Four index Einstein field equation

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July 24, 2025

Abstract

In this work I present possible extension of Einstein field equations into four index equation. It takes both Ricci and Weyl tensor [1] parts into account so whole curvature. It does differ from General Relativity as it does have stress momentum tensor for vacuum or saying other wise for gravity field itself. Still I do not define this tensor only it's properties.

Contents

1 Einstein field equation

Einstein field equations are basis of modern gravity theory. Still in some sense they are not fully finished as they lead to singularities both at black holes and moment of big bang. Most ideas about what is next is a quantum gravity theory that unites gravity and quantum mechanics. My approach will be trying to limit those equations by changing them into four index equations. That will have in some sense extend them but in truth will limit their form. Now whole curvature both Ricci and Weyl will be defined in field equations. Einstein field equations can be written using Einstein tensor simply as:

$$G_{ab} = \kappa T_{ab} \tag{1}$$

Where I skip convention of using Greek letters, as later there will be a lot more indexes manipulations and it will be easier to write it just in Latin letters, so meaning is still that those indexes represent space and time. In those equations there is another component to them, objects follow geodesics , from it follows those equations need another equations that say how object move, those can be written as:

$$\frac{d^2x^a}{ds^2} + \Gamma^a{}_{ij}\frac{dx^i}{ds}\frac{dx^j}{ds} = 0 \tag{2}$$

Geodesic equation is consequence of taking action of spacetime interval, or more precise it is a solution to equation:

$$\delta \int ds = 0 \tag{3}$$

It means that not only field equations are need to fully have working General Relativity but geodesic equation. One can argue that one follows from another and that equivalence principle is where geodesic equation comes from but for now I will just take as a fact that those two equations are need to make general relativity work. Still those equations are most tested theory of gravity , but from them follows that at center of a black hole and moment of big bang those fail as they lead to singularities as mention before. So question is are there a final form of classical gravity equations? Let me take into account simple case where matter field vanish:

$$G_{ab} = 0 \tag{4}$$

Those are so called vacuum solutions, where matter field vanishes. Only condition that arises from it them is that Ricci tensor does vanish. It still lacks full curvature picture as those equations contain only Ricci tensor and no Weyl tensor [1]. It means that there is no gravitational energy tensor in those equations. My approach will be that there is a gravity energy tensor that later will be equal to Weyl tensor [1]. But before arriving at those conclusions let's go back to Bianachi identities [2] .

2 Bianachi identity and conservation of stress momentum tensor

Base of General Relativity is that locally stress momentum tensor is conserved as same goes with right side of equations so Einstein tensor. I can write that tensor as:

$$G_{ab} = R_{ab} - \frac{1}{2} g_{ab} R \tag{5}$$

It's total covariant derivative vanishes , same happens with stress momentum tensor:

$$\nabla^a G_{ab} = \nabla^a \left(R_{ab} - \frac{1}{2} g_{ab} R \right) = 0 \tag{6}$$

$$\nabla^a T_{ab} = 0 \tag{7}$$

It means that in Einstein field equations both side vanish when taking covariant derivative, this gives important clue about four index equation. In general I'm searching for a four index tensor that when taken covariant derivative will vanish. Next it has to reduce to normal two index equations when taken contraction. So in general im searching for a tensor that has this properties:

$$\nabla^a G_{abcd} = 0 \tag{8}$$

$$g^{ac} G_{abcd} = G_{bd} = R_{bd} - \frac{1}{2} g_{bd} R \tag{9}$$

But same must happen for four index stress momentum tensor, it has to vanish when taking covariant derivative and it has to reduce to stress momentum tensor upon contraction:

