Non-Differential Geometry: Mathematical Tools Abandoning the Differential Framework

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Abstract

This paper proposes and systematically elaborates a novel geometric framework—non-differential geometry—whose core lies in entirely abandoning the reliance on smoothness (C^1 or higher continuity) required by traditional differential geometry, demanding only C^0 continuity for geometric objects. By introducing new mathematical tools based on limits and infinite series, non-differential geometry overcomes the smoothness constraints in calculating geometric quantities such as curvature. Furthermore, this paper constructs a unique "integration tool" (distinct from classical integration theory) specific to non-differential geometry, providing a novel approach for analyzing geometric objects. Current research focuses on Euclidean space, but the theoretical framework itself is not confined to any specific spatial structure and is applicable across low- to high-dimensional spaces. Non-differential geometry completely resolves the contradiction between continuity and differentiability and offers potential support for theoretical innovations in fields such as physics.

Keywords: Non-differential geometry; C^0 continuity; curvature calculation; limit methods; geometric foundations

1 Introduction

As a cornerstone of modern mathematics, differential geometry has long relied on smoothness assumptions (e.g., differentiability and derivability) for geometric objects, significantly limiting its applicability and theoretical universality. This paper proposes a groundbreaking geometric framework—non-differential geometry—which requires only C^0 continuity for geometric objects, thereby entirely eliminating the dependence on smoothness. By developing new mathematical tools based on limits and infinite series, non-differential geometry not only enables non-smooth calculations of geometric quantities like curvature but also establishes a distinctive "integration tool," opening new avenues for geometric analysis.

Current research is centered on Euclidean space, yet the theoretical framework of non-differential geometry is not inherently tied to any specific spatial structure, exhibiting potential for generalization to broader spaces. Its key advantage lies in its universality and generality: from low- to high-dimensional spaces, geometric objects can be analyzed without any differentiability conditions, resolving the inherent conflict between continuity and differentiability in traditional geometry. Although still in its pioneering stages, non-differential geometry demonstrates the potential to reshape the foundational framework of geometry and provides new theoretical tools for disciplines such as physics. This paper systematically presents its fundamental concepts and preliminary results, laying the groundwork for future research.

2 Construction of Core Mathematical Tools

2.1 Development of mathematical tools for non-differential geometry

In the one-dimensional coordinate system on the x-axis of Euclidean space, let O denote the origin, with point A located on the positive half-axis satisfying $|OA| = 1$, as shown in Figure 2.1.

Figure 2.1: The line segment OA lies on the x-axis

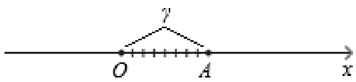

Figure 2.2: Partition the length of line segment OA into γ equal parts

As shown in Figure 2.2, the length of line segment OA is divided into γ equal parts $\gamma(\gamma \in \mathbb{N}^{+})$, with division points sequentially marked from O to A as $i = -\gamma, -\gamma+1, \cdots, -2, -1, 1, 2, \cdots, \gamma-1, \gamma$, Since the essence of divisibility is equipartition, we now establish the following formula:

$$\begin{aligned} |OA| &= 1 \\ &= \frac{1}{\gamma} \times \gamma \\ &= \frac{1}{\gamma} \times (\underbrace{1 + 1 + \cdots + 1}_{\gamma \text{ ones}}) \\ &= \frac{1}{\gamma} \sum_{i=1}^{\gamma} 1\,. \end{aligned} \tag{2.1}$$

By introducing the limit concept and denoting the coordinate of point A on the x-axis as x_A, we refine Equation (2.1) as:

$$x_A = \lim_{\gamma \to \infty} \frac{1}{\gamma} \sum_{i=1}^{\gamma} 1\,. \tag{2.2}$$

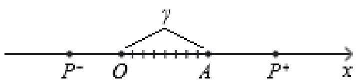

Figure 2.3: Point P^{-} lies on the negative x-axis, P^{+} on the positive x-axis

As shown in Figure 2.3, let point P^{+} on the positive x-axis have coordinate $x_{P^{+}}$ $(x_{P^{+}} > 0)$, and point P^{-} on the negative x-axis have coordinate $x_{P^{-}}$ $(x_{P^{-}} < 0)$. Along the positive x-direction, define:

$$\frac{x_{P^+}}{x_A} = \tau_1 \,. \tag{2.3}$$

By transforming Equation (2.3), we obtain:

$$\begin{aligned} x_{P^+} &= x_A \tau_1 && (2.4) \\ &= 1 \times \tau_1 \\ &= \frac{1}{\gamma} \times (\tau_1 \gamma) \\ &= \lim_{\gamma \to \infty} \frac{1}{\gamma} \sum_{i=1}^{\lfloor \tau_1 \gamma \rfloor} 1 \,. && (2.5) \end{aligned}$$

From Equation (2.4), τ_1 is a scaling parameter where $\tau_1 = 0$ is mathematically admissible, corresponding to point P^+ being located at the coordinate origin. We now examine the mathematical significance of Equation (2.5) for the case $\tau_1 = 0$.

When $\tau_1 = 0$, let RHS and LHS denote the right-hand side and left-hand side of Equation (2.5) respectively. The geometric interpretation is then:

$$\begin{aligned} RHS &= \lim_{\gamma \to \infty} \frac{1}{\gamma} \sum_{i=1}^{0} 1 \,, && (2.6) \\ LHS &= 0 \,. \end{aligned}$$

When considered in isolation, Equation (2.6) is mathematically meaningless but corresponds to a zero value geometrically. To address this problem, we introduce a nonzero real variable μ, from which it follows that:

$$\lim_{\substack{\mu \to 0 \\ \gamma \to \infty}} \frac{1}{\gamma} \sum_{i=1}^{\lfloor \mu \gamma \rfloor} 1 = 0 \,. \tag{2.7}$$

We now analyze the case where point P^- lies on the negative x-axis. Taking the orientation of line segment OA (not vector $\overrightarrow{OA}$) as reference Similarly, define:

$$\frac{x_{P^-}}{x_A} = \tau_2 \,. \tag{2.8}$$

Transform Equation (2.8) into:

$$\begin{aligned} x_{P^-} &= x_A \tau_2 \\ &= \frac{1}{\gamma} \times (\tau_2 \gamma) \\ &= -\lim_{\gamma \to \infty} \frac{1}{\gamma} \sum_{i=-1}^{\lceil \tau_2 \gamma \rceil} 1 \,. && (2.9) \end{aligned}$$

In this paper, series summations are by default ordered by ascending absolute values, as demonstrated in Equation (2.9).

To describe the position of any point on the x-axis, we introduce a scaling coefficient τ. For this purpose, it is necessary to unify Equations (2.5), (2.7) and (2.8) and define the following formula:

Let $\tau \in \mathbb{R}$, define:

$$\llcorner v \lrcorner = \begin{cases} \lfloor v \rfloor \,, & \text{if } v > 0 \,, \\ 0 \,, & \text{if } v = 0 \,, \\ \lceil v \rceil \,, & \text{if } v < 0 \,. \end{cases}$$

We further introduce a nonzero real variable λ.

$$\iota \langle \tau \rangle = \lim_{\lambda \to 0} \frac{\tau + \lambda}{|\tau| + \lambda} \,.$$

Hence, we obtain:

$$\begin{aligned} x &= \iota \langle \tau \rangle \lim_{\substack{\mu \to \tau \\ \gamma \to \infty}} \frac{1}{\gamma} \sum_{i = \iota \langle \tau \rangle}^{\llcorner \mu\gamma \lrcorner} 1 \\ &= \iota \langle \tau \rangle \lim_{\substack{\mu \to \tau \\ \gamma \to \infty}} \frac{1}{\gamma} \sum_{i = \iota \langle \tau \rangle}^{\llcorner \mu\gamma \lrcorner} \cos 0 \,. \end{aligned} \tag{2.10}$$

At this stage, the embryonic form of the mathematical tool has emerged - though currently it merely characterizes the x-axis coordinates. As for the intentional design of Equation (2.10)'s specific form, its geometric significance will be revealed in the next chapter's curve study.

Regarding the rounding operation rule in Equation (2.10): when the object to be rounded is a known real number (e.g., 4.52), its integer part is taken directly (i.e., $\llcorner 4.52 \lrcorner = 4$); when the object is an unspecified variable v, $\llcorner v \lrcorner$ represents the definitional expression for performing the rounding operation on variable v, in which case its value must be determined through the specific formula.

Regarding the computational method of the truncation function, trigonometric function expansion may be employed for processing: By utilizing the characteristic periodicity of trigonometric functions, after adjusting the period parameter to integer units, the rounding operation is achieved by subtracting the fractional part from the original numerical value. The precise formulation is presented below:

Let $\delta \in \mathbb{R}$. We now define:

$$\tau\gamma = u \,.$$

We obtain:

$$\lfloor u \rfloor = \lim_{\delta \to u^+} \left\langle \delta - \frac{1}{\pi} \arctan \left\{ \tan \left[\pi \left(\delta - \frac{1}{2} \right) \right] \right\} - \frac{1}{2} \right\rangle,$$

$$\lceil u \rceil = \lim_{\delta \to u^-} \left\langle \delta - \frac{1}{\pi} \arctan \left\{ \tan \left[\pi \left(\delta - \frac{1}{2} \right) \right] \right\} + \frac{1}{2} \right\rangle.$$

2.2 Temporal and spatial parametric representations of planar continuous curves

We now extend the framework to two dimensions by introducing an infinitely extended curve α, which is only required to satisfy C^0 continuity without any assumptions of smoothness, differentiability, or weak differentiability. On curve α, select an arbitrary point B as the origin, then designate one side of B as the positive direction along the curve and the opposite side as the negative direction. Let point B^+ be the point at curve length 1 from B in the positive direction, and point B^- the point at curve length 1 from B in the negative direction. This geometric configuration is illustrated in Figure 2.4.

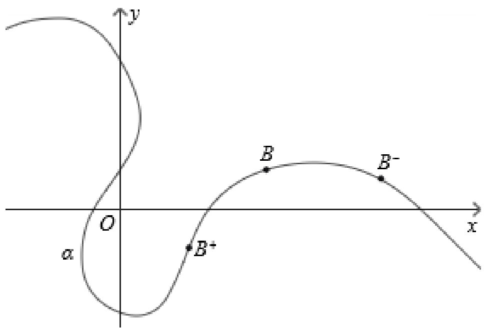

Figure 2.4: Curve α with starting point B

Using the curve segment BB^+ as the baseline, insert $\gamma - 1$ equidistant points along BB^+, dividing it into γ curve segments of equal length. The remaining part of curve α is proportionally partitioned accordingly. Connecting the endpoints of these partitioned segments forms line segments of length ℓ_i, where the direction toward BB^+ is defined as the terminal end and the opposite direction as the initial end. The segments make an angle θ_i with the x-axis in the counterclockwise direction, $\theta_i \in [0, 2\pi)$, as illustrated in Figure 2.5 (for clarity, the diagram shows the case $\gamma = 2$).

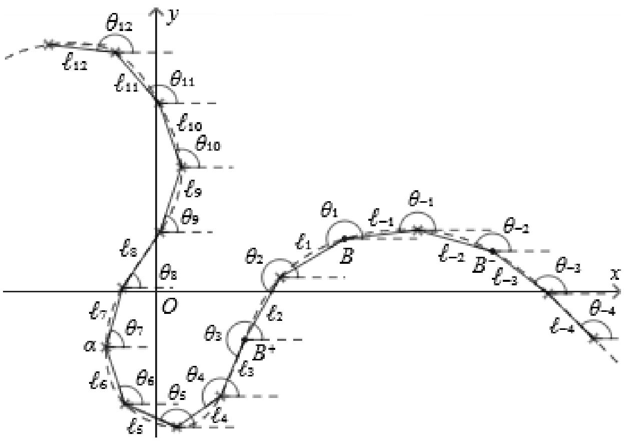

Figure 2.5: Partition the curve α into equal segments

Therefore, we obtain:

$$\lim_{\gamma\to\infty} \frac{\frac{1}{\gamma}}{\ell_i} = 1\,.$$

Let τ be the proportionality coefficient, and let the coordinates of point B be (x_0, y_0). Then, the expression for curve α can be derived as follows:

$$\begin{cases} x = \iota\langle\tau\rangle \lim\limits_{\substack{\mu\to\tau\\ \gamma\to\infty}} \dfrac{1}{\gamma} \sum\limits_{i=\iota\langle\tau\rangle}^{\lfloor\mu\gamma\rfloor} \cos\theta_i + x_0\,, \\ y = \iota\langle\tau\rangle \lim\limits_{\substack{\mu\to\tau\\ \gamma\to\infty}} \dfrac{1}{\gamma} \sum\limits_{i=\iota\langle\tau\rangle}^{\lfloor\mu\gamma\rfloor} \sin\theta_i + y_0\,. \end{cases} \tag{2.11}$$

Since Formula (2.11) inherently lacks any capability for differential or derivative operations, it achieves perfect representation of planar continuous curves.

