$Collatz\ Sequence\ Proof\ (1st\ Way)$

$Abstract$: A Collatz sequence is a sequence of numbers generated by starting with a positive integer and repeatedly applying two rules: • If the number is even, divide it by two • If the number is odd, multiply it by 3 and add 1

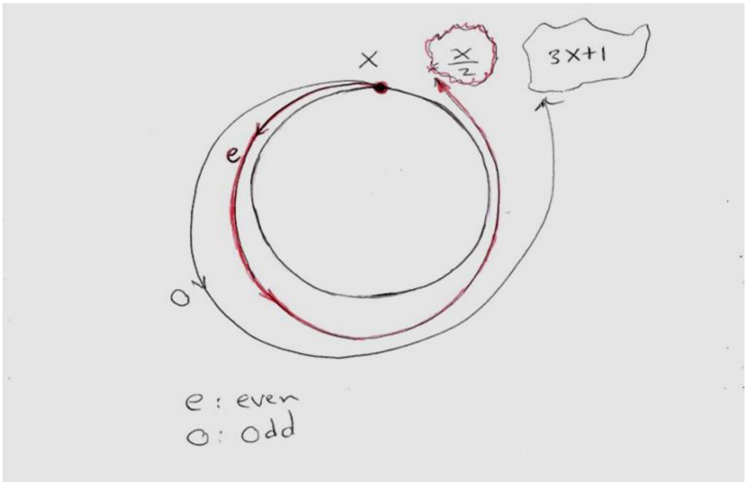
$let\ Collatz\ Sequence\ (n) = S(n), Loop\ of\ Collatz\ Sequence\ (n) = lS(n),$

$let\ r = number\ of\ elements\ of\ lS(n), \&\ x, y, z, t, r, k, h, g, m, n \in N_+$

$$\left[n\ even \rightarrow \frac{n}{2}, or\ n\ odd \rightarrow (3n+1)\right] \Leftrightarrow S(n) = \left\{\frac{n}{2} or\ (3n+1), \ldots, 4,2,1\right\} \Leftrightarrow lS(n) = \{4,2,1\}$$

$Proof$

$A)\ let\ lS(n) = \{x\}\ \forall n \in N_+,$

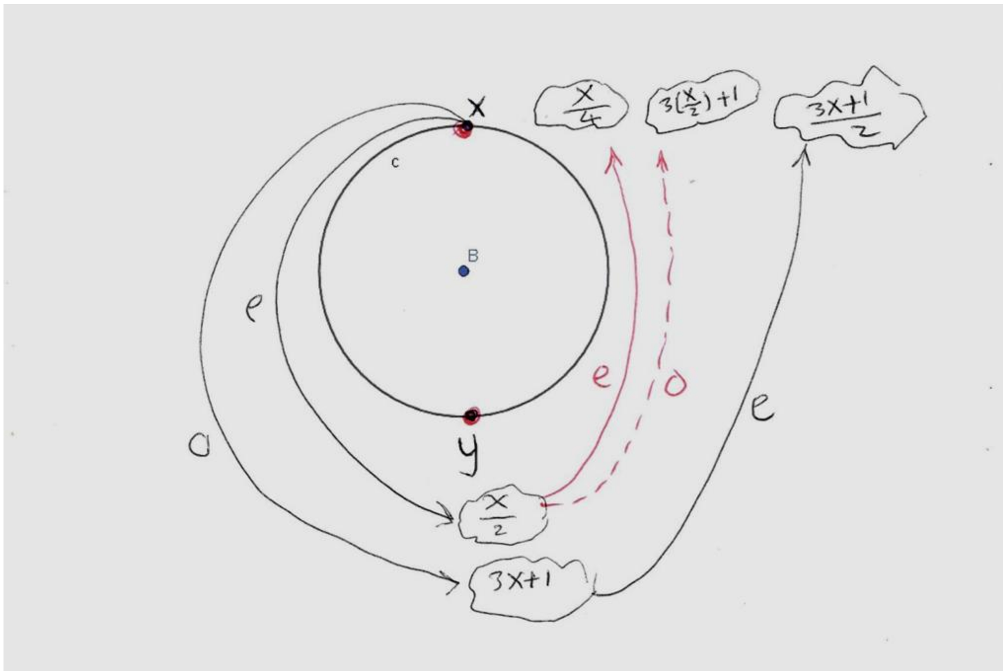


$Taha's\ Sketch\ of\ lS(n)\ to\ find\ equivalent\ expresion\ (Cloud)\ to\ x$

$Cloud$	$Cloud = x$	x
$\frac{x}{2}$	$\frac{x}{2} = x \Rightarrow x = 0 \notin lS(\mathrm{n})$	----
$3x+1$	$3x+1 = x \Rightarrow x = -\frac{1}{2} \notin lS(\mathrm{n})$	----

