)}{(h_1-h_2)^2}\log\frac{h_1+h_2}{2\sqrt{h_1h_2}}.\end{aligned}$$

Thus, the theorem is proved. For $h_1=h_2=h$,

$$s = \frac{l^2-4h^2}{4h}\log\frac{l+2h}{l-2h}.$$

If either h_1 or h_2 is 0, say $h_2=0$ and $h_1=h$,

$$s = \frac{l^2-h^2}{2h}\log\frac{l+h}{l-h}.$$

Due to the difficulty in determining the minimum point from the curve ’ s appearance, the practical applicability of this theorem is limited.

6 Proof of Theorems 3 and 4

6.1 General Approach

Define

$$\begin{aligned} l + h_1 + h_2 &= A, \\ l - h_1 - h_2 &= B, \\ \sqrt{l^2 - (h_1 - h_2)^2} + 2\sqrt{h_1 h_2} &= C, \\ \sqrt{l^2 - (h_1 - h_2)^2} - 2\sqrt{h_1 h_2} &= D, \end{aligned}$$

to simplify the expressions. Here,

$$AB = CD = l^2 - (h_1 + h_2)^2,$$

and

$$\begin{aligned} \frac{A}{C} &= \frac{B}{D}, \\ \frac{C}{A} &= \frac{B}{D}, \\ \frac{A}{D} &= \frac{C}{B}, \\ \frac{D}{A} &= \frac{B}{C}. \end{aligned}$$

Also,

$$\begin{aligned} l &= \frac{A + B}{2}, \\ h_1 + h_2 &= \frac{A - B}{2}, \\ \sqrt{l^2 - (h_1 - h_2)^2} &= \frac{C + D}{2}, \\ 2\sqrt{h_1 h_2} &= \frac{C - D}{2}. \end{aligned}$$

6.2 Proof of Theorem 3

Expressing

$$s = \frac{l^2 - (h_1 + h_2)^2}{h_1 + h_2 + 2l\sqrt{\frac{h_1 h_2}{l^2 - (h_1 - h_2)^2}}} \log \frac{\sqrt{l^2 - (h_1 - h_2)^2} + 2\sqrt{h_1 h_2}}{l - h_1 - h_2},$$

in terms of A, B, C,

$$\begin{aligned}
s &= \frac{AB}{\frac{A-B}{2} + \frac{A+B}{2}\frac{C-D}{C+D}} \log \frac{C}{B}, \\
&= \frac{2AB(C+D)}{(A-B)(C+D) + (A+B)(C-D)} \log \frac{C}{B}, \\
&= \frac{2AB(C+D)}{AC+AD-BC-BD+AC-AD+BC-BD} \log \frac{C}{B}, \\
&= \frac{2AB(C+D)}{2AC-2BD} \log \frac{C}{B}, \\
&= \frac{C+D}{\frac{C}{B} - \frac{D}{A}} \log \frac{C}{B}, \\
&= \frac{C+D}{\frac{C}{B} - \frac{B}{C}} \log \frac{C}{B}.
\end{aligned}$$

Letting $\log \frac{C}{B} = \log \frac{\sqrt{l^2-(h_1-h_2)^2}+2\sqrt{h_1h_2}}{l-h_1-h_2} = X$,

$$\begin{aligned}
s &= \frac{C+D}{2\sinh X} \cdot X, \\
&= \frac{2\sqrt{l^2-(h_1-h_2)^2}}{2\sinh X} \cdot X, \\
&= \frac{X}{\sinh X}\sqrt{l^2-(h_1-h_2)^2}.
\end{aligned}$$

Thus, the theorem is proved.

6.3 Proof of Theorem 4

Expressing

$$s = \frac{l^2-(h_1+h_2)^2}{h_1+h_2-2l\sqrt{\frac{h_1h_2}{l^2-(h_1-h_2)^2}}} \log \frac{\sqrt{l^2-(h_1-h_2)^2}-2\sqrt{h_1h_2}}{l-h_1-h_2},$$

in terms of A, B, C, D,

$$\begin{aligned}
s &= \frac{AB}{\frac{A-B}{2} - \frac{A+B}{2}\frac{C-D}{C+D}} \log \frac{D}{B}, \\
&= \frac{2AB(C+D)}{(A-B)(C+D) - (A+B)(C-D)} \log \frac{A}{C}, \\
&= \frac{2AB(C+D)}{AC+AD-BC-BD-AC+AD-BC+BD} \log \frac{A}{C}, \\
&= \frac{2AB(C+D)}{2AD-2BC} \log \frac{A}{C}, \\
&= \frac{C+D}{\frac{D}{B} - \frac{C}{A}} \log \frac{A}{C}, \\
&= \frac{C+D}{\frac{A}{C} - \frac{C}{A}} \log \frac{A}{C}.
\end{aligned}$$

Letting $\log \frac{A}{C} = \log \frac{l+h_1+h_2}{\sqrt{l^2-(h_1-h_2)^2+2\sqrt{h_1 h_2}}} = X$,

$$
\begin{aligned}
s &= \frac{C+D}{2\sinh X} \cdot X, \\
&= \frac{2\sqrt{l^2-(h_1-h_2)^2}}{2\sinh X} \cdot X, \\
&= \frac{X}{\sinh X}\sqrt{l^2-(h_1-h_2)^2}.
\end{aligned}
$$

Thus, the theorem is proved.

7 Conclusion

This paper derives formulas for the horizontal span s of a catenary curve using the endpoint heights h_1, h_2 and the curve length l. We distinguish between cases where the minimum point lies between the endpoints and where it does not, presenting corresponding formulas in Theorems 1 and 2. These formulas involve complex logarithmic expressions, while Theorems 3 and 4 provide more concise representations using hyperbolic functions. These results enable explicit computation of the horizontal span, previously unavailable as a single formula, and are applicable to engineering designs such as suspension bridges and power lines. However, the practicality of Theorem 2 is limited when identifying the minimum point ' s position is challenging. Future work includes developing methods to easily determine the minimum point ' s position and simplifying formulas under specific boundary conditions. Additionally, applying these results to real-world engineering problems to verify their effectiveness is anticipated.

References

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