In Equation (2.11), if θ_i is identically equal to a constant θ_0, then it represents a straight line, namely:

$$\begin{cases} x = \tau\cos\theta_0 + x_0\,, \\ y = \tau\sin\theta_0 + y_0\,. \end{cases}$$

For a more intuitive understanding of non-linear curves, we introduce an example - curve α_1 - where we substitute n for i, as shown below:

$$\begin{cases} x = \iota\langle\tau\rangle \lim\limits_{\substack{\mu\to\tau\\ \gamma\to\infty}} \dfrac{1}{\gamma} \sum\limits_{n=\iota\langle\tau\rangle}^{\lfloor\mu\gamma\rfloor} \left(\left(\dfrac{2\left(\left(\left(\frac{27n}{8\gamma}+1\right)^{\frac{2}{3}}-1\right)^{\frac{3}{2}} - \left(\left(\frac{27(n-1)}{8\gamma}+1\right)^{\frac{2}{3}}-1\right)^{\frac{3}{2}}\right)}{3\left(\left(\frac{27n}{8\gamma}+1\right)^{\frac{2}{3}} - \left(\left(\frac{27(n-1)}{8\gamma}+1\right)^{\frac{2}{3}}\right)\right)}\right)^2 + 1\right)^{-\frac{1}{2}}\,, \\ y = \iota\langle\tau\rangle \lim\limits_{\substack{\mu\to\tau\\ \gamma\to\infty}} \dfrac{1}{\gamma} \sum\limits_{n=\iota\langle\tau\rangle}^{\lfloor\mu\gamma\rfloor} \left(\left(\dfrac{3\left(\left(\frac{27n}{8\gamma}+1\right)^{\frac{2}{3}} - \left(\left(\frac{27(n-1)}{8\gamma}+1\right)^{\frac{2}{3}}\right)\right)}{2\left(\left(\left(\frac{27n}{8\gamma}+1\right)^{\frac{2}{3}}-1\right)^{\frac{3}{2}} - \left(\left(\frac{27(n-1)}{8\gamma}+1\right)^{\frac{2}{3}}-1\right)^{\frac{3}{2}}\right)}\right)^2 + 1\right)^{-\frac{1}{2}}\,. \end{cases} \tag{2.12}$$

Given the series terms of curve α, since γ and n cannot be arbitrarily combined, we introduce the symbol '$\wr$' to represent their constrained composition into curve elements. The series terms are respectively denoted by $\Psi_{\mathrm{c}}\langle\gamma \wr n\rangle$ and $\Psi_{\mathrm{s}}\langle\gamma \wr n\rangle$ respectively, the curve α can be expressed as:

$$
\begin{cases}
x = \iota\left\langle \tau \right\rangle \lim\limits_{\substack{\mu \to \tau \\ \gamma \to \infty}} \dfrac{1}{\gamma} \sum\limits_{n=\iota\langle \tau \rangle}^{\lfloor \mu\gamma \rfloor} \Psi_{\mathrm{c}}\left\langle \gamma \wr n \right\rangle + x_0\,, \\
y = \iota\left\langle \tau \right\rangle \lim\limits_{\substack{\mu \to \tau \\ \gamma \to \infty}} \dfrac{1}{\gamma} \sum\limits_{n=\iota\langle \tau \rangle}^{\lfloor \mu\gamma \rfloor} \Psi_{\mathrm{s}}\left\langle \gamma \wr n \right\rangle + y_0\,.
\end{cases}
\tag{2.13}
$$

Equation (2.13) represents the novel mathematical tool we have developed, although it is applicable only to planar continuous curves. Here, $|\tau|$ denotes the length of the curve segment from the initial point to the corresponding point, indicating that Equation (2.13) gives the spatial parametric representation of curve α.

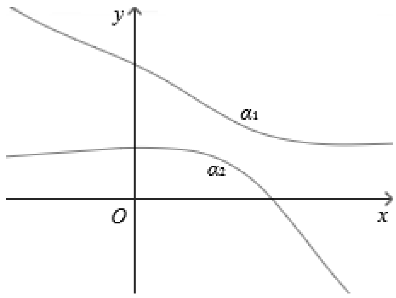

Figure 2.6: Curve α_1 and curve α_2

We now present the temporal parametric representation of curve α. Consider two infinitely extending continuous curves α_1 and α_2, both satisfying the condition of having exactly one intersection point with any line parallel to the y-axis (as shown in Figure 2.6). Following the methodology from the previous chapter, we equidistantly partition the point at unit distance from the origin on the positive x-axis, thereby obtaining the analytical expressions for curves α_1 and α_2.

$$
y = \iota\left\langle \tau \right\rangle \lim_{\substack{\mu \to \tau \\ \gamma \to \infty}} \frac{1}{\gamma} \sum_{i=\iota\langle \tau \rangle}^{\lfloor \mu\gamma \rfloor} \tan\theta_{1,i} + y'_1\,, \tag{2.14}
$$

$$
y = \iota\left\langle \tau \right\rangle \lim_{\substack{\mu \to \tau \\ \gamma \to \infty}} \frac{1}{\gamma} \sum_{i=\iota\langle \tau \rangle}^{\lfloor \mu\gamma \rfloor} \tan\theta_{2,i} + y'_2\,. \tag{2.15}
$$

Equation (2.14) provides the analytic expression for curve α_1, where y'_1 denotes the y-coordinate of its intersection with the y-axis; Equation (2.15) gives the analytic expression for curve α_2, with y'_2 representing the corresponding y-intercept coordinate.

Given that curve α is composed of curves α_1 and α_2, and provided its general series term, replacing the subscript i with n yields the expression for curve α as:

$$
\begin{cases}
x = \iota\left\langle \tau \right\rangle \lim\limits_{\substack{\mu \to \tau \\ \gamma \to \infty}} \dfrac{1}{\gamma} \sum\limits_{n=\iota\langle \tau \rangle}^{\lfloor \mu\gamma \rfloor} \Psi_1\left\langle \gamma \wr n \right\rangle + x_1\,, \\
y = \iota\left\langle \tau \right\rangle \lim\limits_{\substack{\mu \to \tau \\ \gamma \to \infty}} \dfrac{1}{\gamma} \sum\limits_{n=\iota\langle \tau \rangle}^{\lfloor \mu\gamma \rfloor} \Psi_2\left\langle \gamma \wr n \right\rangle + y_1\,.
\end{cases}
\tag{2.16}
$$

Equation (2.16) gives the temporal parametric expression of curve α, where (x_1, y_1) denotes the coordinates corresponding to initial time $\tau = 0$. Here τ represents relative time with respect to the coordinate origin and may take negative values.

2.3 Vectorized representation of planar continuous curves

Curve α possesses a spatiotemporal parametric representation, hence the vectorized representation of its continuous curve carries significant geometric implications.

We define:

$$\begin{cases} \Phi_{\mathrm{c}} \langle \tau \rangle = \iota \langle \tau \rangle \lim\limits_{\substack{\mu \to \tau \\ \gamma \to \infty}} \dfrac{1}{\gamma} \sum\limits_{n=\iota\langle \tau \rangle}^{\lfloor \mu\gamma \rfloor} \Psi_{\mathrm{c}} \langle \gamma \wr n \rangle + x_0 \,, \\ \Phi_{\mathrm{s}} \langle \tau \rangle = \iota \langle \tau \rangle \lim\limits_{\substack{\mu \to \tau \\ \gamma \to \infty}} \dfrac{1}{\gamma} \sum\limits_{n=\iota\langle \tau \rangle}^{\lfloor \mu\gamma \rfloor} \Psi_{\mathrm{s}} \langle \gamma \wr n \rangle + y_0 \,. \end{cases}$$

$$\begin{cases} \Phi_1 \langle \tau \rangle = \iota \langle \tau \rangle \lim\limits_{\substack{\mu \to \tau \\ \gamma \to \infty}} \dfrac{1}{\gamma} \sum\limits_{n=\iota\langle \tau \rangle}^{\lfloor \mu\gamma \rfloor} \Psi_1 \langle \gamma \wr n \rangle + x_1 \,, \\ \Phi_2 \langle \tau \rangle = \iota \langle \tau \rangle \lim\limits_{\substack{\mu \to \tau \\ \gamma \to \infty}} \dfrac{1}{\gamma} \sum\limits_{n=\iota\langle \tau \rangle}^{\lfloor \mu\gamma \rfloor} \Psi_2 \langle \gamma \wr n \rangle + y_1 \,. \end{cases}$$

In the k-dimensional Euclidean space with a given coordinate system $\{O; \vec{e}_1, \vec{e}_2, \cdots, \vec{e}_k\}$, the curve can be represented by the vector $\vec{\xi} \langle \tau \rangle$ and $\vec{\zeta} \langle \tau \rangle$, namely:

$$\begin{aligned} \vec{\xi} \langle \tau \rangle &= \Phi_{\mathrm{c}} \langle \tau \rangle \, \vec{e}_1 + \Phi_{\mathrm{s}} \langle \tau \rangle \, \vec{e}_2 \,, \\ \vec{\zeta} \langle \tau \rangle &= \Phi_1 \langle \tau \rangle \, \vec{e}_1 + \Phi_2 \langle \tau \rangle \, \vec{e}_2 \,. \end{aligned}$$

To ensure the meaning of τ is explicitly defined, let $j = 1, 2, ..., k$, and strictly require that $\vec{e}_j$ forms an standard orthonormal basis. For geometric objects in non-Euclidean spaces, the specification of an standard orthonormal basis is not required.

2.4 Geometric analysis of planar continuous curves

Since this paper does not involve the calculation of differentials or derivatives, it is necessary to redefine concepts such as tangents and curvature. Given that the curve may be non-smooth and traditional tangent lines might not exist, the focus here is primarily on one-sided tangents.

Taking the spatial parametric representation of curve α as an example, let $D(\Phi_{\mathrm{c}} \langle \tau_0 \rangle, \Phi_{\mathrm{s}} \langle \tau_0 \rangle)$ be a fixed point on the curve, and Q be a moving point on one side of D. When Q approaches D unilaterally along the curve indefinitely, if the line DQ connecting D and Q has a limiting position, then this limiting line is called the unilateral tangent to the curve α at point D (see Figure 2.7).

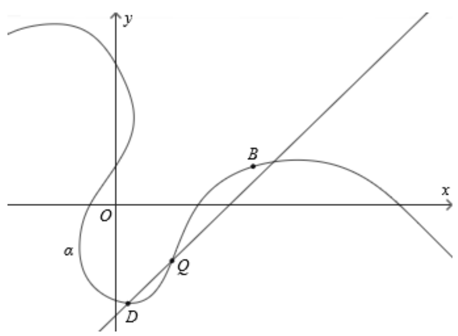

Figure 2.7: Definition of a one-sided tangent line

As the point Q approaches point D, let ω_0 be the angle between line QD and the positive x-axis measured counterclockwise. We now derive the equation for the one-sided tangent line:

$$\begin{cases} x = \tau \cos \omega_0 + \Phi_{\mathrm{c}} \langle \tau_0 \rangle \ , \\ y = \tau \sin \omega_0 + \Phi_{\mathrm{s}} \langle \tau_0 \rangle \ . \end{cases}$$

Whereas:

$$\cos \omega_0 = \lim_{\substack{\mu' \to \tau \\ \mu \to \tau_0 \\ Q \to D}} \frac{\Phi_{\mathrm{c}} \langle \mu' \rangle - \Phi_{\mathrm{c}} \langle \mu \rangle}{\mu' - \mu} \, ,$$

$$\sin \omega_0 = \lim_{\substack{\mu' \to \tau \\ \mu \to \tau_0 \\ Q \to D}} \frac{\Phi_{\mathrm{s}} \langle \mu' \rangle - \Phi_{\mathrm{s}} \langle \mu \rangle}{\mu' - \mu} \, .$$

Let:

$$\lim_{\mu' \to \tau} \mu' = \lim_{\mu \to \tau_0} \mu + \Delta\tau \, .$$

Thus:

$$\cos \omega_0 = \lim_{\substack{\mu \to \tau_0 \\ \Delta\tau \to 0 \\ Q \to D}} \frac{\Phi_{\mathrm{c}} \langle \mu + \Delta\tau \rangle - \Phi_{\mathrm{c}} \langle \mu \rangle}{\Delta\tau} \, , \tag{2.17}$$

$$\sin \omega_0 = \lim_{\substack{\mu \to \tau_0 \\ \Delta\tau \to 0 \\ Q \to D}} \frac{\Phi_{\mathrm{s}} \langle \mu + \Delta\tau \rangle - \Phi_{\mathrm{s}} \langle \mu \rangle}{\Delta\tau} \, . \tag{2.18}$$

Equations (2.17) and (2.18) are not derivatives and do not support any differentiation operations. It employs a distinct computational methodology.

Since $\gamma \gg 1$, let $c \in \mathbb{N}^{+}$.