$\therefore lS(\mathrm{n}) \neq \{x\}\ \forall n \in N_+$

$B)\ let\ lS(n) = \{x, y\}\ \forall n \in N_+$

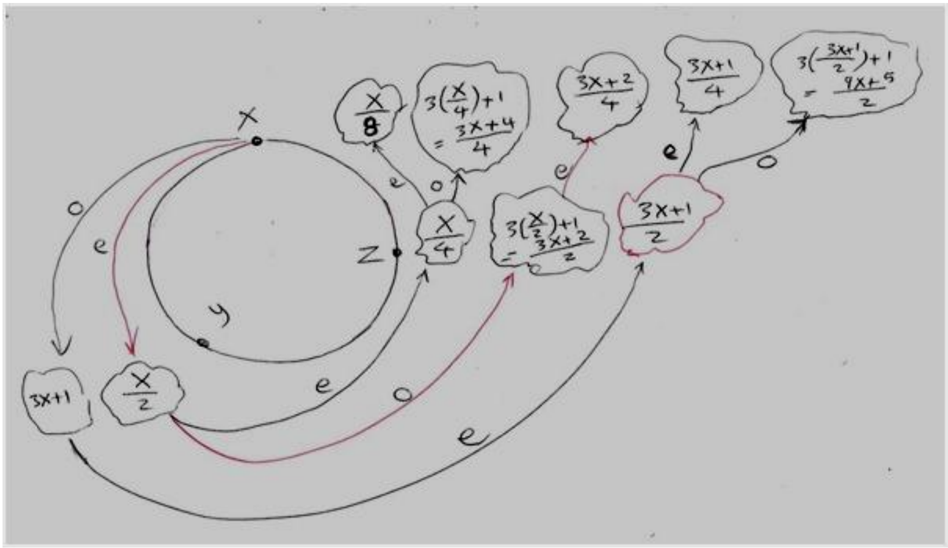


$Taha's\ Sketch\ of\ lS(n)\ to\ find\ equivalent\ expresions\ (Cloud)\ to\ x$

$Cloud$	$Cloud = x$	x	y
$\frac{x}{4}$	$\frac{x}{4} = x \Rightarrow x = 0 \notin lS(n)$	….	….
$\frac{3x+2}{2}$	$\frac{3x+2}{2} = x \Rightarrow 3x + 2 = 2x \Rightarrow x = -2 \notin lS(n)$	….	….
$\frac{3x+1}{2}$	$\frac{3x+1}{2} = x \Rightarrow x = -1 \notin lS(n)$	….	….