$$\nabla^a T_{abcd} = 0 \tag{10}$$

$$g^{ac} T_{abcd} = T_{bd} \tag{11}$$

That will lead to equation of field that equals those two tensors times Einstein constant on side of stress momentum part. And here comes key change this generalized tensor will have energy of a gravity field as there will be Weyl tensor [1] part equal to trace-less part of stress momentum tensor. Or in other words I need to define whole curvature and it's energy, not only Ricci part. Equation in form of $G_{abcd} = \kappa T_{abcd}$ needs to define whole spacetime curvature as consequence of presence of energy both gravitational and matter. In this work I will derive simplest case of this equation that works only in four dimensional spacetime. It's simplest in this way that is fairly easy to derive and stands on simple assumptions and grounds in general. All those assumptions are in those statement from above and additionally that I can use only Ricci and Weyl tensor [1] without full Riemann tensor as it makes those equations in their simplest possible form. To do so I need to write covariant derivative of this tensor and connect it to stress momentum part, that is long calculation but in truth a simple one.

3 Is there a simple way to generalize Einstein?

Let me start by writing covariant derivative of Weyl tensor [1] and Bianachi identities [2]:

$$\nabla^a C_{abcd} = \frac{n-3}{n-2}\left(\nabla_d R_{bc} - \nabla_c R_{bd} - \frac{1}{2(n-1)}\left(g_{bc}\nabla_d - g_{bd}\nabla_c\right) R\right) \tag{12}$$

$$\nabla^a R_{ab} = \frac{1}{2}\nabla^a g_{ab} R \tag{13}$$

$$\nabla^a R_{ab} = \frac{1}{2}\nabla_b R \tag{14}$$

Now all I need to to do is find a tensor that reduces to Einstein tensor and is build from Ricci tensor and metric tensors. Then calculate it's covariant derivative. It's pretty simple task. I can write this kind of tensor as:

$$\frac{1}{n-2}\left(R_{ac}g_{bd} - R_{ad}g_{bc} + R_{bd}g_{ac} - R_{bc}g_{ad}\right) - \frac{n}{2\left(n-2\right)\left(n-1\right)}\left(g_{ac}g_{bd} - g_{ad}g_{bc}\right) R \tag{15}$$

Now what is a bit harder is to calculate covariant derivative of this tensor, but doing it step by step and using Bianachi identity [2]:

$$\frac{1}{n-2}\nabla^a\left(R_{ac}g_{bd} - R_{ad}g_{bc} + R_{bd}g_{ac} - R_{bc}g_{ad}\right) \tag{16}$$

$$\frac{1}{n-2}\left(\nabla^a R_{ac}g_{bd} - \nabla^a R_{ad}g_{bc} + \nabla_c R_{bd} - \nabla_d R_{bc}\right) \tag{17}$$

$$\frac{1}{n-2}\left(\frac{1}{2}\nabla_c R g_{bd} - \frac{1}{2}\nabla_d R g_{bc} + \nabla_c R_{bd} - \nabla_d R_{bc}\right) \tag{18}$$

$$\tag{19}$$

Now I can add part with Ricci scalar:

$$\frac{1}{n-2}\left(\frac{1}{2}\nabla_c R g_{bd} - \frac{1}{2}\nabla_d R g_{bc} + \nabla_c R_{bd} - \nabla_d R_{bc}\right) - \frac{n}{2\left(n-2\right)\left(n-1\right)}\left(\nabla_c g_{bd} - \nabla_d g_{bc}\right) R \tag{20}$$

Take our parts with Ricci scalar and add them to rest:

$$\frac{1}{n-2}\left(\nabla_c R_{bd} - \nabla_d R_{bc}\right) - \frac{n}{2\left(n-2\right)\left(n-1\right)}\left(\nabla_c g_{bd} - \nabla_d g_{bc}\right) R + \frac{1}{2\left(n-2\right)}\left(\nabla_c g_{bd} - \nabla_d g_{bc}\right) R \tag{21}$$

That will give me when summing Ricci parts:

$$\frac{1}{n-2}\left(\nabla_c R_{bd} - \nabla_d R_{bc}\right) - \frac{1}{2\left(n-1\right)\left(n-2\right)}\left(\nabla_c g_{bd} - \nabla_d g_{bc}\right) R \tag{22}$$

And finally taking out common part in both equations:

$$\frac{1}{n-2}\left(\left(\nabla_c R_{bd} - \nabla_d R_{bc}\right) - \frac{1}{2\left(n-1\right)}\left(\nabla_c g_{bd} - \nabla_d g_{bc}\right) R\right) \tag{23}$$

That is exactly minus Weyl tensor [1] covariant derivative with only change is the constant that is missing $n-3$.

