Part 2 of Guide to Hestenes's Geometric Algebra Treatment of Constant-Acceleration (Parabolic) Motion

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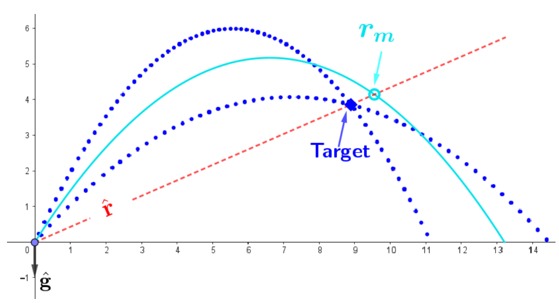
Abstract

As an aid to teachers and students who are learning to apply Geometric Algebra to high-school-level physics, we provide this second installment in our guide guide to Hestenes's treatment of constant-acceleration motion. Specifically, we present a more-detailed version of Hestenes's solution to the problem of finding the time and distance at which a projectile will reach a specific point along a given line of sight. We begin by reviewing the GA ideas that we will use, and finish by verifying the solution via a GeoGebra worksheet.



The open turquoise point is at the maximum range (r_m) that can be reached along the line of sight $\hat{\mathbf{r}}$ by a projectile whose initial velocity is v_o. The turquoise curve is the trajectory of such a projectile. The blue curves are the two trajectories that hit a target at range $r < r_{max}$ along $\hat{\mathbf{r}}$.

Contents

1 Introduction

This document continues our presentation and explanation of Hestenes's "GA" treatment of constant-acceleration motion. Like the introduction that we presented in a previous document ([1]), the present one has been prepared in the spirit of Hestenes's observation that students will need "judicious guidance" to get through his book *New Foundations for Classical Mechanics.*[1]. Therefore, this document is intended to be understandable by students and teachers who are still in the process of learning the basics of GA.

Our approach differs from what the reader may have experienced in mass-market textbooks, whose authors (because of length restrictions imposed by publishers) tend to present the most efficient possible derivations possible for their formulas. This approach can be intimidating for students, who are seldom aware that those formulas were almost never found via these efficient routes. Instead, someone in the past had an insight, then "followed her nose", thus arriving at a useful result that she (or others) later found a way to derive more efficiently. Indeed, that process is more or less the same way in which good problem-solvers (including students) often work. For that reason, our approach here will have a similar "exploratory" spirit.

The examples that are usually presented when teaching constant-acceleration motion concern the trajectories of projectiles. That is the language that will be used here, but the analyses, equations, and solutions hold for any situations in which the acceleration is constant.

Please note that we don't use the terms "division of vectors" or "division of bivectors". Those terms (as well as equations that are written in terms of such divisions) can be ambiguous in ways that may confuse the student. Therefore, we will use the multiplicative inverses that those "divisions" actually represent.

2 Comments on the Scope of this Document, and on the Section Entitled "Maximum Range" in Hestenes's *New Foundations for Classical Mechanics (NFCM)*

Because Hestenes's "Maximum Range" section also treats many other aspects of constant-acceleration motion, the material in that section might, with benefit, have been structured differently, by dividing it into two or three shorter sections, with a stronger "thread" to indicate the goal toward which the work was proceeding at each point. One of our purposes here is to provide such structure.

In the present document, we treat only the portions that deal with maximum range *per se*. Specifically, we will cover as far as Hestenes's Eq. 2.14, p. 130. The remainder of the material in the "Maximum range" section will be covered

[1]"Though my book has been a continual best seller in the series for well over a decade, it is still unknown to most teachers of mechanics in the U.S. To be suitable for the series, I had to design it as a multipurpose book, including a general introduction to GA and material of interest to researchers, as well as problem sets for students. It is not what I would have written to be a mechanics textbook alone. Most students need judicious guidance by the instructor to get through it." [3]

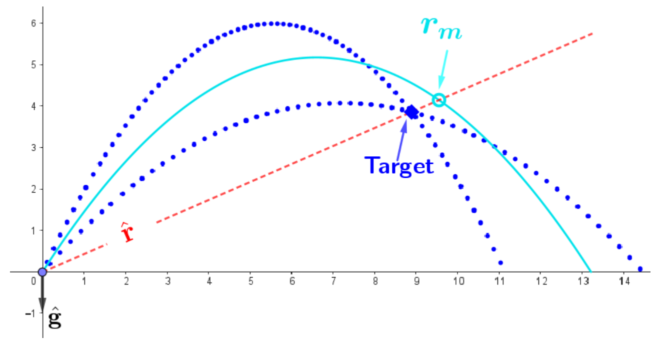


Figure 1: The open turquoise point is at the maximum range (r_m) that can be reached along the line of sight $\hat{\mathbf{r}}$ by a projectile whose initial velocity is v_o. The turquoise curve is the trajectory of such a projectile. The blue curves are the two trajectories that hit a target at range $r < r_{max}$ along $\hat{\mathbf{r}}$.

in a subsequent document.

3 What We Will See in this Document

- Discussion of Hestenes's derivation and transformation of an equation for the range r at which the projectile will cross a line of sight $\hat{\mathbf{r}}$ if launched with an initial velocity v_o in the direction $\hat{\mathbf{v}}_o$.

- Use of the transformed equation to solve the following problems.

  - With reference to Figs. 1 and 2:

    - What is the launch direction $\hat{\mathbf{v}}_o$ that gives the greatest range r along a given direction $\hat{\mathbf{r}}$?
    - What is that maximum r?
    - How does the maximum attainable r vary with $\hat{\mathbf{r}}$?
    - What is the region in space in which targets may be reached by projectiles that are launched at initial velocity v_o?

  - (A calculation not treated by Hestenes) For a given initial velocity v_o, what is the direction $\hat{\mathbf{v}}_o$ in which a projectile must be launched in order to hit a target at distance $r < r_{max}$ along the direction $\hat{\mathbf{r}}$?

  - What are the flight times for the projectiles in each of the above problems?

- "Sanity checks" of intermediate results.

- Validations of solutions via a GeoGebra construction.

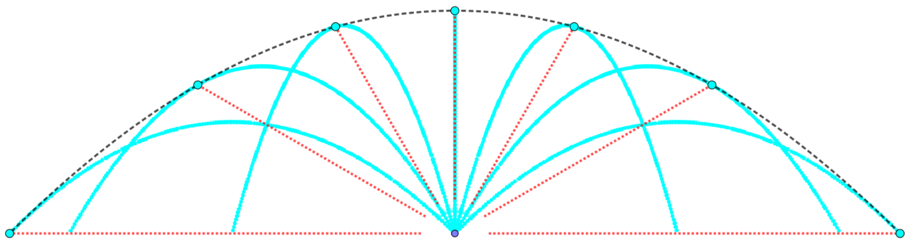


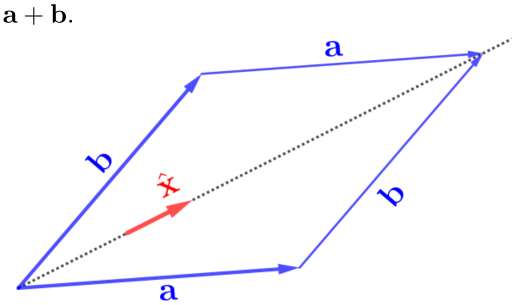
Figure 2: The black dashed curve is the boundary of the region of space in which a target can be hit by a projectile whose initial velocity is v_o. Open circles along that curve show the most-distant targets that can be reached along their respective lines of sight (dotted red lines). Turquoise curves show the trajectories that reach of those most-distant points.