We define:

$$\pm \frac{c}{\gamma} \leqslant \Delta\tau \leqslant \pm \frac{c+1}{\gamma} \, .$$

Therefore,

$$\lim_{\substack{\Delta\tau \to 0 \\ c \to 1 \\ \gamma \to \infty}} \frac{\Delta\tau}{\pm \frac{c}{\gamma}} = 1 \, .$$

Replace Equations (2.17) and (2.18) with:

$$
\begin{aligned}
\cos\omega_0 &= \lim_{\substack{\mu\to\tau_0\\ \gamma\to\infty\\ Q\to D}} \frac{\Phi_\mathrm{c}\left\langle \mu \pm \frac{1}{\gamma}\right\rangle - \Phi_\mathrm{c}\left\langle \mu\right\rangle}{\pm\frac{1}{\gamma}} \\
&= \pm\iota\left\langle \tau_0\right\rangle \lim_{\substack{\mu\to\tau_0\\ \gamma\to\infty\\ Q\to D}} \left(\sum_{n=\iota\langle\tau_0\rangle}^{\lfloor\mu\gamma\pm1\rfloor} \Psi_\mathrm{c}\left\langle \gamma \wr n\right\rangle - \sum_{n=\iota\langle\tau_0\rangle}^{\lfloor\mu\gamma\rfloor} \Psi_\mathrm{c}\left\langle \gamma \wr n\right\rangle \right) , \\
\sin\omega_0 &= \lim_{\substack{\mu\to\tau_0\\ \gamma\to\infty\\ Q\to D}} \frac{\Phi_\mathrm{s}\left\langle \mu \pm \frac{1}{\gamma}\right\rangle - \Phi_\mathrm{s}\left\langle \mu\right\rangle}{\pm\frac{1}{\gamma}} \\
&= \pm\iota\left\langle \tau_0\right\rangle \lim_{\substack{\mu\to\tau_0\\ \gamma\to\infty\\ Q\to D}} \left(\sum_{n=\iota\langle\tau_0\rangle}^{\lfloor\mu\gamma\pm1\rfloor} \Psi_\mathrm{s}\left\langle \gamma \wr n\right\rangle - \sum_{n=\iota\langle\tau_0\rangle}^{\lfloor\mu\gamma\rfloor} \Psi_\mathrm{s}\left\langle \gamma \wr n\right\rangle \right) .
\end{aligned}
$$

Therefore, the analytical expression for the one-sided tangent can be obtained as:

$$
\begin{cases}
x = \pm\tau\iota\left\langle \tau_0\right\rangle \lim\limits_{\substack{\mu\to\tau_0\\ \gamma\to\infty\\ Q\to D}} \left(\sum\limits_{n=\iota\langle\tau_0\rangle}^{\lfloor\mu\gamma\pm1\rfloor} \Psi_\mathrm{c}\left\langle \gamma \wr n\right\rangle - \sum\limits_{n=\iota\langle\tau_0\rangle}^{\lfloor\mu\gamma\rfloor} \Psi_\mathrm{c}\left\langle \gamma \wr n\right\rangle \right) + \iota\left\langle \tau_0\right\rangle \lim\limits_{\substack{\mu\to\tau_0\\ \gamma\to\infty}} \dfrac{1}{\gamma} \sum\limits_{n=\iota\langle\tau_0\rangle}^{\lfloor\mu\gamma\rfloor} \Psi_\mathrm{c}\left\langle \gamma \wr n\right\rangle + x_0 , \\
y = \pm\tau\iota\left\langle \tau_0\right\rangle \lim\limits_{\substack{\mu\to\tau_0\\ \gamma\to\infty\\ Q\to D}} \left(\sum\limits_{n=\iota\langle\tau_0\rangle}^{\lfloor\mu\gamma\pm1\rfloor} \Psi_\mathrm{s}\left\langle \gamma \wr n\right\rangle - \sum\limits_{n=\iota\langle\tau_0\rangle}^{\lfloor\mu\gamma\rfloor} \Psi_\mathrm{s}\left\langle \gamma \wr n\right\rangle \right) + \iota\left\langle \tau_0\right\rangle \lim\limits_{\substack{\mu\to\tau_0\\ \gamma\to\infty}} \dfrac{1}{\gamma} \sum\limits_{n=\iota\langle\tau_0\rangle}^{\lfloor\mu\gamma\rfloor} \Psi_\mathrm{s}\left\langle \gamma \wr n\right\rangle + y_0 .
\end{cases}
$$

If the curve α is expressed via temporal parametrization, the coordinates of point D are given as $\left(\Phi_1\left\langle \tau_0\right\rangle, \Phi_2\left\langle \tau_0\right\rangle\right)$, with all other given conditions remaining invariant.

At this stage, the expression for the one-sided tangent is:

$$
\begin{cases}
x = \tau\cos\omega_0 + \Phi_1\left\langle \tau_0\right\rangle , \\
y = \tau\sin\omega_0 + \Phi_2\left\langle \tau_0\right\rangle .
\end{cases}
$$

Now given:

$$
\begin{aligned}
\cos\omega_0 &= \lim_{\substack{\mu''\to\tau\\ \mu\to\tau_0\\ Q\to D}} \frac{\Phi_1\left\langle \mu''\right\rangle - \Phi_1\left\langle \mu\right\rangle}{\sqrt{\left(\Phi_1\left\langle \mu''\right\rangle - \Phi_1\left\langle \mu\right\rangle\right)^2 + \left(\Phi_2\left\langle \mu''\right\rangle - \Phi_2\left\langle \mu\right\rangle\right)^2}} \\
&= \lim_{\substack{\mu''\to\tau\\ \mu\to\tau_0\\ Q\to D}} \frac{\iota\left\langle \Phi_1\left\langle \mu''\right\rangle - \Phi_1\left\langle \mu\right\rangle\right\rangle}{\sqrt{1 + \left(\dfrac{\Phi_2\left\langle \mu''\right\rangle - \Phi_2\left\langle \mu\right\rangle}{\Phi_1\left\langle \mu''\right\rangle - \Phi_1\left\langle \mu\right\rangle}\right)^2}} \\
&= \lim_{\substack{\mu''\to\tau\\ \mu\to\tau_0\\ Q\to D}} \iota\left\langle \Phi_1\left\langle \mu''\right\rangle - \Phi_1\left\langle \mu\right\rangle\right\rangle \left(\left(\frac{\Phi_2\left\langle \mu''\right\rangle - \Phi_2\left\langle \mu\right\rangle}{\Phi_1\left\langle \mu''\right\rangle - \Phi_1\left\langle \mu\right\rangle}\right)^2 + 1 \right)^{-\frac{1}{2}} .
\end{aligned}
$$

$$
\begin{aligned}
\sin\omega_0 &= \lim_{\substack{\mu''\to\tau\\ \mu\to\tau_0\\ Q\to D}} \frac{\Phi_2\left\langle\mu''\right\rangle-\Phi_2\left\langle\mu\right\rangle}{\sqrt{\left(\Phi_1\left\langle\mu''\right\rangle-\Phi_1\left\langle\mu\right\rangle\right)^2+\left(\Phi_2\left\langle\mu''\right\rangle-\Phi_2\left\langle\mu\right\rangle\right)^2}}\\
&= \lim_{\substack{\mu''\to\tau\\ \mu\to\tau_0\\ Q\to D}} \frac{\iota\left\langle\Phi_2\left\langle\mu''\right\rangle-\Phi_2\left\langle\mu\right\rangle\right\rangle}{\sqrt{\left(\dfrac{\Phi_1\left\langle\mu''\right\rangle-\Phi_1\left\langle\mu\right\rangle}{\Phi_2\left\langle\mu''\right\rangle-\Phi_2\left\langle\mu\right\rangle}\right)^2+1}}\\
&= \lim_{\substack{\mu''\to\tau\\ \mu\to\tau_0\\ Q\to D}} \iota\left\langle\Phi_2\left\langle\mu''\right\rangle-\Phi_2\left\langle\mu\right\rangle\right\rangle\left(\left(\frac{\Phi_1\left\langle\mu''\right\rangle-\Phi_1\left\langle\mu\right\rangle}{\Phi_2\left\langle\mu''\right\rangle-\Phi_2\left\langle\mu\right\rangle}\right)^2+1\right)^{-\frac{1}{2}}\ .
\end{aligned}
$$

Similarly,

$$
\lim_{\mu''\to\tau}\mu'' = \lim_{\mu\to\tau_0}\mu+\Delta\tau'\ ,
$$

$$
\pm\frac{c'}{\gamma}\leqslant\Delta\tau'\leqslant\pm\frac{c'+1}{\gamma}\ ,
$$

$$
\lim_{\substack{\Delta\tau'\to0\\ c'\to1\\ \gamma\to\infty}}\frac{\Delta\tau'}{\pm\frac{c'}{\gamma}}=1\ .
$$

Therefore,

$$
\begin{aligned}
\cos\omega_0 &= \lim_{\substack{\mu\to\tau_0\\ \Delta\mu'\to0\\ Q\to D}} \iota\left\langle\Phi_1\left\langle\mu+\Delta\mu'\right\rangle-\Phi_1\left\langle\mu\right\rangle\right\rangle\left(\left(\frac{\Phi_2\left\langle\mu+\Delta\mu'\right\rangle-\Phi_2\left\langle\mu\right\rangle}{\Phi_1\left\langle\mu+\Delta\mu'\right\rangle-\Phi_1\left\langle\mu\right\rangle}\right)^2+1\right)^{-\frac{1}{2}}\\
&= \lim_{\substack{\mu\to\tau_0\\ \gamma\to\infty\\ Q\to D}} \iota\left\langle\Phi_1\left\langle\mu\pm\frac{1}{\gamma}\right\rangle-\Phi_1\left\langle\mu\right\rangle\right\rangle\left(\left(\frac{\Phi_2\left\langle\mu\pm\frac{1}{\gamma}\right\rangle-\Phi_2\left\langle\mu\right\rangle}{\Phi_1\left\langle\mu\pm\frac{1}{\gamma}\right\rangle-\Phi_1\left\langle\mu\right\rangle}\right)^2+1\right)^{-\frac{1}{2}}\\
&= \lim_{\substack{\mu\to\tau_0\\ \gamma\to\infty\\ Q\to D}} \iota\left\langle\frac{1}{\gamma}\left(\sum_{n=\iota\langle\tau\rangle}^{\lfloor\mu\gamma\pm1\rfloor}\Psi_1\left\langle\gamma\wr n\right\rangle-\sum_{n=\iota\langle\tau\rangle}^{\lfloor\mu\gamma\rfloor}\Psi_1\left\langle\gamma\wr n\right\rangle\right)\right\rangle\left(\left(\frac{\left(\displaystyle\sum_{n=\iota\langle\tau\rangle}^{\lfloor\mu\gamma\pm1\rfloor}\Psi_2\left\langle\gamma\wr n\right\rangle-\sum_{n=\iota\langle\tau\rangle}^{\lfloor\mu\gamma\rfloor}\Psi_2\left\langle\gamma\wr n\right\rangle\right)}{\left(\displaystyle\sum_{n=\iota\langle\tau\rangle}^{\lfloor\mu\gamma\pm1\rfloor}\Psi_1\left\langle\gamma\wr n\right\rangle-\sum_{n=\iota\langle\tau\rangle}^{\lfloor\mu\gamma\rfloor}\Psi_1\left\langle\gamma\wr n\right\rangle\right)}\right)^2+1\right)^{-\frac{1}{2}}\ ,
\end{aligned}
$$

$$
\begin{aligned}
\sin \omega_0 &= \lim_{\substack{\mu \to \tau_0 \\ \Delta\mu' \to 0 \\ Q \to D}} \iota \left\langle \Phi_2 \left\langle \mu + \Delta\mu' \right\rangle - \Phi_2 \left\langle \mu \right\rangle \right\rangle \left(\left(\frac{\Phi_1 \left\langle \mu + \Delta\mu' \right\rangle - \Phi_1 \left\langle \mu \right\rangle}{\Phi_2 \left\langle \mu + \Delta\mu' \right\rangle - \Phi_2 \left\langle \mu \right\rangle} \right)^2 + 1 \right)^{-\frac{1}{2}} \\
&= \lim_{\substack{\mu \to \tau_0 \\ \gamma \to \infty \\ Q \to D}} \iota \left\langle \Phi_1 \left\langle \mu \pm \frac{1}{\gamma} \right\rangle - \Phi_1 \left\langle \mu \right\rangle \right\rangle \left(\left(\frac{\Phi_2 \left\langle \mu \pm \frac{1}{\gamma} \right\rangle - \Phi_2 \left\langle \mu \right\rangle}{\Phi_1 \left\langle \mu \pm \frac{1}{\gamma} \right\rangle - \Phi_1 \left\langle \mu \right\rangle} \right)^2 + 1 \right)^{-\frac{1}{2}} \\
&= \lim_{\substack{\mu \to \tau_0 \\ \gamma \to \infty \\ Q \to D}} \iota \left\langle \frac{1}{\gamma} \left(\sum_{n=\iota\langle\tau\rangle}^{\lfloor \mu\gamma \pm 1 \rfloor} \Psi_2 \left\langle \gamma \wr n \right\rangle - \sum_{n=\iota\langle\tau\rangle}^{\lfloor \mu\gamma \rfloor} \Psi_2 \left\langle \gamma \wr n \right\rangle \right) \right\rangle \left(\left(\frac{\left(\sum_{n=\iota\langle\tau\rangle}^{\lfloor \mu\gamma \pm 1 \rfloor} \Psi_1 \left\langle \gamma \wr n \right\rangle - \sum_{n=\iota\langle\tau\rangle}^{\lfloor \mu\gamma \rfloor} \Psi_1 \left\langle \gamma \wr n \right\rangle \right)}{\left(\sum_{n=\iota\langle\tau\rangle}^{\lfloor \mu\gamma \pm 1 \rfloor} \Psi_2 \left\langle \gamma \wr n \right\rangle - \sum_{n=\iota\langle\tau\rangle}^{\lfloor \mu\gamma \rfloor} \Psi_2 \left\langle \gamma \wr n \right\rangle \right)} \right)^2 + 1 \right)^{-\frac{1}{2}} .
\end{aligned}
$$