$\therefore lS(n) \neq \{x, y\}\ \forall n \in N_+$

$C)$ $let\ lS(n) = \{x, y, z\}\ \ \forall n \in N_+$

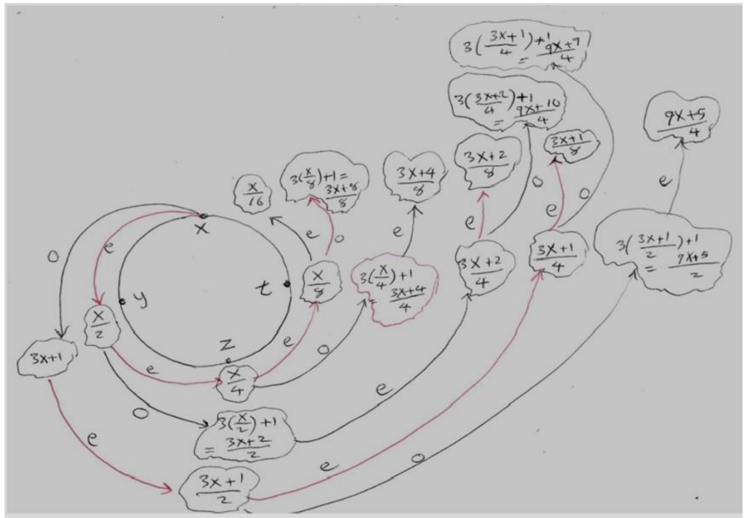


$Taha's\ Sketch\ of\ lS(n)\ to\ find\ equivalent\ expresions\ (Cloud) to\ x$

$Cloud$	$Solve\ equation: Cloud = x$	x	y	z
$\frac{x}{8}$	$\frac{x}{8} = x \Rightarrow x = 0 \notin lS(n)$	---	---	---
$\frac{3x+4}{4}$	$\frac{3x+4}{4} = x \Rightarrow x = 4 \in lS(n)$	4	$\frac{x}{2} = 2$	$\frac{x}{4}$ =1
$\frac{3x+2}{4}$	$\frac{3x+2}{4} = x \Rightarrow x = 2 \in lS(n)$	2	$\frac{x}{2} = 1$	$\frac{3x+2}{2} = 4$
$\frac{3x+1}{4}$	$\frac{3x+1}{4} = x \Rightarrow x = 1 \in lS(n)$	1	$3x + 1$ =3(1)+1=4	$\frac{3x+1}{2}$ =2
$\frac{9x+5}{2}$	$\frac{9x+5}{2} = x \Rightarrow x = -\frac{5}{7} \notin lS(n)$	---	---	---

$\therefore lS(n) = \{x, y, z\} = \{4,2,1\}\ , \forall n \in N_+$

$D)\ let\ lS(n) = \{x, y, z, t\}\ \forall n \in N_+$



$Cloud$	$Solve\ equation: Cloud = x$	x	y	z	t
$\frac{x}{16}$	$\frac{x}{16} = x \Rightarrow x = 0 \notin lS(n)$	---	---	---	---
$\frac{3x+8}{8}$	$\frac{3x+8}{8} = x \Rightarrow x \notin lS(n)$	---	---	---	---
$\frac{3x+4}{8}$	$\frac{3x+4}{8} = x \Rightarrow x \notin lS(n)$	---	---	---	---
$\frac{3x+2}{8}$	$\frac{3x+2}{8} = x \Rightarrow x \notin lS(n)$	---	---	---	---
$\frac{3x+1}{8}$	$\frac{3x+1}{8} = x \Rightarrow x \notin lS(n)$	---	---	---	---
$\frac{9x+10}{4}$	$\frac{9x+10}{4} = x \Rightarrow x \notin lS(n)$	---	---	---	---
$\frac{9x+7}{4}$	$\frac{9x+7}{4} = x \Rightarrow x \notin lS(n)$	---	---	---	---
$\frac{9x+5}{4}$	$\frac{9x+5}{4} = x \Rightarrow x \notin lS(n)$	---	---	---	---

$\therefore lS(n) \neq \{x, y, z, t\}\ \forall n \in N_+$

$r\ is\ number\ of\ elements\ of\ lS(n), \&\ let\ x \in lS(n)$

x = e or odd →

$r = 1$	→ $\frac{x}{2}$ = e or odd →	→ $3x + 1$ = e →

$r = 2$	→ $\frac{x}{4}$ =e or odd →	→ $\frac{3x+2}{2}$ =e →	→ $\frac{3x+1}{2}$ =e or odd →

$r = 3$	→ $\frac{x}{8}$ =e or odd →	→ $\frac{3x+4}{4}$ =e →	→ $\frac{3x+2}{4}$ =e or odd →	→ $\frac{3x+1}{4}$ =e or odd →	→ $\frac{9x+5}{2}$ =e →

$r = 4$	→ $\frac{x}{16}$ =e or odd →	→ $\frac{3x+8}{8}$ =e →	→ $\frac{3x+4}{8}$ =e or odd →	→ $\frac{3x+2}{8}$ =e or odd →	→ $\frac{3x+1}{8}$ =e or odd →	→ $\frac{9x+10}{4}$ =e →

→ $\frac{9x+7}{4}$ =e →	→ $\frac{9x+5}{4}$ =e or odd →

$r=5$	→ $\frac{x}{32}$ =e or odd *	→ $\frac{3x+16}{16}$ =e or odd $	→ $\frac{3x+8}{16}$ =e or odd !	→ $\frac{3x+4}{16}$ =e or odd @	→ $\frac{3x+2}{16}$ =e or odd #	→ $\frac{3x+1}{16}$ =e or odd %
→ $\frac{9x+20}{8}$ =e @@	→ $\frac{9x+14}{8}$ =e &	→ $\frac{9x+11}{8}$ =e ((	→ $\frac{9x+10}{8}$ =e or odd {{	→ $\frac{9x+7}{8}$ =e or odd [[	→ $\frac{9x+5}{8}$ =e or odd **	→ $\frac{27x+19}{4}$ =e ***