4 Ideas that We Will Use

- Hestenes's Eq. (2.8) ([2], p.128):

$$r = \frac{(\mathbf{g} \wedge \mathbf{v}_o)\,(\hat{\mathbf{r}} \wedge \mathbf{v}_o)}{(\mathbf{g} \wedge \hat{\mathbf{r}})^2} = \frac{(\mathbf{g} \wedge \mathbf{v}_o) \cdot (\mathbf{v}_o \wedge \hat{\mathbf{r}})}{\|\mathbf{g} \wedge \hat{\mathbf{r}}\|^2}.$$

- The product $\hat{\mathbf{w}}\mathbf{a}\hat{\mathbf{w}}$ evaluates to the reflection of $\mathbf{a}$ with respect to $\mathbf{w}$. Therefore, in the equation $\hat{\mathbf{x}}\mathbf{a}\hat{\mathbf{x}} = \mathbf{b}$, the vector $\mathbf{x}$ is parallel to the vector $\mathbf{a} + \mathbf{b}$.



- Any vector $\mathbf{u}$ can be written as the sum of its components with respect to a second vector $\hat{\mathbf{a}}$ and the vector $\hat{\mathbf{a}}\mathbf{i}$. Specifically, $\mathbf{u} = (\mathbf{u} \cdot \hat{\mathbf{a}})\,\hat{\mathbf{a}} + [\mathbf{u} \cdot (\hat{\mathbf{a}}\mathbf{i})]\,\hat{\mathbf{a}}\mathbf{i}$ (Fig. 3). Note that $u^2 = (\mathbf{u} \cdot \hat{\mathbf{a}})^2 + [\mathbf{u} \cdot (\hat{\mathbf{a}}\mathbf{i})]^2$.
  - Consequently, if we know the magnitude of $\mathbf{u}$, and if we know the value of $\mathbf{u} \cdot \mathbf{a}$, then we can identify the two possible vectors "$\mathbf{u}$". We start by noting that $\mathbf{u} \cdot \hat{\mathbf{a}} = [\mathbf{u} \cdot \mathbf{a}]\,/a$, after which the two possible values of $\mathbf{u} \cdot (\hat{\mathbf{a}}\mathbf{i})$ are (Fig. 4)

$$\mathbf{u} \cdot (\hat{\mathbf{a}}\mathbf{i}) = \pm\sqrt{u^2 - (\mathbf{u} \cdot \hat{\mathbf{a}})^2}. \qquad (1)$$

    This idea is often useful when solving problems that involve circles ([4]) .

- Three GA identities:

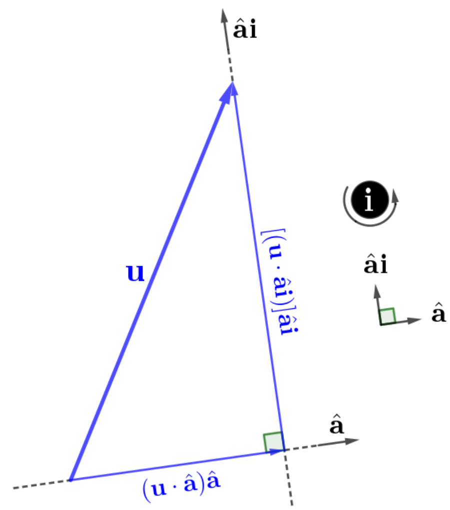


Figure 3: An example of expressing a vector $\mathbf{u}$ as the sum of its projections upon two mutually perpendicular unit vectors: $\mathbf{u} = (\mathbf{u} \cdot \hat{\mathbf{a}})\,\hat{\mathbf{a}} + [\mathbf{u} \cdot (\hat{\mathbf{a}}\mathbf{i})]\,\hat{\mathbf{a}}\mathbf{i}$.

- $2\,(\mathbf{a} \wedge \mathbf{b}) \cdot (\mathbf{b} \wedge \mathbf{c}) = b^2\mathbf{a} \cdot \mathbf{c} - \mathbf{a} \cdot (\mathbf{bcb})$ ([2], p. 71, Exercise 4.8d)
- $\mathbf{a} \wedge \mathbf{b} = [(\mathbf{ai}) \cdot \mathbf{b}]\,\mathbf{i} = {}^{-}[\mathbf{a} \cdot (\mathbf{bi})]\,\mathbf{i}$ ([5])
- $\|\mathbf{a} \wedge \mathbf{b}\|^2 = a^2b^2 - (\mathbf{a} \cdot \mathbf{b})^2$. This identity can be derived as follows:

$$
\begin{aligned}
\|\mathbf{a} \wedge \mathbf{b}\|^2 &= (\mathbf{a} \wedge \mathbf{b})\,(\mathbf{b} \wedge \mathbf{a}) \quad \text{(by definition)} \\
&= (\mathbf{ab} - \mathbf{a} \cdot \mathbf{b})\,(\mathbf{ba} - \mathbf{a} \cdot \mathbf{b}) \\
&= \mathbf{abba} - (\mathbf{a} \cdot \mathbf{b})\,\underbrace{[\mathbf{ab} + \mathbf{ba}]}_{=2\mathbf{a}\cdot\mathbf{b}} + (\mathbf{a} \cdot \mathbf{b})^2 \\
&= a^2b^2 - (\mathbf{a} \cdot \mathbf{b})^2 \,.
\end{aligned}
$$

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5 The $\hat{\mathbf{v}}_o$ that Gives the Maximum Range (r_{max}) Along a Given Direction $\hat{\mathbf{r}}$

5.1 Preliminary Observations

It would be interesting to know how Hestenes found the key ideas that he needed in order to derive the results that he presents in this section. Here, we will follow what appears to have been his thought process. First, we recall that in the previous section of *NFCM* (i.e., on p. 128), Hestenes had derived his Eq. 2.8, which allowed him to calculate the distance r at which a projectile that was