Consequently, the analytic expression for the unilateral tangent is derived as follows:

$$
\begin{cases}
x = \tau \lim\limits_{\substack{\mu \to \tau_0 \\ \gamma \to \infty \\ Q \to D}} \iota \left\langle \frac{1}{\gamma} \left(\sum\limits_{n=\iota\langle\tau\rangle}^{\lfloor \mu\gamma \pm 1 \rfloor} \Psi_1 \left\langle \gamma \wr n \right\rangle - \sum\limits_{n=\iota\langle\tau\rangle}^{\lfloor \mu\gamma \rfloor} \Psi_1 \left\langle \gamma \wr n \right\rangle \right) \right\rangle \left(\left(\frac{\left(\sum\limits_{n=\iota\langle\tau\rangle}^{\lfloor \mu\gamma \pm 1 \rfloor} \Psi_2 \left\langle \gamma \wr n \right\rangle - \sum\limits_{n=\iota\langle\tau\rangle}^{\lfloor \mu\gamma \rfloor} \Psi_2 \left\langle \gamma \wr n \right\rangle \right)}{\left(\sum\limits_{n=\iota\langle\tau\rangle}^{\lfloor \mu\gamma \pm 1 \rfloor} \Psi_1 \left\langle \gamma \wr n \right\rangle - \sum\limits_{n=\iota\langle\tau\rangle}^{\lfloor \mu\gamma \rfloor} \Psi_1 \left\langle \gamma \wr n \right\rangle \right)} \right)^2 + 1 \right)^{-\frac{1}{2}} \\
+\iota \left\langle \tau_0 \right\rangle \lim\limits_{\substack{\mu \to \tau_0 \\ \gamma \to \infty}} \frac{1}{\gamma} \sum\limits_{n=\iota\langle\tau_0\rangle}^{\lfloor \mu\gamma \rfloor} \Psi_1 \left\langle \gamma \wr n \right\rangle + x_1 , \\
y = \tau \lim\limits_{\substack{\mu \to \tau_0 \\ \gamma \to \infty \\ Q \to D}} \iota \left\langle \frac{1}{\gamma} \left(\sum\limits_{n=\iota\langle\tau\rangle}^{\lfloor \mu\gamma \pm 1 \rfloor} \Psi_2 \left\langle \gamma \wr n \right\rangle - \sum\limits_{n=\iota\langle\tau\rangle}^{\lfloor \mu\gamma \rfloor} \Psi_2 \left\langle \gamma \wr n \right\rangle \right) \right\rangle \left(\left(\frac{\left(\sum\limits_{n=\iota\langle\tau\rangle}^{\lfloor \mu\gamma \pm 1 \rfloor} \Psi_1 \left\langle \gamma \wr n \right\rangle - \sum\limits_{n=\iota\langle\tau\rangle}^{\lfloor \mu\gamma \rfloor} \Psi_1 \left\langle \gamma \wr n \right\rangle \right)}{\left(\sum\limits_{n=\iota\langle\tau\rangle}^{\lfloor \mu\gamma \pm 1 \rfloor} \Psi_2 \left\langle \gamma \wr n \right\rangle - \sum\limits_{n=\iota\langle\tau\rangle}^{\lfloor \mu\gamma \rfloor} \Psi_2 \left\langle \gamma \wr n \right\rangle \right)} \right)^2 + 1 \right)^{-\frac{1}{2}} \\
+\iota \left\langle \tau_0 \right\rangle \lim\limits_{\substack{\mu \to \tau_0 \\ \gamma \to \infty}} \frac{1}{\gamma} \sum\limits_{n=\iota\langle\tau_0\rangle}^{\lfloor \mu\gamma \rfloor} \Psi_2 \left\langle \gamma \wr n \right\rangle + y_1 .
\end{cases}
$$

We now investigate the curvature at point D.

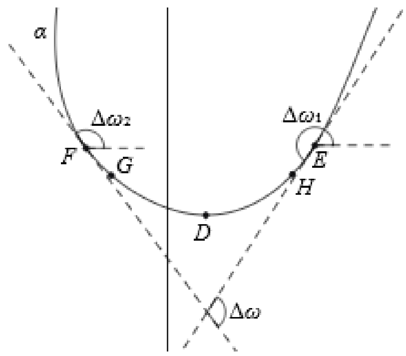

Figure 2.8: Determination of curvature

As illustrated in Figure 2.8, If the current curve α is represented using spatial parameterization, let points F and E be moving points on opposite sides of point D along the curve α, satisfying that both curve lengths FD and DE equal $|\Delta\tau''|$. As point G approaches F asymptotically along the curve DF and point H approaches E along the curve DE, we define the curve lengths FG and EH as second-order infinitesimals $|\Delta\left(\Delta\tau''\right)|$. Here, the inclination angles of the one-sided tangent at E (along the curve DE direction) and at F (along the curve DF direction) are $\Delta\omega_1$ and $\Delta\omega_2$, respectively. $\Delta\omega$ denotes the directional angular variation of the one-sided tangent.

Define the curvature κ at point D, which yields the following relation:

$$
\begin{aligned}
\kappa &= \left|\lim_{\Delta\tau''\to 0} \frac{\Delta\omega}{2\Delta\tau''}\right| \\
&= \frac{1}{2}\lim_{\Delta\tau''\to 0} \frac{|\Delta\omega_1 - \Delta\omega_2|}{|\Delta\tau''|} .
\end{aligned}
$$

Analogously, let:

$$
\pm\frac{c}{\gamma^2} \leqslant \Delta\left(\Delta\tau''\right) \leqslant \pm\frac{c+1}{\gamma^2} .
$$

Consequently:

$$
\lim_{\substack{\Delta(\Delta\tau'')\to 0 \\ c\to 1 \\ \gamma\to\infty}} \frac{\Delta\left(\Delta\tau''\right)}{\pm\frac{c}{\gamma^2}} = 1 .
$$

Hence,

$$
\begin{aligned}
\cos\Delta\omega_1 &= \lim_{\substack{\mu\to\tau_0 \\ \Delta(\Delta\tau)''\to 0 \\ H\to E}} \frac{\Phi_{\mathrm{c}}\left\langle \mu + \Delta\tau'' + \Delta(\Delta\tau'')\right\rangle - \Phi_{\mathrm{c}}\left\langle \mu + \Delta\tau''\right\rangle}{\Delta\left(\Delta\tau''\right)} \\
&= \lim_{\substack{\mu\to\tau_0 \\ \gamma\to\infty \\ H\to E}} \frac{\Phi_{\mathrm{c}}\left\langle \mu \pm \frac{1}{\gamma} \pm \frac{1}{\gamma^2}\right\rangle - \Phi_{\mathrm{c}}\left\langle \mu \pm \frac{1}{\gamma}\right\rangle}{\pm\frac{1}{\gamma^2}} \\
&= \pm\iota\left\langle\tau_0\right\rangle \lim_{\substack{\mu\to\tau_0 \\ \gamma\to\infty \\ H\to E}} \gamma\left(\sum_{n=\iota\langle\tau_0\rangle}^{\lfloor\mu\gamma\pm 1\pm\frac{1}{\gamma}\rfloor} \Psi_{\mathrm{c}}\left\langle\gamma \wr n\right\rangle - \sum_{n=\iota\langle\tau_0\rangle}^{\lfloor\mu\gamma\pm 1\rfloor} \Psi_{\mathrm{c}}\left\langle\gamma \wr n\right\rangle\right) , \\
\cos\Delta\omega_2 &= \lim_{\substack{\mu\to\tau_0 \\ \gamma\to\infty \\ G\to F}} \frac{\Phi_{\mathrm{c}}\left\langle \mu \mp \frac{1}{\gamma} \mp \frac{1}{\gamma^2}\right\rangle - \Phi_{\mathrm{c}}\left\langle \mu \mp \frac{1}{\gamma}\right\rangle}{\mp\frac{1}{\gamma^2}} \\
&= \mp\iota\left\langle\tau_0\right\rangle \lim_{\substack{\mu\to\tau_0 \\ \gamma\to\infty \\ G\to F}} \gamma\left(\sum_{n=\iota\langle\tau_0\rangle}^{\lfloor\mu\gamma\mp 1\mp\frac{1}{\gamma}\rfloor} \Psi_{\mathrm{c}}\left\langle\gamma \wr n\right\rangle - \sum_{n=\iota\langle\tau_0\rangle}^{\lfloor\mu\gamma\mp 1\rfloor} \Psi_{\mathrm{c}}\left\langle\gamma \wr n\right\rangle\right) ,
\end{aligned}
$$

$$
\begin{aligned}
\sin \Delta\omega_1 = & \lim_{\substack{\mu\to\tau_0 \\ \Delta(\Delta\tau)''\to 0 \\ H\to E}} \frac{\Phi_{\mathrm{s}}\left\langle \mu + \Delta\tau'' + \Delta(\Delta\tau'')\right\rangle - \Phi_{\mathrm{s}}\left\langle \mu + \Delta\tau''\right\rangle}{\Delta\left(\Delta\tau''\right)} \\
= & \lim_{\substack{\mu\to\tau_0 \\ \gamma\to\infty \\ H\to E}} \frac{\Phi_{\mathrm{s}}\left\langle \mu \pm \frac{1}{\gamma} \pm \frac{1}{\gamma^2}\right\rangle - \Phi_{\mathrm{s}}\left\langle \mu \pm \frac{1}{\gamma}\right\rangle}{\pm\frac{1}{\gamma^2}} \\
= & \pm\iota\left\langle\tau_0\right\rangle \lim_{\substack{\mu\to\tau_0 \\ \gamma\to\infty \\ H\to E}} \gamma\left(\sum_{n=\iota\langle\tau_0\rangle}^{\lfloor\mu\gamma\pm 1\pm\frac{1}{\gamma}\rfloor} \Psi_{\mathrm{s}}\left\langle\gamma \wr n\right\rangle - \sum_{n=\iota\langle\tau_0\rangle}^{\lfloor\mu\gamma\pm 1\rfloor} \Psi_{\mathrm{s}}\left\langle\gamma \wr n\right\rangle\right), \\
\sin \Delta\omega_2 = & \lim_{\substack{\mu\to\tau_0 \\ \gamma\to\infty \\ G\to F}} \frac{\Phi_{\mathrm{s}}\left\langle \mu \mp \frac{1}{\gamma} \mp \frac{1}{\gamma^2}\right\rangle - \Phi_{\mathrm{s}}\left\langle \mu \mp \frac{1}{\gamma}\right\rangle}{\mp\frac{1}{\gamma^2}} \\
= & \mp\iota\left\langle\tau_0\right\rangle \lim_{\substack{\mu\to\tau_0 \\ \gamma\to\infty \\ G\to F}} \gamma\left(\sum_{n=\iota\langle\tau_0\rangle}^{\lfloor\mu\gamma\mp 1\mp\frac{1}{\gamma}\rfloor} \Psi_{\mathrm{s}}\left\langle\gamma \wr n\right\rangle - \sum_{n=\iota\langle\tau_0\rangle}^{\lfloor\mu\gamma\mp 1\rfloor} \Psi_{\mathrm{s}}\left\langle\gamma \wr n\right\rangle\right).
\end{aligned}
$$

Regarding the angle between the unilateral tangent and the x-axis in the counterclockwise direction, there exist at least two distinct computational approaches.

If $\eta_1, \eta_2 \in \mathbb{R}$, let:

$$
\begin{aligned}
\iota_1\left\langle\eta_1, \eta_2\right\rangle &= \lim_{\lambda\to 0} \frac{\dfrac{\eta_1+\lambda}{|\eta_1|+\lambda} - \dfrac{\eta_2+\lambda}{|\eta_2|+\lambda} + 1}{\left|\dfrac{\eta_1+\lambda}{|\eta_1|+\lambda} - \dfrac{\eta_2+\lambda}{|\eta_2|+\lambda} + 1\right|}, \\
\iota_2\left\langle\eta_1, \eta_2\right\rangle &= \lim_{\lambda\to 0} \frac{1 - \dfrac{\eta_1+\lambda}{|\eta_1|+\lambda} - \dfrac{\eta_2+\lambda}{|\eta_2|+\lambda}}{\left|\dfrac{\eta_1+\lambda}{|\eta_1|+\lambda} + \dfrac{\eta_2+\lambda}{|\eta_2|+\lambda} - 1\right|}.
\end{aligned}
$$

Taking $\Delta\omega_1$ as an example,

$$
\Delta\omega_1 = \iota\left\langle\cos\Delta\omega_1\right\rangle\left(\arcsin\left(\sin\Delta\omega_1\right) + \pi\right) + \iota_1\left\langle\sin\Delta\omega_1, \cos\Delta\omega_1\right\rangle\cdot\frac{\pi}{2} + \iota_2\left\langle\sin\Delta\omega_1, \cos\Delta\omega_1\right\rangle\cdot\frac{3\pi}{2}, \tag{2.19}
$$

$$
\Delta\omega_1 = \pi + \iota\left\langle\sin\Delta\omega_1\right\rangle\left(\arccos\left(\cos\Delta\omega_1\right) - \pi\right). \tag{2.20}
$$