4 Four index Einstein tensor

Now it's pretty straightforward to create four index Einstein tensor, I just need to divide Weyl tensor [1] by $n-3$ and add tensor with Ricci tensors and metric tensors defined before. That written in whole form gives:

$$G_{abcd} = \frac{1}{n-3} C_{abcd} + \frac{1}{n-2} \left(R_{ac} g_{bd} - R_{ad} g_{bc} + R_{bd} g_{ac} - R_{bc} g_{ad} \right) - \frac{n}{2\left(n-2\right)\left(n-1\right)} \left(g_{ac} g_{bd} - g_{ad} g_{bc} \right) R \tag{24}$$

Now I can calculate covariant derivative of this whole tensor:

$$\nabla^a G_{abcd} = \frac{1}{n-2} \left(\nabla_d R_{bc} - \nabla_c R_{bd} - \frac{1}{2(n-1)} \left(g_{bc} \nabla_d - g_{bd} \nabla_c \right) R \right) + \frac{1}{n-2} \left(\left(\nabla_c R_{bd} - \nabla_d R_{bc} \right) - \frac{1}{2\left(n-1\right)} \left(\nabla_c g_{bd} - \nabla_d g_{bc} \right) R \right) = 0 \tag{25}$$

That is as expected equal to zero. Last thing to check is contraction of this tensor is equal to Einstein tensor that will be total six contractions, two of them will give Einstein tensor and two minus it last two will give zero:

$$g^{ab} G_{abcd} = 0 \tag{26}$$
$$g^{ac} G_{abcd} = G_{bd} \tag{27}$$
$$g^{ad} G_{abcd} = -G_{bc} \tag{28}$$
$$g^{bc} G_{abcd} = -G_{ad} \tag{29}$$
$$g^{bd} G_{abcd} = G_{ac} \tag{30}$$
$$g^{cd} G_{abcd} = 0 \tag{31}$$

So this tensor has symmetries of Riemann tensor. Weyl tensor [1] part always vanishes as it's trace-less for all indexes. Next step is to find stress momentum tensor that has same properties. That means that it will fulfill:

$$g^{ab} T_{abcd} = 0 \tag{32}$$
$$g^{ac} T_{abcd} = T_{bd} \tag{33}$$
$$g^{ad} T_{abcd} = -T_{bc} \tag{34}$$
$$g^{bc} T_{abcd} = -T_{ad} \tag{35}$$
$$g^{bd} T_{abcd} = T_{ac} \tag{36}$$
$$g^{cd} T_{abcd} = 0 \tag{37}$$

And from it follows that it's covariant derivative will vanish, this tensor has to be split into two parts just like Einstein tensor parts, one for vacuum one for matter field. In total it will give energy of both gravity field and matter field.

5 Four index stress momentum tensor

I can use analogy from before but this time need to calculate covariant derivative of new stress momentum tensor, that as Ricci part will be equal to, where for now I skip part with vacuum energy:

$$\frac{1}{n-2}\left(T_{ac}g_{bd}-T_{ad}g_{bc}+T_{bd}g_{ac}-T_{bc}g_{ad}\right)-\frac{1}{\left(n-2\right)\left(n-1\right)}\left(g_{ac}g_{bd}-g_{ad}g_{bc}\right)T \tag{38}$$