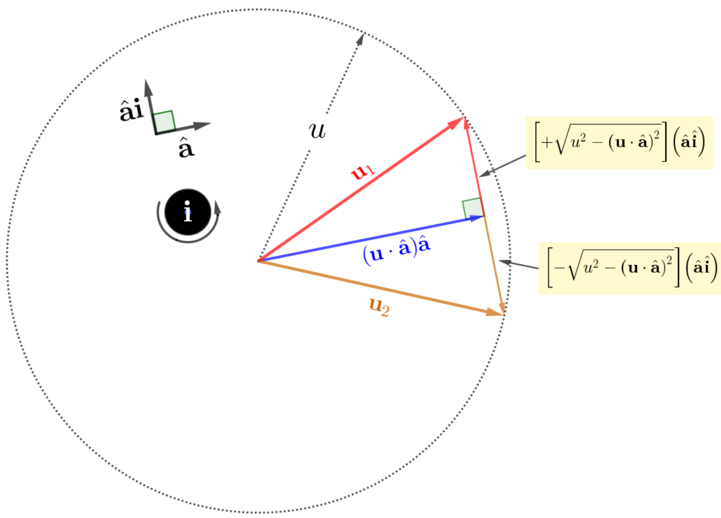

Figure 4: If we know the inner product of an unknown vector $\mathbf{u}$ with a known vector $\mathbf{a}$, then the two possible vectors "$\mathbf{u}$" are $\mathbf{u}_1 = (\mathbf{u}\cdot\hat{\mathbf{a}})\,\hat{\mathbf{a}} + \left[\sqrt{u^2-(\mathbf{u}\cdot\hat{\mathbf{a}})^2}\right]\hat{\mathbf{a}}\mathbf{i}$ and $\mathbf{u}_2 = (\mathbf{u}\cdot\hat{\mathbf{a}})\,\hat{\mathbf{a}} - \left[\sqrt{u^2-(\mathbf{u}\cdot\hat{\mathbf{a}})^2}\right]\hat{\mathbf{a}}\mathbf{i}$.

launched with velocity vector $\mathbf{v}$ would cross the line of sight $\hat{\mathbf{r}}$:

$$r = \frac{2\,(\mathbf{g}\wedge\mathbf{v}_o)\,(\hat{\mathbf{r}}\wedge\mathbf{v}_o)}{(\mathbf{g}\wedge\hat{\mathbf{r}})^2}. \tag{2}$$

Now, at the beginning of his "Maximum range" section, Hestenes notes that although his Eq. 2.8 (which is our Eq. (2)) allowed him to calculate r for a given $\hat{\mathbf{v}}_o$, that equation is not suitable for understanding how r will be affected by variations in $\hat{\mathbf{v}}_o$. This is because an increase in $\hat{\mathbf{g}}\wedge\hat{\mathbf{v}}_o$ will be accompanied by a decrease in $\hat{\mathbf{r}}\wedge\hat{\mathbf{v}}_o$, so their product might either increase or decrease. Therefore, Hestenes sought to transform his Eq. (2.8) into a version that would make clear the functional relationship between r and $\hat{\mathbf{v}}_o$.

5.2 Transforming Hestenes's Eq. 2.8 (Our Eq. (2)) to Find r_{max} and the $\hat{\mathbf{v}}_o$ Necessary to Reach It

Where might we find ideas for effecting the necessary transformation? Many possibilities are found in the extensive collection of GA identities that Hestenes presented earlier in NFCM. The identity that Hestenes chose is

$$2\,(\mathbf{a}\wedge\mathbf{b})\cdot(\mathbf{b}\wedge\mathbf{c}) = b^2\mathbf{a}\cdot\mathbf{c} - \mathbf{a}\cdot(\mathbf{bcb})\,. \tag{3}$$

The numerator of Eq. (2) will coincide with the left-hand side of this identity under two conditions: (1) If it is true that $(\mathbf{g}\wedge\mathbf{v}_o)\,(\hat{\mathbf{r}}\wedge\mathbf{v}_o) = (\mathbf{g}\wedge\mathbf{v}_o)\cdot(\hat{\mathbf{r}}\wedge\mathbf{v}_o)$, and (2) if we change $\hat{\mathbf{r}}\wedge\mathbf{v}_o$ to $-(\mathbf{v}_o\wedge\hat{\mathbf{r}})$. Fortunately, it is indeed true that

$(\mathbf{g} \wedge \mathbf{v}_o)(\hat{\mathbf{r}} \wedge \mathbf{v}_o) = (\mathbf{g} \wedge \mathbf{v}_o) \cdot (\hat{\mathbf{r}} \wedge \mathbf{v}_o)$ because vectors $\mathbf{g}$, $\mathbf{v}_o$, and $\hat{\mathbf{r}}$ are all parallel ("coplanar"). Making use of this fact, and that $(\mathbf{g} \wedge \hat{\mathbf{r}})^2 = -\|\mathbf{g} \wedge \hat{\mathbf{r}}\|^2$, Hestenes took the preliminary step of putting Eq. (2) into the form that is necessary for transforming it via the identity in Eq. (3). That is, he rewrote Eq. (2) as

$$r = \frac{2\,(\mathbf{g} \wedge \mathbf{v}_o)\,(\hat{\mathbf{r}} \wedge \mathbf{v}_o)}{(\mathbf{g} \wedge \hat{\mathbf{r}})^2} = \frac{2\,(\mathbf{g} \wedge \mathbf{v}_o) \cdot (\mathbf{v}_o \wedge \hat{\mathbf{r}})}{\|\mathbf{g} \wedge \hat{\mathbf{r}}\|^2}. \tag{4}$$

Now, applying Eq. (3),

$$\begin{aligned} r &= \frac{2\,(\mathbf{g} \wedge \mathbf{v}_o) \cdot (\mathbf{v}_o \wedge \hat{\mathbf{r}})}{\|\mathbf{g} \wedge \hat{\mathbf{r}}\|^2} \\ &= \frac{v_0^2\,[(g\hat{\mathbf{g}}) \cdot \hat{\mathbf{r}}] - (g\hat{\mathbf{g}}) \cdot [(v_o\hat{\mathbf{v}}_0)\,\hat{\mathbf{r}}]\,(v_o\hat{\mathbf{v}}_0)}{\|\,(g\hat{\mathbf{g}}) \wedge \hat{\mathbf{r}}\|^2} \\ &= \left[\frac{v_o^2}{g\|\hat{\mathbf{g}} \wedge \hat{\mathbf{r}}\|^2}\right]\left[\hat{\mathbf{g}} \cdot \hat{\mathbf{r}} - \hat{\mathbf{g}} \cdot (\hat{\mathbf{v}}_0\hat{\mathbf{r}}\hat{\mathbf{v}}_0)\right]. \end{aligned} \tag{5}$$

To identify the $\hat{\mathbf{v}}_o$ that gives the maximum r, Hestenes needed only to note that because $\hat{\mathbf{r}}$, v_0, $\hat{\mathbf{g}}$, and g are all constant, the value of r is determined exclusively by the value of $\hat{\mathbf{g}} \cdot \hat{\mathbf{r}}$. Furthermore, the product $\hat{\mathbf{v}}_0\hat{\mathbf{r}}\hat{\mathbf{v}}_0$ evaluates to a unit vector, so the possible values of $\hat{\mathbf{g}} \cdot (\hat{\mathbf{v}}_0\hat{\mathbf{r}}\hat{\mathbf{v}}_0)$ are $-1 \leq \hat{\mathbf{g}} \cdot (\hat{\mathbf{v}}_0\hat{\mathbf{r}}\hat{\mathbf{v}}_0) \leq 1$. Of these, "$-1$" is the one that maximizes r.

At this, point we could plunge in and calculate the (supposedly) maximum value of r by simply substituting -1 for $\hat{\mathbf{g}} \cdot (\hat{\mathbf{v}}_0\hat{\mathbf{r}}\hat{\mathbf{v}}_0)$ in Eq. (5). But before we do so, it is prudent to find out whether "$\hat{\mathbf{g}} \cdot (\hat{\mathbf{v}}_0\hat{\mathbf{r}}\hat{\mathbf{v}}_0) = -1$" is physically possible.