Since Equation (2.20) is significantly more concise than Equation (2.19), we consequently discard Equation (2.20). Therefore,

$$
\Delta\omega_1 - \Delta\omega_2 = \iota\left\langle\sin\Delta\omega_1\right\rangle\left(\arccos\left(\cos\Delta\omega_1\right) - \pi\right) - \iota\left\langle\sin\Delta\omega_2\right\rangle\left(\arccos\left(\cos\Delta\omega_2\right) - \pi\right).
$$

$$
\begin{aligned}
\kappa = \frac{1}{2} \lim_{\substack{\mu\to\tau_0\\ \gamma\to\infty}} \gamma^2 \Bigg| \iota \left\langle \pm\iota\left\langle\tau_0\right\rangle \lim_{H\to E} \gamma \left(\sum_{n=\iota\langle\tau_0\rangle}^{\lfloor \mu\gamma\pm 1\pm\frac{1}{\gamma} \rfloor} \Psi_{\mathrm{s}}\left\langle\gamma \wr n\right\rangle - \sum_{n=\iota\langle\tau_0\rangle}^{\lfloor \mu\gamma\pm 1 \rfloor} \Psi_{\mathrm{s}}\left\langle\gamma \wr n\right\rangle \right) \right\rangle \times \\
\left(\arccos\left(\pm\iota\left\langle\tau_0\right\rangle \lim_{H\to E} \gamma \left(\sum_{n=\iota\langle\tau_0\rangle}^{\lfloor \mu\gamma\pm 1\pm\frac{1}{\gamma} \rfloor} \Psi_{\mathrm{c}}\left\langle\gamma \wr n\right\rangle - \sum_{n=\iota\langle\tau_0\rangle}^{\lfloor \mu\gamma\pm 1 \rfloor} \Psi_{\mathrm{c}}\left\langle\gamma \wr n\right\rangle \right)\right) - \pi \right) - \\
\iota \left\langle \mp\iota\left\langle\tau_0\right\rangle \lim_{G\to F} \gamma \left(\sum_{n=\iota\langle\tau_0\rangle}^{\lfloor \mu\gamma\mp 1\mp\frac{1}{\gamma} \rfloor} \Psi_{\mathrm{s}}\left\langle\gamma \wr n\right\rangle - \sum_{n=\iota\langle\tau_0\rangle}^{\lfloor \mu\gamma\mp 1 \rfloor} \Psi_{\mathrm{s}}\left\langle\gamma \wr n\right\rangle \right) \right\rangle \times \\
\left(\arccos\left(\mp\iota\left\langle\tau_0\right\rangle \lim_{G\to F} \gamma \left(\sum_{n=\iota\langle\tau_0\rangle}^{\lfloor \mu\gamma\mp 1\mp\frac{1}{\gamma} \rfloor} \Psi_{\mathrm{c}}\left\langle\gamma \wr n\right\rangle - \sum_{n=\iota\langle\tau_0\rangle}^{\lfloor \mu\gamma\mp 1 \rfloor} \Psi_{\mathrm{c}}\left\langle\gamma \wr n\right\rangle \right)\right) - \pi \right) \Bigg| .
\end{aligned}
$$

For the temporally parametrized curve α, after substituting $\Delta\tau''$ with $\Delta\tau^*$: the original condition indicating equal segment lengths now denotes equal traversal time along curve segments.

By the same token,

$$
\pm\frac{c''}{\gamma^2} \leqslant \Delta\left(\Delta\tau^*\right) \leqslant \pm\frac{c''+1}{\gamma^2} .
$$

Therefore,

$$
\lim_{\substack{\Delta(\Delta\tau^*)\to 0\\ c''\to 1\\ \gamma\to\infty}} \frac{\Delta\left(\Delta\tau^*\right)}{\pm\frac{c''}{\gamma^2}} = 1 .
$$

Namely:

$$
\begin{aligned}
\kappa &= \left| \lim_{\substack{\mu\to\tau_0\\ \Delta\tau^*\to 0}} \frac{\Delta\omega}{\sqrt{\sum_{i=1}^{2}\left(\Phi_i\left\langle\mu+\Delta\tau^*\right\rangle - \Phi_i\left\langle\mu\right\rangle\right)^2} + \sqrt{\sum_{i=1}^{2}\left(\Phi_i\left\langle\mu-\Delta\tau^*\right\rangle - \Phi_i\left\langle\mu\right\rangle\right)^2}} \right| \\
&= \lim_{\substack{\mu\to\tau_0\\ \Delta\tau^*\to 0}} \frac{\left|\Delta\omega_1 - \Delta\omega_2\right|}{\sqrt{\sum_{i=1}^{2}\left(\Phi_i\left\langle\mu+\Delta\tau^*\right\rangle - \Phi_i\left\langle\mu\right\rangle\right)^2} + \sqrt{\sum_{i=1}^{2}\left(\Phi_i\left\langle\mu-\Delta\tau^*\right\rangle - \Phi_i\left\langle\mu\right\rangle\right)^2}} \\
&= \lim_{\substack{\mu\to\tau_0\\ \Delta\tau^*\to 0}} \frac{\left|\iota\left\langle\sin\Delta\omega_1\right\rangle\left(\arccos\left(\cos\Delta\omega_1\right)-\pi\right) - \iota\left\langle\sin\Delta\omega_2\right\rangle\left(\arccos\left(\cos\Delta\omega_2\right)-\pi\right)\right|}{\sqrt{\sum_{i=1}^{2}\left(\Phi_i\left\langle\mu+\Delta\tau^*\right\rangle - \Phi_i\left\langle\mu\right\rangle\right)^2} + \sqrt{\sum_{i=1}^{2}\left(\Phi_i\left\langle\mu-\Delta\tau^*\right\rangle - \Phi_i\left\langle\mu\right\rangle\right)^2}} \\
&= \lim_{\substack{\mu\to\tau_0\\ \gamma\to\infty}} \frac{\left|\iota\left\langle\sin\Delta\omega_1\right\rangle\left(\arccos\left(\cos\Delta\omega_1\right)-\pi\right) - \iota\left\langle\sin\Delta\omega_2\right\rangle\left(\arccos\left(\cos\Delta\omega_2\right)-\pi\right)\right|}{\sqrt{\sum_{i=1}^{2}\left(\Phi_i\left\langle\mu\pm\frac{1}{\gamma}\right\rangle - \Phi_i\left\langle\mu\right\rangle\right)^2} + \sqrt{\sum_{i=1}^{2}\left(\Phi_i\left\langle\mu\mp\frac{1}{\gamma}\right\rangle - \Phi_i\left\langle\mu\right\rangle\right)^2}} .
\end{aligned}
$$

Whereas,

$$\cos\Delta\omega_1 = \lim_{\substack{\mu\to\tau_0\\ \gamma\to\infty\\ H\to E}} \iota\left\langle\frac{1}{\gamma}\left(\sum_{n=\iota\langle\tau\rangle}^{\lfloor\mu\gamma\pm1\pm\frac{1}{\gamma}\rfloor}\Psi_1\langle\gamma\wr n\rangle-\sum_{n=\iota\langle\tau\rangle}^{\lfloor\mu\gamma\pm1\rfloor}\Psi_1\langle\gamma\wr n\rangle\right)\right\rangle\left(\left(\frac{\left(\sum_{n=\iota\langle\tau\rangle}^{\lfloor\mu\gamma\pm1\pm\frac{1}{\gamma}\rfloor}\Psi_2\langle\gamma\wr n\rangle-\sum_{n=\iota\langle\tau\rangle}^{\lfloor\mu\gamma\pm1\rfloor}\Psi_2\langle\gamma\wr n\rangle\right)}{\left(\sum_{n=\iota\langle\tau\rangle}^{\lfloor\mu\gamma\pm1\pm\frac{1}{\gamma}\rfloor}\Psi_1\langle\gamma\wr n\rangle-\sum_{n=\iota\langle\tau\rangle}^{\lfloor\mu\gamma\pm1\rfloor}\Psi_1\langle\gamma\wr n\rangle\right)}\right)^2+1\right)^{-\frac{1}{2}},$$

$$\sin\Delta\omega_1 = \lim_{\substack{\mu\to\tau_0\\ \gamma\to\infty\\ H\to E}} \iota\left\langle\frac{1}{\gamma}\left(\sum_{n=\iota\langle\tau\rangle}^{\lfloor\mu\gamma\pm1\pm\frac{1}{\gamma}\rfloor}\Psi_2\langle\gamma\wr n\rangle-\sum_{n=\iota\langle\tau\rangle}^{\lfloor\mu\gamma\pm1\rfloor}\Psi_2\langle\gamma\wr n\rangle\right)\right\rangle\left(\left(\frac{\left(\sum_{n=\iota\langle\tau\rangle}^{\lfloor\mu\gamma\pm1\pm\frac{1}{\gamma}\rfloor}\Psi_1\langle\gamma\wr n\rangle-\sum_{n=\iota\langle\tau\rangle}^{\lfloor\mu\gamma\pm1\rfloor}\Psi_1\langle\gamma\wr n\rangle\right)}{\left(\sum_{n=\iota\langle\tau\rangle}^{\lfloor\mu\gamma\pm1\pm\frac{1}{\gamma}\rfloor}\Psi_2\langle\gamma\wr n\rangle-\sum_{n=\iota\langle\tau\rangle}^{\lfloor\mu\gamma\pm1\rfloor}\Psi_2\langle\gamma\wr n\rangle\right)}\right)^2+1\right)^{-\frac{1}{2}},$$

$$\cos\Delta\omega_2 = \lim_{\substack{\mu\to\tau_0\\ \gamma\to\infty\\ H\to E}} \iota\left\langle\frac{1}{\gamma}\left(\sum_{n=\iota\langle\tau\rangle}^{\lfloor\mu\gamma\mp1\mp\frac{1}{\gamma}\rfloor}\Psi_1\langle\gamma\wr n\rangle-\sum_{n=\iota\langle\tau\rangle}^{\lfloor\mu\gamma\mp1\rfloor}\Psi_1\langle\gamma\wr n\rangle\right)\right\rangle\left(\left(\frac{\left(\sum_{n=\iota\langle\tau\rangle}^{\lfloor\mu\gamma\mp1\mp\frac{1}{\gamma}\rfloor}\Psi_2\langle\gamma\wr n\rangle-\sum_{n=\iota\langle\tau\rangle}^{\lfloor\mu\gamma\mp1\rfloor}\Psi_2\langle\gamma\wr n\rangle\right)}{\left(\sum_{n=\iota\langle\tau\rangle}^{\lfloor\mu\gamma\mp1\mp\frac{1}{\gamma}\rfloor}\Psi_1\langle\gamma\wr n\rangle-\sum_{n=\iota\langle\tau\rangle}^{\lfloor\mu\gamma\mp1\rfloor}\Psi_1\langle\gamma\wr n\rangle\right)}\right)^2+1\right)^{-\frac{1}{2}},$$

$$\sin\Delta\omega_2 = \lim_{\substack{\mu\to\tau_0\\ \gamma\to\infty\\ H\to E}} \iota\left\langle\frac{1}{\gamma}\left(\sum_{n=\iota\langle\tau\rangle}^{\lfloor\mu\gamma\mp1\mp\frac{1}{\gamma}\rfloor}\Psi_2\langle\gamma\wr n\rangle-\sum_{n=\iota\langle\tau\rangle}^{\lfloor\mu\gamma\mp1\rfloor}\Psi_2\langle\gamma\wr n\rangle\right)\right\rangle\left(\left(\frac{\left(\sum_{n=\iota\langle\tau\rangle}^{\lfloor\mu\gamma\mp1\mp\frac{1}{\gamma}\rfloor}\Psi_1\langle\gamma\wr n\rangle-\sum_{n=\iota\langle\tau\rangle}^{\lfloor\mu\gamma\mp1\rfloor}\Psi_1\langle\gamma\wr n\rangle\right)}{\left(\sum_{n=\iota\langle\tau\rangle}^{\lfloor\mu\gamma\mp1\mp\frac{1}{\gamma}\rfloor}\Psi_2\langle\gamma\wr n\rangle-\sum_{n=\iota\langle\tau\rangle}^{\lfloor\mu\gamma\mp1\rfloor}\Psi_2\langle\gamma\wr n\rangle\right)}\right)^2+1\right)^{-\frac{1}{2}}.$$

To avoid overly lengthy formulas, the full expression of curvature κ is not expanded here. Through a theoretical breakthrough that completely abandons differentiability requirements, this study has successfully established a universal curvature computation framework for arbitrary planar continuous curves (the complete expressions are omitted here due to excessive length).

3 Characterization of Multidimensional Geometric Objects

3.1 Metric characterization of continuous surfaces

Building upon the τ-proportional coefficient for the x-axis established in Section 2.1, we now extend the space to a two-dimensional xOy coordinate system, where the x-axis retains τ as its proportional coefficient, the y-axis adopts ν as its proportional coefficient, and an additional non-zero variable υ is introduced. As shown in Figure 3.1, a unit circle centered at the origin O is constructed, with the radius $r\langle x, y\rangle$ denoting the distance from any point to the origin. An arbitrary point A' is selected on the unit circle, and the line segment OA' is connected.

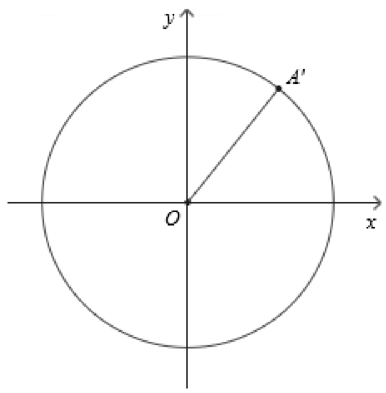

Figure 3.1: The unit circle and its radius OA'

Analogous to the operation in Section 2.1, divide OA' into γ equal parts, which yields:

$$x = \iota\left\langle \tau \right\rangle \lim_{\substack{\mu\to\tau \\ \gamma\to\infty}} \frac{1}{\gamma} \sum_{i=\iota\langle\tau\rangle}^{\lfloor\mu\gamma\rfloor} 1\,, \tag{3.1}$$

$$y = \iota\left\langle \nu \right\rangle \lim_{\substack{\upsilon\to\nu \\ \gamma\to\infty}} \frac{1}{\gamma} \sum_{i=\iota\langle\nu\rangle}^{\lfloor\upsilon\gamma\rfloor} 1\,. \tag{3.2}$$

To construct the foundation for continuous surfaces, we integrate Formula 3.1 and Formula 3.2 to obtain:

$$r\left\langle x, y \right\rangle = \lim_{\substack{\mu\to\tau \\ \upsilon\to\nu \\ \gamma\to\infty}} \frac{1}{\gamma} \sum_{i=1}^{\lfloor\sqrt{\mu^2+\upsilon^2}\gamma\rfloor} 1\,. \tag{3.3}$$

Equation (3.3) fully characterizes the entire plane.