$r=6$	* $\frac{x}{64}$	* $\frac{3x+32}{32}$	$ $\frac{3x+16}{32}$	! $\frac{3x+8}{32}$	@ $\frac{3x+4}{32}$	# $\frac{3x+2}{32}$	% $\frac{9x+19}{16}$
% $\frac{3x+1}{32}$	$ $\frac{9x+64}{16}$	{{ $\frac{27x+38}{8}$	@ $\frac{9x+28}{16}$	# $\frac{9x+22}{16}$	% $\frac{3x+1}{32}$	$ $\frac{9x+64}{16}$	! $\frac{9x+40}{16}$
@@ $\frac{9x+20}{16}$		& $\frac{9x+14}{16}$	(($\frac{9x+11}{16}$	{{ $\frac{9x+10}{16}$	[[$\frac{9x+7}{16}$	[[$\frac{27x+29}{16}$	** $\frac{9x+5}{16}$
** $\frac{27x+23}{8}$		*** $\frac{27x+19}{8}$					

Taha's Table to find lS(n) Pattern

$r = 1 = 3(0) + 1$ $k = 0$	$x = \frac{3^0 x}{2^{1-0}} \Rightarrow x = 0 \notin N_+, 0 \notin lS(n) \Rightarrow lS(n) \neq \{x\},\ \forall n \in N_+$
$r = 2 = 3(0) + 2$ $k = 0$	$x = \frac{3^0 x}{2^{2-0}} \Rightarrow x = \frac{1}{3} \notin N_+ \Rightarrow x, y \notin lS(n) \Rightarrow lS(n) \neq \{\, x, y\},\ \forall n \in N_+$
$r = 3 = 3(1)$ $k = 1$	$x = \frac{3^1 x + 3^0 2^0}{2^{3-1}} \Rightarrow x = 1, y = 4, z = 2 \Rightarrow lS(n) = \{4,2,1\},\ \forall n \in N_+$
$r = 4 = 3(1) + 1$ $k = 1$	$x = \frac{3^1 x + 3^0 2^0}{2^{4-1}} \Rightarrow\ x \notin N_+ \Rightarrow\ lS(n) = \emptyset, \forall n \in N_+$
$r = 5 = 3(1) + 2$ $k = 1$	$x = \frac{3^1 x + 3^0 2^0}{2^{5-1}} \Rightarrow\ x \notin N_+ \Rightarrow\ lS(n) = \emptyset, \forall n \in N_+$
$r = 6 = 3(2)$ $k = 2$	$x = \frac{3^2 x + 3(2^0) + 3^0 2^2}{2^{6-2}} \Rightarrow x = 1, y = 4, z = 2 \Rightarrow\ lS(n) = \{1,4,2\},\ \forall n \in N_+$
$r = 7 = 3(2) + 1$ $k = 2$	$x = \frac{3^2 x + 3(2^0) + 3^0 2^2}{2^{7-2}} \Rightarrow x = \frac{7}{23} \notin N_+ \Rightarrow\ lS(n) = \emptyset, \forall n \in N_+$
$r = 9 = 3(3)$ $k = 3$	$x = \frac{3^3 x + 3^2(2^0) + 3^1(2^2) + 3^0 2^4}{2^{9-3}} \Rightarrow x = 1, y = 4, z = 2 \Rightarrow lS(n) = \{1,4,2\},$ $\forall n \in N_+$
$r = 12 = 3(4)$ $k = 4$	$x = \frac{3^4 x + 3^3(2^0) + 3^2(2^2) + 3(2^4) + 3^0 2^6}{2^{12-4}} \Rightarrow x = 1 \Rightarrow$ $lS(n) = \{1,4,2\}, \forall n \in N_+$

$$Taha's\ Pattern\ to\ find\ lS(n)$$

$Part\ A$:

$i)\ if\ r = 3(k),$

$$is\ x = \frac{3^k x + 3^{k-1}(2^0) + 3^{k-2}(2^2) + 3^{k-3}(2^4) + \cdots + 3^{k-k+1}\left(2^{2(k-2)}\right) + (3^0)2^{2(k-1)}}{2^{r-k}} = 1?$$

Proof:

$$\therefore x = \frac{3^k x + 3^{k-1}(2^0) + 3^{k-2}(2^2) + 3^{k-3}(2^4) + \cdots + 3^{k-k+1}\left(2^{2(k-2)}\right) + (3^0)2^{2(k-1)}}{2^{3k-k}} \Rightarrow$$