What is the specific $\hat{\mathbf{v}}o$ that gives r_{max}? Because $\hat{\mathbf{v}}_o$ bisects the angle between $\hat{\mathbf{r}}$ and $-\hat{\mathbf{g}}$, $\hat{\mathbf{v}}_o$ is parallel to the vector $(\hat{\mathbf{r}} + -\hat{\mathbf{g}})$. Also, $\hat{\mathbf{v}}_o$ is a unit vector, so $\hat{\mathbf{v}}_o = \pm(\hat{\mathbf{r}} - \hat{\mathbf{g}})/\|\hat{\mathbf{r}} - \hat{\mathbf{g}}\|$. For the problem that we are treating here, only the "$+$" solution is valid. (Why is $\hat{\mathbf{v}}_o = -(\hat{\mathbf{r}} - \hat{\mathbf{g}})/\|\hat{\mathbf{r}} - \hat{\mathbf{g}}\|$ not a solution to our problem?)

To find out, we begin by noting that in order for $\hat{\mathbf{g}} \cdot (\hat{\mathbf{v}}_0\hat{\mathbf{r}}\hat{\mathbf{v}}_0)$ to equal -1, it is necessary that $\hat{\mathbf{v}}_0\hat{\mathbf{r}}\hat{\mathbf{v}}_0 = -\hat{\mathbf{g}}$. From the Appendix, we can infer that $\hat{\mathbf{v}}_o$ must bisect the angle between $\hat{\mathbf{r}}$ and $-\hat{\mathbf{g}}$. That is, it must bisect the angle between $\hat{\mathbf{r}}$ and "vertically upward" (Fig. 5). Clearly, this is indeed physically possible. So, yes, we are indeed justified in substituting -1 for $\hat{\mathbf{g}} \cdot (\hat{\mathbf{v}}_0\hat{\mathbf{r}}\hat{\mathbf{v}}_0)$ in Eq. (5):

$$\begin{aligned} r &= \left[\frac{v_o^2}{g\|\hat{\mathbf{g}} \wedge \hat{\mathbf{r}}\|^2}\right]\left[\hat{\mathbf{g}} \cdot \hat{\mathbf{r}} - \hat{\mathbf{g}} \cdot (\hat{\mathbf{v}}_0\hat{\mathbf{r}}\hat{\mathbf{v}}_0)\right]; \\ \therefore r_{max} &= \left[\frac{v_o^2}{g\|\hat{\mathbf{g}} \wedge \hat{\mathbf{r}}\|^2}\right]\left[\hat{\mathbf{g}} \cdot \hat{\mathbf{r}} - (-1)\right] \\ &= \left[\frac{v_o^2}{g\|\hat{\mathbf{g}} \wedge \hat{\mathbf{r}}\|^2}\right]\left[1 + \hat{\mathbf{g}} \cdot \hat{\mathbf{r}}\right]. \end{aligned} \tag{6}$$

We've now answered our first two questions, which were "What is the $\hat{\mathbf{v}}_o$ that gives the greatest range r along a given direction $\hat{\mathbf{r}}$?, and "What is r_{max}?" But we can do a bit more by noting that $\|\hat{\mathbf{g}} \wedge \hat{\mathbf{r}}\|^2$ is the square of the magnitude of bivector. Therefore, we transform Eq. (6) by making use of the very useful GA identity that

$$\|\mathbf{a} \wedge \mathbf{b}\|^2 = a^2b^2 - (\mathbf{a} \cdot \mathbf{b})^2\,.$$

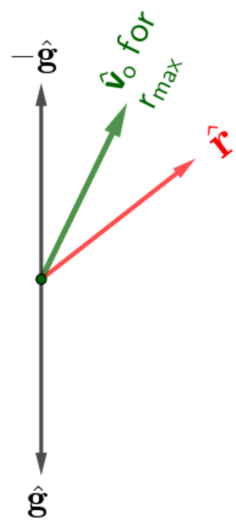

Figure 5: The $\hat{\mathbf{v}}_o$ that gives the maximum range along $\hat{\mathbf{r}}$ bisects the angle between $\hat{\mathbf{r}}$ and "vertically upward" ($-\hat{\mathbf{g}}$).

Specifically,

$$\begin{aligned} r_{max} &= \left[\frac{v_o^2}{g\|\hat{\mathbf{g}}\wedge\hat{\mathbf{r}}\|^2}\right][1+\hat{\mathbf{g}}\cdot\hat{\mathbf{r}}] \\ &= \left[\frac{v_o^2}{g\left[(\hat{\mathbf{g}})^2(\hat{\mathbf{r}})^2-(\hat{\mathbf{g}}\cdot\hat{\mathbf{r}})^2\right]}\right][1+\hat{\mathbf{g}}\cdot\hat{\mathbf{r}}] \\ &= \left(\frac{v_o^2}{g}\right)\left[\frac{1+\hat{\mathbf{g}}\cdot\hat{\mathbf{r}}}{1-(\hat{\mathbf{g}}\cdot\hat{\mathbf{r}})^2}\right] \\ &= \left(\frac{v_o^2}{g}\right)\left[\frac{1}{1-\hat{\mathbf{g}}\cdot\hat{\mathbf{r}}}\right]. \end{aligned} \qquad (7)$$

Hestenes arrives at our Eq. (7) (his Eq. 2.12) by substituting $(\hat{\mathbf{r}}-\hat{\mathbf{g}})/\|\hat{\mathbf{r}}-\hat{\mathbf{g}}\|$ for $\hat{\mathbf{v}}_o$ in his Eq. 2.8 (our Eq. (4)) .

Time for a "sanity check". Can we test Eq. (7) by comparing its predictions to known cases? For example, what is the maximum range for a projectile that is launched vertically upward? In this case, $\hat{\mathbf{g}}\cdot\hat{\mathbf{r}}=-1$, so Eq. gives $r_{max}=\frac{v_o^2}{2g}$. By comparison, we know from introductory physics that at the projectile's maximum height (i.e., its r_{max}), its velocity is zero. Thus, from the familiar formula $v^2-v_o^2=2as$, $0^2-v_o^2=2(-g)\,r_{max}$, from which $r_{max}=\frac{v_o^2}{2g}$. ✓

What about r_{max} for a horizontal line of sight? In this case, $\hat{\mathbf{g}}\cdot\hat{\mathbf{r}}=0$, so Eq. gives $r_{max}=\frac{v_o^2}{g}$. By comparison, the conventional treatment of this case uses the equations $x=v_ot\cos\theta$ and $y=v_ot\sin\theta+\frac{1}{2}gt^2$, in which θ is the launch angle, x is the horizontal distance of the projectile from the launch point at time t after launch, and y is the vertical distance. When the projectile strikes the ground, $y=0$, giving t (at that instant)$=\frac{2v_o\sin\theta}{g}$. Substituting that expression for t in the 'x" equation, $x=v_o\cos\theta\frac{2v_o\sin\theta}{g}$, which simplifies to $x=\frac{v_o^2}{g}\sin 2\theta$. The maximum value if x is for $\sin 2\theta=1$ (which occurs when $\theta=\pi/4$). Thus,r_{max}