We now introduce a z-axis to extend the space to three dimensions, redefining the angle θ in Section 2.2 as the counterclockwise angle from the positive y-axis, while adding a new counterclockwise angle φ from the positive x-axis, as illustrated in Figure 3.2.

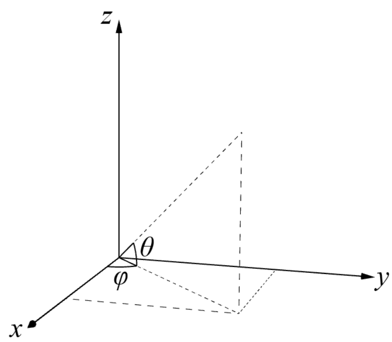

Figure 3.2: Angle θ and angle φ

Analogous to Formula 2.11, by reducing the two-dimensional case to a one-dimensional scenario, we obtain:

$$
\begin{cases}
x = \iota\left\langle\tau\right\rangle \lim\limits_{\substack{\mu\to\tau\\ \upsilon\to\nu\\ \gamma\to\infty}} \dfrac{1}{\gamma} \sum\limits_{i=\iota\langle\tau\rangle}^{\lfloor \iota\langle\tau\rangle\sqrt{\mu^2+\upsilon^2}\gamma \rfloor} \cos\theta_i \cos\varphi_i + x_0' \,, \\
y = \iota\left\langle\nu\right\rangle \lim\limits_{\substack{\mu\to\tau\\ \upsilon\to\nu\\ \gamma\to\infty}} \dfrac{1}{\gamma} \sum\limits_{i=\iota\langle\nu\rangle}^{\lfloor \iota\langle\nu\rangle\sqrt{\mu^2+\upsilon^2}\gamma \rfloor} \cos\theta_i \sin\varphi_i + y_0' \,, \\
z = \iota\left\langle\nu\right\rangle \lim\limits_{\substack{\mu\to\tau\\ \upsilon\to\nu\\ \gamma\to\infty}} \dfrac{1}{\gamma} \sum\limits_{i=\iota\langle\nu\rangle}^{\lfloor \iota\langle\nu\rangle\sqrt{\mu^2+\upsilon^2}\gamma \rfloor} \sin\theta_i + z_0' \,.
\end{cases}
$$

Given an infinitely extended continuous surface S (where S satisfies only C^0 continuity), we establish its parametric representation as follows:

$$
\begin{cases}
x = \iota\left\langle\tau\right\rangle \lim\limits_{\substack{\mu\to\tau\\ \upsilon\to\nu\\ \gamma\to\infty}} \dfrac{1}{\gamma} \sum\limits_{n=\iota\langle\tau\rangle}^{\lfloor \iota\langle\tau\rangle\sqrt{\mu^2+\upsilon^2}\gamma \rfloor} \Psi_x \left\langle\gamma \wr n\right\rangle + x_0' \,, \\
y = \iota\left\langle\nu\right\rangle \lim\limits_{\substack{\mu\to\tau\\ \upsilon\to\nu\\ \gamma\to\infty}} \dfrac{1}{\gamma} \sum\limits_{n=\iota\langle\nu\rangle}^{\lfloor \iota\langle\nu\rangle\sqrt{\mu^2+\upsilon^2}\gamma \rfloor} \Psi_y \left\langle\gamma \wr n\right\rangle + y_0' \,, \\
z = \iota\left\langle\nu\right\rangle \lim\limits_{\substack{\mu\to\tau\\ \upsilon\to\nu\\ \gamma\to\infty}} \dfrac{1}{\gamma} \sum\limits_{n=\iota\langle\nu\rangle}^{\lfloor \iota\langle\nu\rangle\sqrt{\mu^2+\upsilon^2}\gamma \rfloor} \Psi_z \left\langle\gamma \wr n\right\rangle + z_0' \,.
\end{cases}
$$

Where (x_0', y_0', z_0') denotes the initial point coordinates.

An alternative parametric representation of surface S is given by:

$$
\begin{cases}
x = \iota\left\langle\tau\right\rangle \lim\limits_{\substack{\mu\to\tau\\ \upsilon\to\nu\\ \gamma\to\infty}} \dfrac{1}{\gamma} \sum\limits_{n=\iota\langle\tau\rangle}^{\lfloor \iota\langle\tau\rangle\sqrt{\mu^2+\upsilon^2}\gamma \rfloor} \Psi_1 \left\langle\gamma \wr n\right\rangle + x_0'' \,, \\
y = \iota\left\langle\nu\right\rangle \lim\limits_{\substack{\mu\to\tau\\ \upsilon\to\nu\\ \gamma\to\infty}} \dfrac{1}{\gamma} \sum\limits_{n=\iota\langle\nu\rangle}^{\lfloor \iota\langle\nu\rangle\sqrt{\mu^2+\upsilon^2}\gamma \rfloor} \Psi_2 \left\langle\gamma \wr n\right\rangle + y_0'' \,, \\
z = \iota\left\langle\nu\right\rangle \lim\limits_{\substack{\mu\to\tau\\ \upsilon\to\nu\\ \gamma\to\infty}} \dfrac{1}{\gamma} \sum\limits_{n=\iota\langle\nu\rangle}^{\lfloor \iota\langle\nu\rangle\sqrt{\mu^2+\upsilon^2}\gamma \rfloor} \Psi_3 \left\langle\gamma \wr n\right\rangle + z_0'' \,.
\end{cases}
$$

with x_0'', y_0'', z_0'' being constants.

3.2 Metric characterization of continuous space curves

For the infinitely extended continuous space curve β (where β satisfies only C^0 continuity) with initial point (x_0''', y_0''', z_0'''), its spatial parametric representation can be directly established based on the construction theory of continuous surfaces as follows:

$$
\begin{cases}
x = \iota\left\langle \tau \right\rangle \lim\limits_{\substack{\mu\to\tau \\ \gamma\to\infty}} \dfrac{1}{\gamma} \sum\limits_{n=\iota\langle\tau\rangle}^{\lfloor\mu\gamma\rfloor} \Psi_x \left\langle \gamma \wr n \right\rangle + x_0''' , \\
y = \iota\left\langle \tau \right\rangle \lim\limits_{\substack{\mu\to\tau \\ \gamma\to\infty}} \dfrac{1}{\gamma} \sum\limits_{n=\iota\langle\tau\rangle}^{\lfloor\mu\gamma\rfloor} \Psi_y \left\langle \gamma \wr n \right\rangle + y_0''' , \\
z = \iota\left\langle \tau \right\rangle \lim\limits_{\substack{\mu\to\tau \\ \gamma\to\infty}} \dfrac{1}{\gamma} \sum\limits_{n=\iota\langle\tau\rangle}^{\lfloor\mu\gamma\rfloor} \Psi_z \left\langle \gamma \wr n \right\rangle + z_0''' .
\end{cases}
$$

We can likewise express curve β in parametric form with respect to time:

$$
\begin{cases}
x = \iota\left\langle \tau \right\rangle \lim\limits_{\substack{\mu\to\tau \\ \gamma\to\infty}} \dfrac{1}{\gamma} \sum\limits_{n=\iota\langle\tau\rangle}^{\lfloor\mu\gamma\rfloor} \Psi_1 \left\langle \gamma \wr n \right\rangle + x_0^* , \\
y = \iota\left\langle \tau \right\rangle \lim\limits_{\substack{\mu\to\tau \\ \gamma\to\infty}} \dfrac{1}{\gamma} \sum\limits_{n=\iota\langle\tau\rangle}^{\lfloor\mu\gamma\rfloor} \Psi_2 \left\langle \gamma \wr n \right\rangle + y_0^* , \\
z = \iota\left\langle \tau \right\rangle \lim\limits_{\substack{\mu\to\tau \\ \gamma\to\infty}} \dfrac{1}{\gamma} \sum\limits_{n=\iota\langle\tau\rangle}^{\lfloor\mu\gamma\rfloor} \Psi_3 \left\langle \gamma \wr n \right\rangle + z_0^* .
\end{cases}
$$

where (x_0^*, y_0^*, z_0^*) denotes the spatial coordinates at $\tau = 0$.

3.3 The metric characterization of geometric object in higher-dimensional spaces

Consider an infinitely extended continuous geometric object $\aleph$ (where $\aleph$ satisfies only C^0 continuity) of dimension $k-1$ embedded in a k-dimensional Euclidean space. Let $x_{(j)}$ denote the j-th coordinate axis, with nonzero variables $\mu_{(j)}$ and their corresponding parameters $\tau_{(j)}$, $x_{0,(j)}$ and $x^*_{0,(j)}$ are constants. Through surface extension, we obtain two parametric representations of the hypersurface $\aleph$:

$$
\begin{cases}
x_{(1)} = \iota\left\langle \tau_{(1)} \right\rangle \lim\limits_{\substack{\mu_{(1)}\to\tau_{(1)} \\ \gamma\to\infty}} \dfrac{1}{\gamma} \sum\limits_{n=\iota\left\langle\tau_{(1)}\right\rangle}^{\left\lfloor \iota\left\langle\tau_{(1)}\right\rangle \sqrt{\sum_{j=1}^{k-1}\mu_{(j)}^2}\,\gamma \right\rfloor} \Psi_{x_{(1)}} \left\langle \gamma \wr n \right\rangle + x_{0,(1)} , \\
x_{(2)} = \iota\left\langle \tau_{(2)} \right\rangle \lim\limits_{\substack{\mu_{(2)}\to\tau_{(2)} \\ \gamma\to\infty}} \dfrac{1}{\gamma} \sum\limits_{n=\iota\left\langle\tau_{(2)}\right\rangle}^{\left\lfloor \iota\left\langle\tau_{(2)}\right\rangle \sqrt{\sum_{j=1}^{k-1}\mu_{(j)}^2}\,\gamma \right\rfloor} \Psi_{x_{(2)}} \left\langle \gamma \wr n \right\rangle + x_{0,(2)} , \\
\quad \cdots\cdots\cdots\cdots \\
x_{(k-1)} = \iota\left\langle \tau_{(k-1)} \right\rangle \lim\limits_{\substack{\mu_{(k-1)}\to\tau_{(k-1)} \\ \gamma\to\infty}} \dfrac{1}{\gamma} \sum\limits_{n=\iota\left\langle\tau_{(k-1)}\right\rangle}^{\left\lfloor \iota\left\langle\tau_{(k-1)}\right\rangle \sqrt{\sum_{j=1}^{k-1}\mu_{(j)}^2}\,\gamma \right\rfloor} \Psi_{x_{(k-1)}} \left\langle \gamma \wr n \right\rangle + x_{0,(k-1)} , \\
x_{(k)} = \iota\left\langle \tau_{(k)} \right\rangle \lim\limits_{\substack{\mu_{(k)}\to\tau_{(k)} \\ \gamma\to\infty}} \dfrac{1}{\gamma} \sum\limits_{n=\iota\left\langle\tau_{(k)}\right\rangle}^{\left\lfloor \iota\left\langle\tau_{(k)}\right\rangle \sqrt{\sum_{j=1}^{k-1}\mu_{(j)}^2}\,\gamma \right\rfloor} \Psi_{x_{(k)}} \left\langle \gamma \wr n \right\rangle + x_{0,(k)} ,
\end{cases}
$$

$$
\begin{cases}
x_{(1)} = \iota\left\langle\tau_{(1)}\right\rangle \lim\limits_{\substack{\mu_{(1)}\to\tau_{(1)}\\ \gamma\to\infty}} \dfrac{1}{\gamma} \sum\limits_{n=\iota\left\langle\tau_{(1)}\right\rangle}^{\left\lfloor \iota\left\langle\tau_{(1)}\right\rangle\sqrt{\sum_{j=1}^{k-1}\mu_{(j)}^2}\gamma\right\rfloor} \Psi_{(1)}\left\langle\gamma \wr n\right\rangle + x_{0,(1)}^{*}\,, \\
x_{(2)} = \iota\left\langle\tau_{(2)}\right\rangle \lim\limits_{\substack{\mu_{(2)}\to\tau_{(2)}\\ \gamma\to\infty}} \dfrac{1}{\gamma} \sum\limits_{n=\iota\left\langle\tau_{(2)}\right\rangle}^{\left\lfloor \iota\left\langle\tau_{(2)}\right\rangle\sqrt{\sum_{j=1}^{k-1}\mu_{(j)}^2}\gamma\right\rfloor} \Psi_{(2)}\left\langle\gamma \wr n\right\rangle + x_{0,(2)}^{*}\,, \\
\quad \cdots\cdots\cdots\cdots \\
x_{(k-1)} = \iota\left\langle\tau_{(2)}\right\rangle \lim\limits_{\substack{\mu_{(k-1)}\to\tau_{(k-1)}\\ \gamma\to\infty}} \dfrac{1}{\gamma} \sum\limits_{n=\iota\left\langle\tau_{(k-1)}\right\rangle}^{\left\lfloor \iota\left\langle\tau_{(k-1)}\right\rangle\sqrt{\sum_{j=1}^{k-1}\mu_{(j)}^2}\gamma\right\rfloor} \Psi_{(k-1)}\left\langle\gamma \wr n\right\rangle + x_{0,(k-1)}^{*}\,, \\
x_{(k)} = \iota\left\langle\tau_{(k-1)}\right\rangle \lim\limits_{\substack{\mu_{(k-1)}\to\tau_{(k-1)}\\ \gamma\to\infty}} \dfrac{1}{\gamma} \sum\limits_{n=\iota\left\langle\tau_{(k-1)}\right\rangle}^{\left\lfloor \iota\left\langle\tau_{(k-1)}\right\rangle\sqrt{\sum_{j=1}^{k-1}\mu_{(j)}^2}\gamma\right\rfloor} \Psi_{(k)}\left\langle\gamma \wr n\right\rangle + x_{0,(k)}^{*}\,.
\end{cases}
$$

Multidimensional geometric objects can be either manifolds or hypersurfaces, but within a non-differential geometric framework, only continuity is required without any smoothness conditions.