$(2^{2k} - 3^k)x = 3^{k-1}(2^0) + 3^{k-2}(2^2) + 3^{k-3}(2^4) + \cdots + 3^{k-k+1}\left(2^{2(k-2)}\right) + (3^0)2^{2(k-1)} \ldots eq1.$

$let\ k = 4$

$\therefore (2^{2k} - 3^k) = (2^8 - 3^4) = 175 \ldots (\ Coefficient\ of\ x\ in\ LHS\ of\ eq1)$

$3^3(2^0) + 3^2(2^2) + 3^1(2^4) + 3^0(2^6) = 175 \ldots (RHS\ of\ eq1)$

$\therefore 175x = 175 \Rightarrow x = 1 \Rightarrow\ lS(n) = \{4,2,1\}, \forall n\ \in N_+\ when\ r = 3(k)$

$ii)\ if\ r = 3(k+1),$

$$is\ x = \frac{3^{k+1}x + 3^k(2^0) + 3^{k-1}(2^2) + 3^{k-2}(2^4) + \cdots + 3^{k-k+1}(2^{2k-2}) + (3^0)2^{2k}}{2^{r-(k+1)}} = 1\ ?$$

$Proof$:

$$x = \frac{3^{k+1}x + 3^k(2^0) + 3^{k-1}(2^2) + 3^{k-2}(2^4) + \cdots + 3^{k-k+1}(2^{2k-2}) + (3^0)2^{2k}}{2^{3(k+1)-(k+1)}} \Rightarrow$$

$(2^{2k+2} - 3^{k+1})x = 3^k(2^0) + 3^{k-1}(2^2) + 3^{k-2}(2^4) + \cdots + 3^{k-k+1}(2^{2k-2}) + (3^0)2^{2k} \ldots eq2.$

$let\ k = 4$

$\therefore (2^{2k+2} - 3^{k+1}) = (2^{10} - 3^5) = 781\ \ldots\ (\ Coefficient\ of\ x\ in\ LHS\ of\ eq2)$

$3^4(2^0) + 3^3(2^2) + 3^2(2^4) + 3^1(2^6) + 3^0(2^8) = 781 \ldots (RHS\ of\ eq2)$

$\therefore 781x = 781 \Rightarrow x = 1 \Rightarrow\ \ lS(n) = \{4,2,1\}, \forall n\ \in N_+\ when\ r = 3(k+1)$

$by\ i\ \&\ ii \Rightarrow\ lS(n) = \{4,2,1\}, \forall n\ \in N_+\ when\ r\ is\ divisible\ by\ 3.$

$Part\ B$:

$i)\ if\ \ r = 3(k) + 1, eq1\ below$:

$$is\ x = \frac{3^k x + 3^{k-1}(2^0) + 3^{k-2}(2^2) + 3^{k-3}(2^4) + \cdots + 3^{k-k+1}\left(2^{2(k-2)}\right) + 3^0 2^{2(k-1)}}{2^{r-k}} \notin N_+?$$

$Proof$:

$$x = \frac{3^k x + 3^{k-1}(2^0) + 3^{k-2}(2^2) + 3^{k-3}(2^4) + \cdots + 3^{k-k+1}\left(2^{2(k-2)}\right) + 3^0 2^{2(k-1)}}{2^{(3k+1)-k}}$$

$(2^{2k+1} - 3^k)x = 3^{k-1}(2^0) + 3^{k-2}(2^2) + 3^{k-3}(2^4) + \cdots + 3^{k-k+1}\left(2^{2(k-2)}\right) + 3^0 2^{2(k-1)} \ldots eq1$

$let\ k = 4$

$\therefore (2^{2k+1} - 3^k) = (2^9 - 3^4) = 431 \ldots (\ Coefficient\ of\ x\ in\ \ LHS\ of\ eq1)$

$RHS\ of\ eq1$: $2^3(2^0) + 3^2(2^2) + 3^1(2^4) + 3^0(2^6) = 156$

$\therefore 431x = 156 \Rightarrow x \notin N_+ \ \Rightarrow\ \nexists\ lS(n), \forall n\ \in N_+\ when\ r = 3(k) + 1$

$ii)\ if\ \ r = 3(k) + 2, is\ eq2\ below$:

$$x = \frac{3^k x + 3^{k-1}(2^0) + 3^{k-2}(2^2) + 3^{k-3}(2^4) + \cdots + 3^{k-k+1}\left(2^{2(k-2)}\right) + 3^0 2^{2(k-1)}}{2^{r-k}} \notin N_+?$$