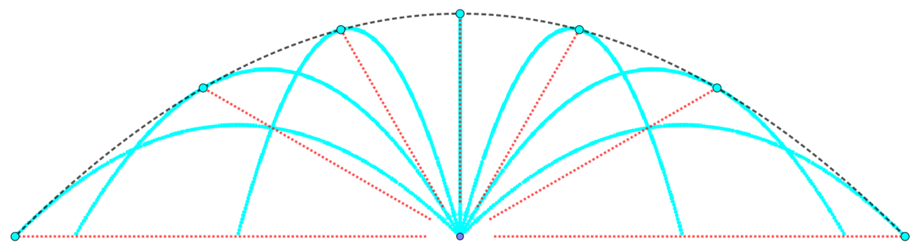

Figure 6: (This is Fig. 2, reproduced here for the reader's convenience.)The black dashed curve is a parabola. It is the boundary of the region of space in which a target can be hit by a projectile whose initial velocity is v_o. Open circles along that curve show the furthest targets that can be reached along their respective lines of sight (dotted red lines). Turquoise curves show the trajectories that reach those most-distant points.

for a horizontal line of sight is indeed $\frac{v_o^2}{g}$. ✓

Because our "sanity checks" have shown that Eq. (7) gives correct values of r_{max} for both extremes of the possible directions $\hat{\mathbf{r}}$ (i.e., for the horizontal and vertical lines of sight), we'll proceed to answer our third question by interpreting that equation geometrically.

5.3 Identifying the Region in Space in which Targets May be Reached by Projectiles that are Launched at Initial Velocity $\mathbf{v}_o$

In Section 2-6 ("Analytic Geometry") of *NFCM*, Hestenes showed that his Eq. 2.12 (our Eq. (7)) is the equation of a paraboloid of revolution. Therefore, the black curve in Fig. 2 is a parabola.

No target that lies outside that paraboloid can be hit by a projectile whose initial speed is v_o. But does that statement mean that *every* target within that paraboloid can be hit? If so, what is the launch direction needed to hit for a target at a given $r < r_{max}$ along a given line of sight $\hat{\mathbf{r}}$?

We will address that question later (in Section 6). But first, we will finish our discussion of Hestenes's treatment of "maximum range", by deriving an equation for the time of flight to reach a given target.

5.4 Time of Flight t for the Projectile to Reach r_{max}

Here, again, Hestenes might have helped students by indicating where he was heading with each step. First, why does he begin by deriving his Eq. 2.13

(*NFCM*, p. 130),

$$\frac{1}{2}t^2 = (\mathbf{r} \wedge \mathbf{v}o)\,(\mathbf{g} \wedge \mathbf{v}_o)^{-1}$$

when he'd already presented a perfectly adequate (and arguably better) equation for t before beginning to treat "maximum range"? That equation was (*NFCM*, p. 128, Eq. 2.6)

$$\frac{1}{2}t\,(\mathbf{g} \wedge \mathbf{r}) = \mathbf{r} \wedge \mathbf{v}_o$$
$$\therefore t = 2\,(\mathbf{r} \wedge \mathbf{v}_o)\,(\mathbf{g} \wedge \mathbf{r})^{-1}.$$

Note that a simple equation for t can be derived from the vector equation

$$\hat{\mathbf{r}} = \mathbf{v}t + \frac{1}{2}\mathbf{g}t^2,$$

by taking the outer product of both sides with $\hat{\mathbf{g}}$. An equivalent equation can be derived by taking the dot product of both sides with $\hat{\mathbf{g}}\hat{\mathbf{i}}$: $t = [\mathbf{r} \cdot (\hat{\mathbf{g}}\mathbf{i})] \,/\, [\mathbf{v} \cdot (\hat{\mathbf{g}}\mathbf{i})]$. That is, that t is equal to the horizontal component of $\mathbf{r}$ divided by the horizontal component of $\mathbf{v}_o$.

As we shall soon see, Hestenes' purpose in deriving his Eq. 2.13 was not to use that equation for calculating t; instead, his intention was to obtain an expression for t that could be transformed into one that would be especially convenient for the special case of $r = r_{max}$. Let's see how he did it. To obtain his Eq. 2.1, he eliminated the product $\mathbf{g} \wedge \mathbf{r}$ between his Eqs. 2.6 and 2.7:

$$Eq.\ 2.6: \quad t = 2\,(\mathbf{r} \wedge \mathbf{v}_o)\,(\mathbf{g} \wedge \mathbf{r})^{-1}$$
$$Eq.\ 2.7: \quad \mathbf{g} \wedge \mathbf{r} = t\mathbf{g} \wedge \mathbf{v}_o$$
$$\therefore t = 2\,(\mathbf{r} \wedge \mathbf{v}_o)\underbrace{\left[\frac{t\mathbf{v_o} \wedge \mathbf{g}}{\|t\mathbf{g} \wedge \mathbf{v}_o\|^2}\right]}_{=(\mathbf{g}\wedge\mathbf{r})^{-1}},\ \text{and}$$
$$t^2 = 2\,(\mathbf{r} \wedge \mathbf{v}_o)\underbrace{\left[\frac{\mathbf{v_o} \wedge \mathbf{g}}{\|\mathbf{g} \wedge \mathbf{v}_o\|^2}\right]}_{=(\mathbf{g}\wedge\mathbf{v}_o)^{-1}}.$$

Proceeding,

$$\begin{aligned} t^2 &= 2\,(\mathbf{r} \wedge \mathbf{v}_o)\,(\mathbf{g} \wedge \mathbf{v}_o)^{-1} \\ &= 2rv_o\,(\hat{\mathbf{r}} \wedge \hat{\mathbf{v}}_o)\,(\mathbf{g} \wedge \mathbf{v}_o)^{-1} \\ &= 2rv_o\,(\hat{\mathbf{r}} \wedge \hat{\mathbf{v}}_o)\left[\frac{v_o g\,(\hat{\mathbf{v}}_o \wedge \hat{\mathbf{g}})}{\|v_o g\,(\hat{\mathbf{v}}_o \wedge \hat{\mathbf{g}})\,\|^2}\right] \\ &= \frac{2r}{g}\,(\hat{\mathbf{r}} \wedge \hat{\mathbf{v}}_o)\,(\hat{\mathbf{g}} \wedge \hat{\mathbf{v}}_o)^{-1}. \end{aligned} \tag{8}$$

Now we can see why Hestenes derived his Eq. 2.13: he'll make a clever substitution for the factor $\hat{\mathbf{r}} \wedge \hat{\mathbf{v}}_o$ in that equation. What substitution might that be? Recall that in the process of deriving his equation for the $\hat{\mathbf{v}}_o$ that gives r_{max}, he had shown that

$$\bar{\hat{\mathbf{g}}} \cdot (\hat{\mathbf{v}}_o\hat{\mathbf{r}}\hat{\mathbf{v}}_o) = 1$$
$$\therefore \bar{\hat{\mathbf{g}}} = \hat{\mathbf{v}}_o\hat{\mathbf{r}}\hat{\mathbf{v}}_o$$
$$\therefore \bar{\hat{\mathbf{g}}}\hat{\mathbf{v}}_o = \hat{\mathbf{v}}_o\hat{\mathbf{r}},$$

from which (again, for the $\hat{\mathbf{v}}_o$ that gives the maximum range),

$$\hat{\mathbf{r}} \wedge \hat{\mathbf{v}}_o = \hat{\mathbf{g}} \wedge \hat{\mathbf{v}}_o.$$