Since the determination of geometric quantities (such as curvature) for continuous surfaces, spatial continuous curves, and higher-dimensional continuous geometric objects requires new axiomatic theories and interdisciplinary collaboration (e.g., with physics), this paper does not address the study of such quantities.

4 Theoretical paradigm breakthrough

4.1 Extensions and breakthroughs in integration theory through non-differential geometric analysis tools

When exploring extensions and breakthroughs in integration theory through non-differential geometric tools, it is necessary to first verify their compatibility with classical integration. This can be demonstrated by performing just two types of constructions on classical planar continuous smooth curves $y = f(x)$, as higher-dimensional geometric objects are inherently extensions of lower-dimensional cases. Selecting the initial point $(s_0, f(s_0))$ and taking the positive x-axis direction as the reference orientation, we now construct the spatial parametric form using non-differential geometry.

Let s be an unknown real number. From the classical arc length formula, we obtain:

$$
\int_{s_0}^{s} \sqrt{1 + f'^2(s_0)}\,\mathrm{d}x = \frac{n}{\gamma}\,. \tag{4.1}
$$

Since the integrand may be non-integrable, and even when integrable typically requires solving transcendental equations, Equation (4.1) is practically intractable for analytical solutions. Here, solving the transcendental equation carries substantive significance — it's not merely mathematical exercise but theoretical necessity. Given that Formula (2.12) has established a paradigm through the construct $y = x^{\frac{3}{2}}$ (leveraging its analytical tractability), we postulate $s_n (n > 0)$ as the solution to Equation (4.1).

Therefore,

$$
\begin{aligned}
\Psi_{\mathrm{c}} \langle \gamma \wr n \rangle &= \frac{s_n - s_{n-1}}{\sqrt{(s_n - s_{n-1})^2 + (f(s_n) - f(s_{n-1}))^2}} \\
&= \iota \langle s_n - s_{n-1} \rangle \left(\left(\frac{f(s_n) - f(s_{n-1})}{s_n - s_{n-1}} \right)^2 + 1 \right)^{-\frac{1}{2}} , \\
\Psi_{\mathrm{s}} \langle \gamma \wr n \rangle &= \frac{f(s_n) - f(s_{n-1})}{\sqrt{(s_n - s_{n-1})^2 + (f(s_n) - f(s_{n-1}))^2}} \\
&= \iota \langle f(s_n) - f(s_{n-1}) \rangle \left(\left(\frac{s_n - s_{n-1}}{f(s_n) - f(s_{n-1})} \right)^2 + 1 \right)^{-\frac{1}{2}} .
\end{aligned}
$$

After the construction, the parameter n now satisfies $n \in \mathbb{Z}$ with $n \neq 0$, from which we obtain the non-differential geometric expression for the curve $y = f(x)$ as:

$$
\begin{cases}
x = \iota \langle \tau \rangle \lim\limits_{\substack{\mu \to \tau \\ \gamma \to \infty}} \dfrac{1}{\gamma} \sum\limits_{n=\iota\langle \tau \rangle}^{\lfloor \mu\gamma \rfloor} \iota \langle s_n - s_{n-1} \rangle \left(\left(\dfrac{f(s_n) - f(s_{n-1})}{s_n - s_{n-1}} \right)^2 + 1 \right)^{-\frac{1}{2}} + s_0 , \\
y = \iota \langle \tau \rangle \lim\limits_{\substack{\mu \to \tau \\ \gamma \to \infty}} \dfrac{1}{\gamma} \sum\limits_{n=\iota\langle \tau \rangle}^{\lfloor \mu\gamma \rfloor} \iota \langle f(s_n) - f(s_{n-1}) \rangle \left(\left(\dfrac{s_n - s_{n-1}}{f(s_n) - f(s_{n-1})} \right)^2 + 1 \right)^{-\frac{1}{2}} + f(s_0) .
\end{cases}
$$

The curve $y = f(x)$ intersects the y-axis at $(0, f(0))$. We now investigate an alternative mathematical representation of this curve based on a non-differential geometric framework.

Similarly,

$$
\begin{aligned}
\Psi_{\mathrm{t}} \langle \gamma \wr n \rangle &= \frac{f\left(\dfrac{n}{\gamma}\right) - f\left(\dfrac{n-1}{\gamma}\right)}{\dfrac{1}{\gamma}} \\
&= \gamma \left(f\left(\frac{n}{\gamma}\right) - f\left(\frac{n-1}{\gamma}\right) \right) .
\end{aligned}
$$

Let $\varkappa$ be the nonzero real variable corresponding to x, from which we obtain another parametric expression:

$$
y = \iota \langle x \rangle \lim_{\substack{\varkappa \to x \\ \gamma \to \infty}} \sum_{n=\iota\langle x \rangle}^{\lfloor \varkappa\gamma \rfloor} \left(f\left(\frac{n}{\gamma}\right) - f\left(\frac{n-1}{\gamma}\right) \right) + f(0) .
$$

It follows that the mathematical tools of non-differential geometry are fully compatible with classical differential geometry, and all problems in classical differential geometry can be transformed into non-differential geometric formulations for solution. Notably, when classical differential geometry only satisfies C^1 continuity yet requires computation of geometric quantities like curvature (which typically demands C^2 continuity), conversion to the non-differential geometric framework becomes advantageous—the latter requires merely C^0 continuity to achieve equivalent computations. However, if classical differential geometry inherently meets computational requirements (e.g., with C^2 continuity), conversion is unnecessary as non-differential geometric methods incur higher computational complexity. Therefore, the non-differential geometric tools in this study are by default applied to cases that cannot be addressed by classical differential geometry.

While differential geometry and non-differential geometry are theoretically compatible, and all conclusions in the differential geometry framework can be translated into non-differential formulations, the converse generally does not hold. Typical counterexamples include:

$$y = \iota \langle x \rangle \lim_{\substack{\varkappa \to x \\ \gamma \to \infty}} \frac{1}{\gamma} \sum_{n=\iota\langle x \rangle}^{\llcorner \varkappa\gamma \lrcorner} \frac{4n^3 + n^2 + 3n + \sin n}{\gamma n^2 + \gamma + n} + 2 \,.$$

In non-differential geometry, the rejection of differential structures necessitates discarding the integral symbol '$\int$'. However, to ensure compatibility when reducing to classical differential geometry, the symbol '$\int\!\!\!\!\sum$' is adopted for two essential reasons: (1) it inherently embodies infinite series (a fundamental feature of non-differential geometry), and (2) its visual similarity maintains conceptual linkage with classical integration theory. It must be expressly noted that while resembling an integral, '$\int\!\!\!\!\sum$' constitutes a fundamentally distinct operation.

Given a constant σ_0, define:

$$\sigma \langle x \rangle = \iota \langle x \rangle \lim_{\substack{\varkappa \to x \\ \gamma \to \infty}} \frac{1}{\gamma} \sum_{n=\iota\langle x \rangle}^{\llcorner \varkappa\gamma \lrcorner} \Psi \langle \gamma \wr n \rangle + \sigma_0 \,.$$

Since the curve $y = \sigma \langle x \rangle$ merely satisfies C^0 continuity, its tangent may not exist, thus precluding the definition of any concept analogous to the indefinite integral. Consequently, the symbol '$\int\!\!\!\!\sum$' must explicitly specify upper and lower bounds, with γ^{-1} defined as a non-differential operator, and its properties studied on the interval $[a, b]$.

There must exist a definite real number T satisfying the following formula:

$$\mathop{\int\!\!\!\!\!\sum}_a^b \sigma \langle x \rangle \, \gamma^{-1} x = T \,. \tag{4.2}$$

Analogous in form to a definite integral but distinct in nature, in Equation (4.2), γ^{-1} acts as a non-differential operator, where its operand $\gamma^{-1}x$ must be treated as an indivisible entity, precluding any internal operations. Since the left-hand side of Equation (4.2) is not a definite integral, and $y = \sigma \langle x \rangle$ does not constitute a function (merely satisfying C^0 continuity), the Newton-Leibniz formula is clearly inapplicable. We therefore must investigate its mathematical properties anew.

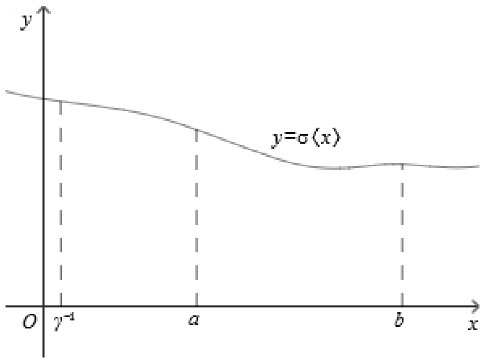

Figure 4.1: To perform computations based on non-differential operators

Distinct from conventional definite integrals (see Figure 4.1), we must employ non-differential operators for analytical computation. Therefore:

$$\sum\!\!\!\!\!\!\int_a^b \sigma\langle x\rangle\,\gamma^{-1}x = \sum\!\!\!\!\!\!\int_0^b \sigma\langle x\rangle\,\gamma^{-1}x - \sum\!\!\!\!\!\!\int_0^a \sigma\langle x\rangle\,\gamma^{-1}x\,.$$

Whereas:

$$\sum\!\!\!\!\!\!\int_0^b \sigma\langle x\rangle\,\gamma^{-1}x = \lim_{\gamma\to\infty}\sum_{n=\iota\langle b\rangle}^{\llcorner b\gamma\lrcorner}\frac{1}{\gamma}\sigma\left\langle\frac{n}{\gamma}\right\rangle\,. \tag{4.3}$$

Let $m=-n,\cdots,-2,-1,1,2,\cdots,n$. It follows that:

$$\sigma\left\langle\frac{n}{\gamma}\right\rangle = \iota\langle n\rangle \lim_{\gamma\to\infty}\frac{1}{\gamma}\sum_{m=\iota\langle n\rangle}^{n}\Psi\langle\gamma\wr m\rangle + \sigma_0\,.$$

Substitute it into Equation (4.3), i.e.,

$$\begin{aligned}
\sum\!\!\!\!\!\!\int_0^b \sigma\langle x\rangle\,\gamma^{-1}x &= \lim_{\gamma\to\infty}\frac{1}{\gamma}\sum_{n=\iota\langle b\rangle}^{\llcorner b\gamma\lrcorner}\left(\iota\langle n\rangle\frac{1}{\gamma}\sum_{m=\iota\langle n\rangle}^{n}\Psi\langle\gamma\wr m\rangle+\sigma_0\right)\\
&= \lim_{\gamma\to\infty}\frac{1}{\gamma^2}\sum_{n=\iota\langle b\rangle}^{\llcorner b\gamma\lrcorner}\iota\langle n\rangle\sum_{m=\iota\langle n\rangle}^{n}\Psi\langle\gamma\wr m\rangle + b\sigma_0\\
&= \lim_{\gamma\to\infty}\frac{1}{\gamma^2}\sum_{n=\iota\langle b\rangle}^{\llcorner b\gamma\lrcorner}\left(\llcorner b\gamma\lrcorner - n + \iota\langle b\rangle\right)\Psi\langle\gamma\wr n\rangle + b\sigma_0\,.
\end{aligned}$$

Therefore,

$$\sum\!\!\!\!\!\!\int_0^a \sigma\langle x\rangle\,\gamma^{-1}x = \lim_{\gamma\to\infty}\frac{1}{\gamma^2}\sum_{n=\iota\langle a\rangle}^{\llcorner a\gamma\lrcorner}\left(\llcorner a\gamma\lrcorner - n + \iota\langle a\rangle\right)\Psi\langle\gamma\wr n\rangle + a\sigma_0\,.$$

We immediately obtain:

$$\mathop{\sum\!\!\!\!\!\!\int}_a^b \sigma\left\langle x\right\rangle \gamma^{-1}x = \lim_{\gamma\to\infty}\frac{1}{\gamma^2}\left(\sum_{n=\iota\langle b\rangle}^{\lfloor b\gamma\rfloor}\left(\lfloor b\gamma\rfloor - n + \iota\left\langle b\right\rangle\right)\Psi\left\langle \gamma \wr n\right\rangle - \sum_{n=\iota\langle a\rangle}^{\lfloor a\gamma\rfloor}\left(\lfloor a\gamma\rfloor - n + \iota\left\langle a\right\rangle\right)\Psi\left\langle \gamma \wr n\right\rangle\right)$$
$$+ (b-a)\,\sigma_0 \,.$$

Let the curve $y = \sigma\langle x\rangle$ have length l over the interval $[a, b]$. We now proceed to determine l, i.e.:

$$l = \lim_{\gamma\to\infty}\left(\iota\left\langle b\right\rangle \sum_{n=\iota\langle b\rangle}^{\lfloor b\gamma\rfloor}\sqrt{\left(\frac{1}{\gamma}\right)^2 + \left(\sigma\left\langle\frac{n}{\gamma}\right\rangle - \sigma\left\langle\frac{n-1}{\gamma}\right\rangle\right)^2} - \iota\left\langle a\right\rangle \sum_{n=\iota\langle a\rangle}^{\lfloor a\gamma\rfloor}\sqrt{\left(\frac{1}{\gamma}\right)^2 + \left(\sigma\left\langle\frac{n}{\gamma}\right\rangle - \sigma\left\langle\frac{n-1}{\gamma}\right\rangle\right)^2}\right).$$