$Proof$:

$$x = \frac{3^k x + 3^{k-1}(2^0) + 3^{k-2}(2^2) + 3^{k-3}(2^4) + \cdots + 3^{k-k+1}\left(2^{2(k-2)}\right) + 3^0 2^{2(k-1)}}{2^{(3k+2)-k}}$$

$(2^{2k+2} - 3^k)x = 3^{k-1}(2^0) + 3^{k-2}(2^2) + 3^{k-3}(2^4) + \cdots + 3^{k-k+1}\left(2^{2(k-2)}\right) + 3^0 2^{2(k-1)} \ldots eq2$

$let\ k = 4$

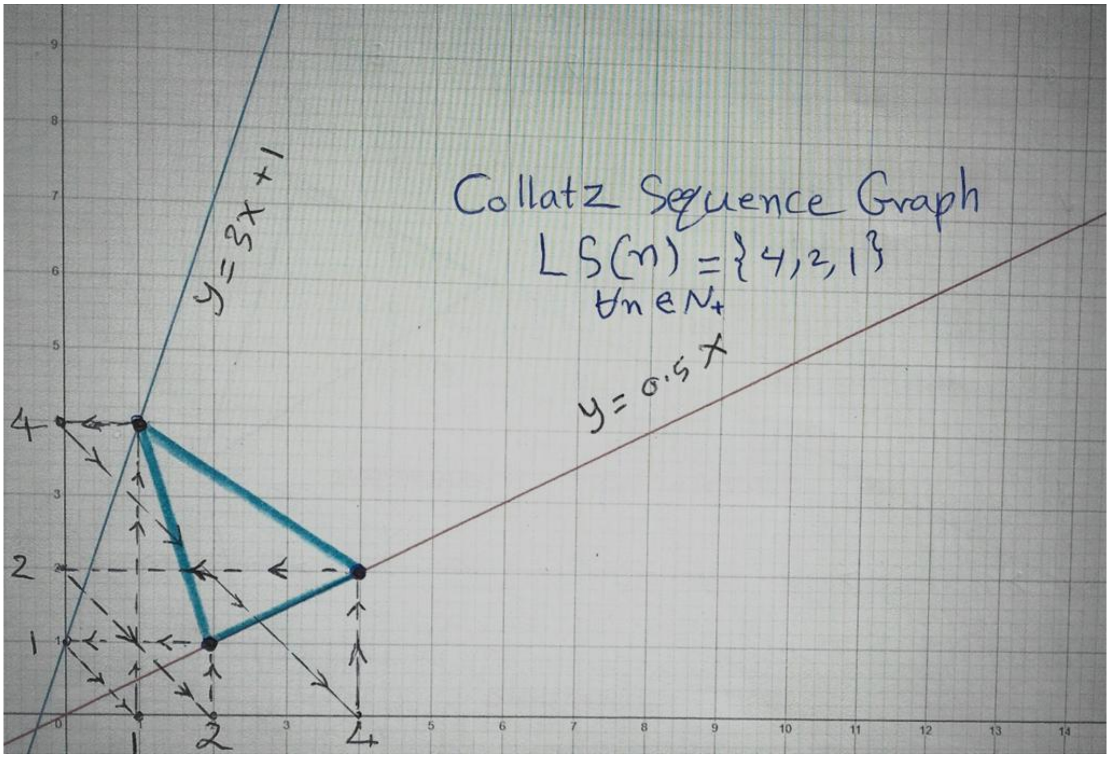
$\therefore (2^{2k+2} - 3^k) = (2^{10} - 3^4) = 943 \ldots (Coefficient\ of\ x\ in\ LHS\ of\ eq2)$

$RHS\ of\ eq2$: $2^3(2^0) + 3^2(2^2) + 3^1(2^4) + 3^0(2^6)$

$\therefore 943x = 156 \Rightarrow x \notin N_+ \ \Rightarrow\ \nexists\ lS(n), \forall n\ \in N_+\ when\ r = 3(k) + 2$

$by\ i\ \&ii$: $\nexists\ lS(n), \forall n\ \in N_+\ r = 3(k) + h, h \in \{1,2\}$

$\therefore Final\ Conclusion\ by\ parts\ A\ \&\ \ B \Rightarrow lS(n) = \{1,2,4\}, \forall n \in N_+$



$Taha's\ Graph\ to\ find\ S(n) and\ lS(n)$