Making that substitution for $\hat{\mathbf{r}} \wedge \hat{\mathbf{v}}_o$ in our Eq. (8),

$$\begin{aligned} t^2 &= \frac{2r_{max}}{g}\underbrace{(\hat{\mathbf{g}} \wedge \hat{\mathbf{v}}_o)}_{=\hat{\mathbf{r}}\wedge\hat{\mathbf{v}}_o}(\hat{\mathbf{g}} \wedge \hat{\mathbf{v}}_o)^{-1} \\ &= \frac{2r_{max}}{g}. \end{aligned}$$

Now, Hestenes makes another substitution: in his Eq. 2.12 (our Eq. (7)), Hestenes showed that

$$r_{max} = \left(\frac{v_o^2}{g}\right)\left[\frac{1}{1 - \hat{\mathbf{g}} \cdot \hat{\mathbf{r}}}\right].$$

Thus we arrive at Hestenes's Eq. 2.14 (*NFCM*, p. 130):

$$t^2 \text{ for } r_{max} \text{ along the direction } \hat{\mathbf{r}} = \left(\frac{2v_o^2}{g^2}\right)\left[\frac{1}{1 - \hat{\mathbf{g}} \cdot \hat{\mathbf{r}}}\right], \quad (9)$$

which is quite convenient for the case of $r = r_{max}$.

The reader is encouraged to "sanity-check" this result, using (for example) $\hat{\mathbf{r}} = \bar{}\hat{\mathbf{g}}$ and $\hat{\mathbf{r}} = 0$.

6 The $\hat{\mathbf{v}}_0$ for a Given $r < r_{max}$ along a Given Direction $\hat{\mathbf{r}}$ (Fig. 1)

Where might we get an idea for solving this problem, which Hestenes does not appear to have treated in *NFCM*? A reasonable candidate for a starting point is Eq. (5), —after all, that equation led us directly enough to an equation for the $\hat{\mathbf{v}}_o$ that gives r_{max}. Here's Eq. (5) again:

$$r = \left[\frac{v_o^2}{g\|\hat{\mathbf{g}} \wedge \hat{\mathbf{r}}\|^2}\right]\left[\hat{\mathbf{g}} \cdot \hat{\mathbf{r}} - \hat{\mathbf{g}} \cdot (\hat{\mathbf{v}}_0 \hat{\mathbf{r}} \hat{\mathbf{v}}_0)\right] \quad \text{(5, revisited)}$$

Unfortunately, the $\hat{\mathbf{v}}_o$ that we need to find is so thoroughly "buried" in this equation that we're unlikely to find a direct way to solve for it. So, let's think about that equation in a different way, bearing in mind the ideas that we reviewed in Section 4. One of those ideas was that $\hat{\mathbf{v}}_0 \hat{\mathbf{r}} \hat{\mathbf{v}}_0$ evaluates to a unit vector. Do we know anything about that vector? Or can we identify anything about it that might prove useful?

Yes, but before we list what we know about that vector, let's represent it via a single, convenient symbol. We'll use "$\hat{\mathbf{w}}$:

$$\hat{\mathbf{w}} = \hat{\mathbf{v}}_0 \hat{\mathbf{r}} \hat{\mathbf{v}}_0.$$

We'll also write that equation "backwards" just in case that version might be more suggestive as we look for an idea:

$$\hat{\mathbf{v}}_0 \hat{\mathbf{r}} \hat{\mathbf{v}}_0 = \hat{\mathbf{w}}.$$

Now, let's list what we know about $\hat{\mathbf{w}}$. First, we know that $\hat{\mathbf{w}}$ is the reflection of $\hat{\mathbf{r}}$ with respect to the $\hat{\mathbf{v}}_o$ that we wish to find. To proceed in our search for an idea, we'll substitute $\hat{\mathbf{w}}$ for $\hat{\mathbf{v}}_0 \hat{\mathbf{r}} \hat{\mathbf{v}}_0$ in Eq. (5):

$$r = \left[\frac{v_o^2}{g\|\hat{\mathbf{g}} \wedge \hat{\mathbf{r}}\|^2}\right]\left[\hat{\mathbf{g}} \cdot \hat{\mathbf{r}} - \hat{\mathbf{g}} \cdot \hat{\mathbf{w}}\right]. \quad (10)$$

Upon examining that equation, one possibility that comes to mind is (as an intermediate step) to find $\hat{\mathbf{r}} - \hat{\mathbf{w}}$ by rewriting the previous equation as

$$r = \left[\frac{v_o^2}{g\|\hat{\mathbf{g}} \wedge \hat{\mathbf{r}}\|^2}\right] [\hat{\mathbf{g}} \cdot (\hat{\mathbf{r}} - \hat{\mathbf{w}})],$$

so that we can find the value of $\hat{\mathbf{g}} \cdot (\hat{\mathbf{r}} - \hat{\mathbf{w}})$, after which we then use another idea from Section 4, and write $\hat{\mathbf{r}} - \hat{\mathbf{w}}$ as

$$\hat{\mathbf{r}} - \hat{\mathbf{w}} = [\hat{\mathbf{g}} \cdot (\hat{\mathbf{r}} - \hat{\mathbf{w}})]\,\hat{\mathbf{g}} \pm \left\{\sqrt{\|\hat{\mathbf{r}} - \hat{\mathbf{w}}\|^2 - [\hat{\mathbf{g}} \cdot (\hat{\mathbf{r}} - \hat{\mathbf{w}})]^2}\right\}\hat{\mathbf{g}}\mathbf{i}.$$

Note that there are two possible vectors "$\hat{\mathbf{r}} - \hat{\mathbf{w}}$", and therefore two $\hat{\mathbf{w}}$'s.

Unfortunately, we can't make use of that idea because we don't know $\|\hat{\mathbf{r}} - \hat{\mathbf{w}}\|$. But we can indeed use that idea if we solve Eq. (10) for $\hat{\mathbf{w}} \cdot \hat{\mathbf{g}}$, then write the two $\hat{\mathbf{w}}$'s as

$$\begin{aligned}
\hat{\mathbf{w}}_1 &= (\hat{\mathbf{w}} \cdot \hat{\mathbf{g}})\,\hat{\mathbf{g}} + \left\{\sqrt{(\hat{\mathbf{w}})^2 - (\hat{\mathbf{w}} \cdot \hat{\mathbf{g}})^2}\right\}\hat{\mathbf{g}}\mathbf{i} \\
&= (\hat{\mathbf{w}} \cdot \hat{\mathbf{g}})\,\hat{\mathbf{g}} + \left\{\sqrt{1 - (\hat{\mathbf{w}} \cdot \hat{\mathbf{g}})^2}\right\}\hat{\mathbf{g}}\mathbf{i}\,; \\
\hat{\mathbf{w}}_2 &= (\hat{\mathbf{w}} \cdot \hat{\mathbf{g}})\,\hat{\mathbf{g}} - \left\{\sqrt{1 - (\hat{\mathbf{w}} \cdot \hat{\mathbf{g}})^2}\right\}\hat{\mathbf{g}}\mathbf{i}.
\end{aligned} \tag{11}$$