Since:

$$\begin{aligned}
\sigma\left\langle\frac{n}{\gamma}\right\rangle - \sigma\left\langle\frac{n-1}{\gamma}\right\rangle &= \iota\left\langle n\right\rangle \lim_{\gamma\to\infty}\frac{1}{\gamma}\sum_{m=\iota\langle n\rangle}^{n}\Psi\left\langle\gamma\wr m\right\rangle + \sigma_0 - \left(\iota\left\langle n\right\rangle \lim_{\gamma\to\infty}\frac{1}{\gamma}\sum_{m=\iota\langle n\rangle}^{n-1}\Psi\left\langle\gamma\wr m\right\rangle + \sigma_0\right)\\
&= \iota\left\langle n\right\rangle \lim_{\gamma\to\infty}\frac{1}{\gamma}\sum_{m=\iota\langle n\rangle}^{n}\Psi\left\langle\gamma\wr m\right\rangle - \iota\left\langle n\right\rangle \lim_{\gamma\to\infty}\frac{1}{\gamma}\sum_{m=\iota\langle n\rangle}^{n}\Psi\left\langle\gamma\wr m\right\rangle + \iota\left\langle n\right\rangle \lim_{\gamma\to\infty}\frac{1}{\gamma}\Psi\left\langle\gamma\wr n\right\rangle\\
&= \iota\left\langle n\right\rangle \lim_{\gamma\to\infty}\frac{1}{\gamma}\Psi\left\langle\gamma\wr n\right\rangle \,.
\end{aligned}$$

Therefore,

$$\begin{aligned}
l &= \lim_{\gamma\to\infty}\left(\iota\left\langle b\right\rangle \sum_{n=\iota\langle b\rangle}^{\lfloor b\gamma\rfloor}\sqrt{\frac{1}{\gamma^2} + \frac{1}{\gamma^2}\Psi^2\left\langle\gamma\wr n\right\rangle} - \iota\left\langle a\right\rangle \sum_{n=\iota\langle a\rangle}^{\lfloor a\gamma\rfloor}\sqrt{\frac{1}{\gamma^2} + \frac{1}{\gamma^2}\Psi^2\left\langle\gamma\wr n\right\rangle}\right)\\
&= \lim_{\gamma\to\infty}\frac{1}{\gamma}\left(\iota\left\langle b\right\rangle \sum_{n=\iota\langle b\rangle}^{\lfloor b\gamma\rfloor}\sqrt{1 + \Psi^2\left\langle\gamma\wr n\right\rangle} - \iota\left\langle a\right\rangle \sum_{n=\iota\langle a\rangle}^{\lfloor a\gamma\rfloor}\sqrt{1 + \Psi^2\left\langle\gamma\wr n\right\rangle}\right).
\end{aligned}$$

Whereas:

$$\begin{aligned}
\mathop{\sum\!\!\!\!\!\!\int}_a^b \sqrt{1 + \Psi^2\left\langle\gamma\wr n\right\rangle}\gamma^{-1}x &= \mathop{\sum\!\!\!\!\!\!\int}_0^b \sqrt{1 + \Psi^2\left\langle\gamma\wr n\right\rangle}\gamma^{-1}x - \mathop{\sum\!\!\!\!\!\!\int}_0^a \sqrt{1 + \Psi^2\left\langle\gamma\wr n\right\rangle}\gamma^{-1}x\\
&= \lim_{\gamma\to\infty}\frac{1}{\gamma}\left(\iota\left\langle b\right\rangle \sum_{n=\iota\langle b\rangle}^{\lfloor b\gamma\rfloor}\sqrt{1 + \Psi^2\left\langle\gamma\wr n\right\rangle} - \iota\left\langle a\right\rangle \sum_{n=\iota\langle a\rangle}^{\lfloor a\gamma\rfloor}\sqrt{1 + \Psi^2\left\langle\gamma\wr n\right\rangle}\right).
\end{aligned}$$

Thus it follows that:

$$l = \mathop{\sum\!\!\!\!\!\!\int}_a^b \sqrt{1 + \Psi^2\left\langle\gamma\wr n\right\rangle}\gamma^{-1}x \,.$$

Let $\sigma_{0,x}$ and $\sigma_{0,y}$ be constants. If the curve is defined by the following expression:

$$
\begin{cases}
\sigma_x \langle \tau \rangle = \iota \langle \tau \rangle \lim\limits_{\substack{\mu \to \tau \\ \gamma \to \infty}} \dfrac{1}{\gamma} \sum\limits_{n=\iota\langle \tau \rangle}^{\lfloor \mu\gamma \rfloor} \Psi_1 \langle \gamma \wr n \rangle + \sigma_{0,x} \,, \\
\sigma_y \langle \tau \rangle = \iota \langle \tau \rangle \lim\limits_{\substack{\mu \to \tau \\ \gamma \to \infty}} \dfrac{1}{\gamma} \sum\limits_{n=\iota\langle \tau \rangle}^{\lfloor \mu\gamma \rfloor} \Psi_2 \langle \gamma \wr n \rangle + \sigma_{0,y} \,.
\end{cases}
$$

We now determine the length l^* of the curve segment over the closed interval $[a, b]$:

$$
\begin{aligned}
l^* &= \lim_{\gamma \to \infty} \sum_{n=\iota\langle b \rangle}^{\lfloor b\gamma \rfloor} \left(\sqrt{\left(\sigma_x \left\langle \frac{n}{\gamma} \right\rangle - \sigma_x \left\langle \frac{n-1}{\gamma} \right\rangle\right)^2 + \left(\sigma_y \left\langle \frac{n}{\gamma} \right\rangle - \sigma_y \left\langle \frac{n-1}{\gamma} \right\rangle\right)^2} \right. \\
&\quad \left. - \sum_{n=\iota\langle a \rangle}^{\lfloor a\gamma \rfloor} \sqrt{\left(\sigma_x \left\langle \frac{n}{\gamma} \right\rangle - \sigma_x \left\langle \frac{n-1}{\gamma} \right\rangle\right)^2 + \left(\sigma_y \left\langle \frac{n}{\gamma} \right\rangle - \sigma_y \left\langle \frac{n-1}{\gamma} \right\rangle\right)^2} \right) \\
&= \lim_{\gamma \to \infty} \left(\sum_{n=\iota\langle b \rangle}^{\lfloor b\gamma \rfloor} \sqrt{\frac{1}{\gamma^2} \Psi_1^2 \langle \gamma \wr n \rangle + \frac{1}{\gamma^2} \Psi_2^2 \langle \gamma \wr n \rangle} - \sum_{n=\iota\langle a \rangle}^{\lfloor a\gamma \rfloor} \sqrt{\frac{1}{\gamma^2} \Psi_1^2 \langle \gamma \wr n \rangle + \frac{1}{\gamma^2} \Psi_2^2 \langle \gamma \wr n \rangle} \right) \\
&= \lim_{\gamma \to \infty} \frac{1}{\gamma} \left(\sum_{n=\iota\langle b \rangle}^{\lfloor b\gamma \rfloor} \sqrt{\Psi_1^2 \langle \gamma \wr n \rangle + \Psi_2^2 \langle \gamma \wr n \rangle} - \sum_{n=\iota\langle a \rangle}^{\lfloor a\gamma \rfloor} \sqrt{\Psi_1^2 \langle \gamma \wr n \rangle + \Psi_2^2 \langle \gamma \wr n \rangle} \right) .
\end{aligned}
$$

Similarly:

$$
\fint_a^b \sqrt{\Psi_1^2 \langle \gamma \wr n \rangle + \Psi_2^2 \langle \gamma \wr n \rangle \gamma^{-1} x} = \lim_{\gamma \to \infty} \frac{1}{\gamma} \left(\sum_{n=\iota\langle b \rangle}^{\lfloor b\gamma \rfloor} \sqrt{\Psi_1^2 \langle \gamma \wr n \rangle + \Psi_2^2 \langle \gamma \wr n \rangle} - \sum_{n=\iota\langle a \rangle}^{\lfloor a\gamma \rfloor} \sqrt{\Psi_1^2 \langle \gamma \wr n \rangle + \Psi_2^2 \langle \gamma \wr n \rangle} \right) .
$$

Therefore,

$$
l^* = \fint_a^b \sqrt{\Psi_1^2 \langle \gamma \wr n \rangle + \Psi_2^2 \langle \gamma \wr n \rangle \gamma^{-1} x} \,.
$$

Since this study is of a pioneering nature, the rigorous definition of "integration" on curved surfaces and higher-dimensional geometric objects (note: the term "integration" here does not refer to the classical theory but to an analogue of definite integrals outside the framework of differential geometry) will most likely require a new axiomatic system for support. Therefore, this paper will refrain from delving into the discussion for now.

4.2 Generalized proof of the shortest property of straight line segments between two points in Euclidean space: application of non-differential geometric tools

In a k-dimensional Euclidean space, consider a point O (the origin) and an arbitrary point W. We aim to prove that among all continuous curves connecting O and W, the straight-line segment is the shortest path. For simplicity in the proof, we assume O to be the coordinate origin.

Let the coordinates of point W be $(w_1, w_2, \ldots, w_k)$. Then:

$$|OW| = \sqrt{\sum_{j=1}^{k} w_j^2} \, .$$

Construct a k-dimensional sphere centered at the origin O with radius $|OW|$. On this sphere, take a point W' such that the vector $\overrightarrow{OW'}$ is parallel to the first coordinate axis x_1, and moreover, $\overrightarrow{OW'}$ is codirectional with the x_1-axis. Let the sphere's radius be $r = |OW| = |OW'|$.

To prove that among all continuous curves connecting O and W, the straight-line segment is the shortest, it is equivalent to prove: For any rectifiable continuous curve Ω connecting O and W', its arc length $L(\Omega)$ satisfies $L(\Omega) \geqslant |OW'|$ (the linear distance). We now parameterize the curve Ω using spatial parameter τ, expressed as:

$$\begin{cases} x_{(1)} = \iota\langle\tau\rangle \lim\limits_{\substack{\mu\to\tau \\ \gamma\to\infty}} \dfrac{1}{\gamma} \sum\limits_{i=\iota\langle\tau\rangle}^{\lfloor\mu\gamma\rfloor} \prod\limits_{j=1}^{k-1} \cos\theta_{(j),i} \, , \\ x_{(2)} = \iota\langle\tau\rangle \lim\limits_{\substack{\mu\to\tau \\ \gamma\to\infty}} \dfrac{1}{\gamma} \sum\limits_{i=\iota\langle\tau\rangle}^{\lfloor\mu\gamma\rfloor} \prod\limits_{j=1}^{k-2} \cos\theta_{(j),i} \sin\theta_{(k-1),i} \, , \\ \quad \cdots\cdots\cdots\cdots \\ x_{(k-1)} = \iota\langle\tau\rangle \lim\limits_{\substack{\mu\to\tau \\ \gamma\to\infty}} \dfrac{1}{\gamma} \sum\limits_{i=\iota\langle\tau\rangle}^{\lfloor\mu\gamma\rfloor} \cos\theta_{(1),i} \sin\theta_{(2),i} \, , \\ x_{(k)} = \iota\langle\tau\rangle \lim\limits_{\substack{\mu\to\tau \\ \gamma\to\infty}} \dfrac{1}{\gamma} \sum\limits_{i=\iota\langle\tau\rangle}^{\lfloor\mu\gamma\rfloor} \sin\theta_{(1),i} \, . \end{cases}$$

Let $\tau_w (\tau_w > 0)$ denote the value of parameter τ at point W'. Then we have:

$$\begin{aligned} L(\Omega) &= \tau_w \\ &= \lim_{\substack{\mu\to\tau_w \\ \gamma\to\infty}} \frac{1}{\gamma} \sum_{i=\iota\langle\tau_w\rangle}^{\lfloor\mu\gamma\rfloor} 1 \, , \\ |OW'| &= \lim_{\substack{\mu\to\tau_w \\ \gamma\to\infty}} \frac{1}{\gamma} \sum_{i=\iota\langle\tau_w\rangle}^{\lfloor\mu\gamma\rfloor} \prod_{j=1}^{k-1} \cos\theta_{(j),i} \, . \end{aligned}$$

Since:

$$\lim_{\substack{\mu\to\tau_w \\ \gamma\to\infty}} \frac{1}{\gamma} \sum_{i=\iota\langle\tau_w\rangle}^{\lfloor\mu\gamma\rfloor} 1 \geqslant \lim_{\substack{\mu\to\tau_w \\ \gamma\to\infty}} \frac{1}{\gamma} \sum_{i=\iota\langle\tau_w\rangle}^{\lfloor\mu\gamma\rfloor} \prod_{j=1}^{k-1} \cos\theta_{(j),i} \, .$$

Hence:

$$L(\Omega) \geqslant |OW'| \, .$$

Thus, by ingeniously employing the properties of multidimensional spheres and non-differential geometric methods, we have concisely and intuitively demonstrated the minimality of straight-line segments between two points in Euclidean space.

References

[1] Hirschman, I. I. *Infinite Series* (pp. 1–3). Dover Publications, 2014. (Reprint of 1962 edition).
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Since this theory differs paradigmatically from existing mathematical systems at the level of foundational axioms, apart from the basic concepts mentioned above, there are no directly citable references available at this time.