Having identified a strategy that's workable (even if not particularly elegant!), let's proceed. First, we solve Eq. (10) for $\hat{\mathbf{w}} \cdot \hat{\mathbf{g}}$:

$$\begin{aligned}
\hat{\mathbf{w}} \cdot \hat{\mathbf{g}} &= \frac{1}{v_0^2}\left[v_0^2 \hat{\mathbf{g}} \cdot \hat{\mathbf{r}} - rg\|\hat{\mathbf{g}} \wedge \hat{\mathbf{r}}\|^2\right] \\
&= \frac{1}{v_0^2 rg}\left[v_0^2 \mathbf{r} \cdot \mathbf{g} - \|\mathbf{r} \wedge \mathbf{g}\|^2\right].
\end{aligned} \tag{12}$$

Next, we find $\sqrt{1 - (\hat{\mathbf{w}} \cdot \hat{\mathbf{g}})^2}$. We'll provide lots of details for the reader's convenience:

$$\begin{aligned}
\sqrt{1 - (\hat{\mathbf{w}} \cdot \hat{\mathbf{g}})^2} &= \sqrt{1 - \left\{\frac{1}{v_0^2 rg}\left[v_0^2 \mathbf{r} \cdot \mathbf{g} - \|\mathbf{r} \wedge \mathbf{g}\|^2\right]\right\}^2} \\
&= \frac{1}{v_0^2 rg}\sqrt{v_o^4 r^2 g^2 - v_o^4 (\mathbf{r} \cdot \mathbf{g})^2 + 2v_o^2 \mathbf{r} \cdot \mathbf{g}\|\mathbf{r} \wedge \mathbf{g}\|^2 - \|\mathbf{r} \wedge \mathbf{g}\|^4} \\
&= \frac{1}{v_0^2 rg}\sqrt{v_o^4 \left[r^2 g^2 - (\mathbf{r} \cdot \mathbf{g})^2\right] + 2v_o^2 \mathbf{r} \cdot \mathbf{g}\|\mathbf{r} \wedge \mathbf{g}\|^2 - \|\mathbf{r} \wedge \mathbf{g}\|^4} \\
&= \frac{1}{v_0^2 rg}\sqrt{v_o^4 \|\mathbf{r} \wedge \mathbf{g}\|^2 + 2v_o^2 \mathbf{g} \cdot \mathbf{r}\|\mathbf{g} \wedge \mathbf{r}\|^2 - \|\mathbf{r} \wedge \mathbf{g}\|^4} \\
&= \frac{1}{v_0^2 rg}\sqrt{\|\mathbf{r} \wedge \mathbf{g}\|^2 \left\{v_o^4 + 2v_o^2 \mathbf{r} \cdot \mathbf{g} - \|\mathbf{r} \wedge \mathbf{g}\|^2\right\}} \\
&= \frac{1}{v_0^2 rg}\sqrt{\|\mathbf{r} \wedge \mathbf{g}\|^2 \left\{v_o^4 + 2v_o^2 \mathbf{r} \cdot \mathbf{g} - \left[r^2 g^2 - (\mathbf{r} \cdot \mathbf{g})^2\right]\right\}} \\
&= \frac{1}{v_0^2 rg}\sqrt{\|\mathbf{r} \wedge \mathbf{g}\|^2 \left\{v_o^4 + 2v_o^2 \mathbf{g} \cdot \mathbf{r} + (\mathbf{r} \cdot \mathbf{g})^2 - r^2 g^2\right\}} \\
&= \frac{1}{v_0^2 rg}\sqrt{\|\mathbf{r} \wedge \mathbf{g}\|^2 \left\{\left[v_o^2 + \mathbf{r} \cdot \mathbf{g}\right]^2 - r^2 g^2\right\}}
\end{aligned} \tag{13}$$

$r^2 g^2 - (\mathbf{g} \cdot \mathbf{r})^2 = \|\mathbf{g} \wedge \mathbf{r}\|^2.$

Thus, the two $\hat{\mathbf{w}}$'s, as defined in Eq. (11), are

$$\begin{aligned}\hat{\mathbf{w}}_1 = &\left\{\frac{1}{v_0^2 rg}\left[v_0^2 \mathbf{r}\cdot\mathbf{g} - \|\mathbf{r}\wedge\mathbf{g}\|^2\right]\right\}\hat{\mathbf{g}} \\ &+ \left\{\frac{1}{v_0^2 rg}\sqrt{\|\mathbf{r}\wedge\mathbf{g}\|^2\left\{\left[v_o^2 + \mathbf{r}\cdot\mathbf{g}\right]^2 - r^2 g^2\right\}}\right\}(\hat{\mathbf{g}}\mathbf{i}) \ ; \\ \hat{\mathbf{w}}_2 = &\left\{\frac{1}{v_0^2 rg}\left[v_0^2 \mathbf{r}\cdot\mathbf{g} - \|\mathbf{r}\wedge\mathbf{g}\|^2\right]\right\}\hat{\mathbf{g}} \\ &- \left\{\frac{1}{v_0^2 rg}\sqrt{\|\mathbf{r}\wedge\mathbf{g}\|^2\left\{\left[v_o^2 + \mathbf{r}\cdot\mathbf{g}\right]^2 - r^2 g^2\right\}}\right\}(\hat{\mathbf{g}}\mathbf{i}) \ . \end{aligned} \tag{14}$$

To find the $\hat{\mathbf{v}}_o$ for each $\hat{\mathbf{w}}$, we use the method that is presented in the Appendix. We'll use the notation $(\hat{\mathbf{v}}_{o,i})$ to represent the $\hat{\mathbf{v}}_o$ that corresponds to vector $\hat{\mathbf{w}}_i$.

$$(\hat{\mathbf{v}}_{o,i})\,(\hat{\mathbf{r}})\,(\hat{\mathbf{v}}_{o,i}) = \hat{\mathbf{w}}_i \quad \rightarrow \ (\hat{\mathbf{v}}_{o,i}) = \frac{\hat{\mathbf{w}}_i + \hat{\mathbf{r}}}{\|\hat{\mathbf{w}}_i + \hat{\mathbf{r}}\|}. \tag{15}$$

Interestingly, $\hat{\mathbf{v}}_{o,1}$ and $\hat{\mathbf{v}}_{o,2}$ are symmetric to each other with respect to the $\hat{\mathbf{v}}_o$ for maximum range (Fig. 7). The proof of that relationship is a corollary of Hestenes's Eq. 2.22 (*NFCM*, p. 132), which is derived in a later part of the "Maximum Range" section that falls outside the scope of the present document.

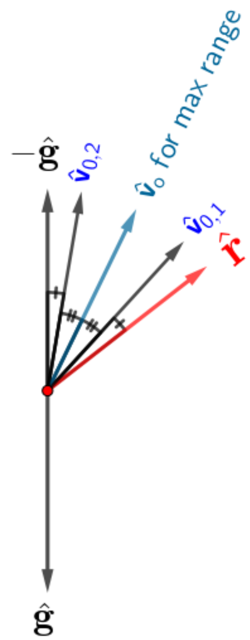

Figure 7: Hestenes demonstrates (*NFCM*, p. 132) that the angle between $\hat{\mathbf{v}}_{0,1}$ and $\hat{\mathbf{r}}$ is equal to the angle between $\hat{\mathbf{v}}_{0,2}$ and $\bar{\ }\hat{\mathbf{g}}$. Because $\hat{\mathbf{v}}_0$ for r_{max} bisects the angle between $\hat{\mathbf{r}}$ and $\bar{\ }\hat{\mathbf{g}}$, $\hat{\mathbf{v}}_{0,1}$ and $\hat{\mathbf{v}}_{0,2}$ are symmetric to each other with respect to the $\hat{\mathbf{v}}_0$ for r_{max}.

The times of flight for each $(\hat{\mathbf{v}}_{o,i})$ can be found via the equation $t = [\mathbf{r} \cdot (\hat{\mathbf{g}}\mathbf{i})] \,/\, [\mathbf{v} \cdot (\hat{\mathbf{g}}\mathbf{i})]$. (See the margin note on p. 11).

7 The GeoGebra Construction for Testing the Formulas

Fig. 8 shows the interactive GeoGebra construction ([6]) for verifying the formulas that have been derived in this document.

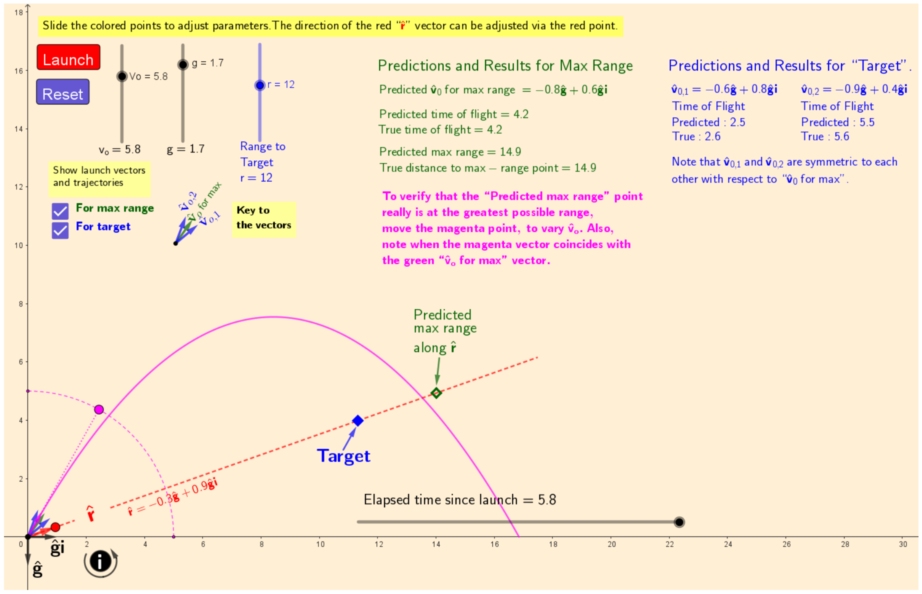

Figure 8: Screenshot of the interactive GeoGebra construction for verifying the formulas that have been derived in this document.

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A Regarding the Equation $\hat{\mathbf{x}}\mathbf{u}\hat{\mathbf{x}} = \mathbf{z}$

A.1 Proof that if $\hat{\mathbf{w}}\mathbf{a}\hat{\mathbf{w}} = \mathbf{b}$, then $\|\mathbf{a}\| = \|\mathbf{b}\|$

$$\begin{aligned}
\hat{\mathbf{w}}\mathbf{a}\hat{\mathbf{w}} &= \mathbf{b} \\
(\hat{\mathbf{w}}\mathbf{a}\hat{\mathbf{w}})^2 &= b^2 \\
(\hat{\mathbf{w}}\mathbf{a}\hat{\mathbf{w}})(\hat{\mathbf{w}}\mathbf{a}\hat{\mathbf{w}}) &= b^2 \\
\hat{\mathbf{w}}\mathbf{a}(\hat{\mathbf{w}}\hat{\mathbf{w}})\mathbf{a}\hat{\mathbf{w}} &= b^2 \\
\hat{\mathbf{w}}\mathbf{a}(1)\mathbf{a}\hat{\mathbf{w}} &= b^2 \\
\hat{\mathbf{w}}\mathbf{a}\mathbf{a}\hat{\mathbf{w}} &= b^2 \\
\hat{\mathbf{w}}a^2\hat{\mathbf{w}} &= b^2 \\
a^2\hat{\mathbf{w}}\hat{\mathbf{w}} &= b^2 \\
a^2 &= b^2 \\
\therefore \|\mathbf{a}\| &= \|\mathbf{b}\|.
\end{aligned}$$

A.2 Proof that if $\hat{\mathbf{u}}\mathbf{a}\hat{\mathbf{u}} = \mathbf{b}$, then $\hat{\mathbf{u}} = \pm\frac{\mathbf{a}+\mathbf{b}}{\|\mathbf{a}+\mathbf{b}\|}$

$$\begin{aligned}
\hat{\mathbf{u}}\mathbf{a}\hat{\mathbf{u}} &= \mathbf{b} \\
\hat{\mathbf{u}}\mathbf{a}\hat{\mathbf{u}}\hat{\mathbf{u}} &= \mathbf{b}\hat{\mathbf{u}} \\
\hat{\mathbf{u}}\mathbf{a} &= \mathbf{b}\hat{\mathbf{u}} \\
\hat{\mathbf{u}}\cdot\mathbf{a} + \hat{\mathbf{u}}\wedge\mathbf{a} &= \hat{\mathbf{b}}\cdot\mathbf{u} + \hat{\mathbf{b}}\wedge\mathbf{u} \\
\therefore \hat{\mathbf{u}}\wedge\mathbf{a} &= \hat{\mathbf{b}}\wedge\mathbf{u} \\
\hat{\mathbf{u}}\wedge\mathbf{a} - \hat{\mathbf{b}}\wedge\mathbf{u} &= 0 \\
\hat{\mathbf{u}}\wedge\mathbf{a} + \hat{\mathbf{u}}\wedge\mathbf{b} &= 0 \\
\hat{\mathbf{u}}\wedge(\mathbf{a}+\mathbf{b}) &= 0 \\
\therefore \hat{\mathbf{u}} \parallel (\mathbf{a}+\mathbf{b}) \rightarrow \hat{\mathbf{u}} &= k(\mathbf{a}+\mathbf{b}); \\
\hat{\mathbf{u}}^2 &= [k(\mathbf{a}+\mathbf{b})]^2; \\
1 &= k^2\|\mathbf{a}+\mathbf{b}\|^2; \\
k &= \pm\frac{1}{\|\mathbf{a}+\mathbf{b}\|}; \\
\hat{\mathbf{u}} &= \pm\frac{\mathbf{a}+\mathbf{b}}{\|\mathbf{a}+\mathbf{b}\|}.
\end{aligned}$$