So I can calculate covariant derivative of this tensor , again split into two parts to make calculation easier:

$$\frac{1}{n-2}\nabla^{a}\left(T_{ac}g_{bd}-T_{ad}g_{bc}+T_{bd}g_{ac}-T_{bc}g_{ad}\right) \tag{39}$$

$$\frac{1}{n-2}\left(\nabla^{a}T_{ac}g_{bd}-\nabla^{a}T_{ad}g_{bc}+\nabla_{c}T_{bd}-\nabla_{d}T_{bc}\right) \tag{40}$$

$$\frac{1}{n-2}\left(\nabla_{c}T_{bd}-\nabla_{d}T_{bc}\right)-\frac{1}{\left(n-2\right)\left(n-1\right)}\left(\nabla_{c}g_{bd}-\nabla_{d}g_{bc}\right)T \tag{41}$$

Now let me take out common denominator:

$$\frac{1}{n-2}\left(\left(\nabla_{c}T_{bd}-\nabla_{d}T_{bc}\right)-\frac{1}{\left(n-1\right)}\left(\nabla_{c}g_{bd}-\nabla_{d}g_{bc}\right)T\right) \tag{42}$$

That is exactly propagation equation from General Relativity when I take out constant $n-3$. Now I will add new term that will define a vacuum, vacuum stress momentum tensor that covariant derivative is equal to minus Weyl tensor [1]:

$$\begin{aligned}T_{abcd}=\frac{1}{n-3}\hat{T}_{abcd}+\frac{1}{n-2}\left(T_{ac}g_{bd}-T_{ad}g_{bc}+T_{bd}g_{ac}-T_{bc}g_{ad}\right)\\-\frac{1}{\left(n-2\right)\left(n-1\right)}\left(g_{ac}g_{bd}-g_{ad}g_{bc}\right)T\end{aligned} \tag{43}$$

Now it can be see that covariant derivative vanishes if propagation equation is fulfilled. That means that it does vanish in general case. So I can write that:

$$\nabla^{a}T_{abcd}=0 \tag{44}$$

$$\begin{aligned}&\frac{1}{n-2}\left(\left(\nabla_{c}T_{bd}-\nabla_{d}T_{bc}\right)-\frac{1}{\left(n-1\right)}\left(\nabla_{c}g_{bd}-\nabla_{d}g_{bc}\right)T\right)\\-&\frac{1}{n-2}\left(\nabla_{d}R_{bc}-\nabla_{c}R_{bd}-\frac{1}{2(n-1)}\left(g_{bc}\nabla_{d}-g_{bd}\nabla_{c}\right)R\right)\end{aligned} \tag{45}$$

$$\begin{aligned}&\frac{1}{n-2}\left(\left(\nabla_{c}T_{bd}-\nabla_{d}T_{bc}\right)-\frac{1}{\left(n-1\right)}\left(\nabla_{c}g_{bd}-\nabla_{d}g_{bc}\right)T\right)\\=&\frac{1}{n-2}\left(\nabla_{d}R_{bc}-\nabla_{c}R_{bd}-\frac{1}{2(n-1)}\left(g_{bc}\nabla_{d}-g_{bd}\nabla_{c}\right)R\right)\end{aligned} \tag{46}$$

Where crucial here is that $\nabla^{a}\hat{T}_{abcd}=-\nabla^{a}C_{abcd}$ that means that covariant derivative of vacuum energy tensor is equal to minus sign of Weyl tensor [1]. It has to in order to for this equations to be fulfilled.

6 Four index field equation

Now I have all the parts to create full field equation. Equation will just state that:

$$G_{abcd} = \kappa T_{abcd} \tag{47}$$

That can be expanded:

$$\begin{aligned} \frac{1}{n-3} C_{abcd} + \frac{1}{n-2} \left(R_{ac} g_{bd} - R_{ad} g_{bc} + R_{bd} g_{ac} - R_{bc} g_{ad} \right) \\ - \frac{n}{2(n-2)(n-1)} \left(g_{ac} g_{bd} - g_{ad} g_{bc} \right) R \\ = \frac{1}{n-3} \hat{T}_{abcd} + \frac{1}{n-2} \left(T_{ac} g_{bd} - T_{ad} g_{bc} + T_{bd} g_{ac} - T_{bc} g_{ad} \right) \\ - \frac{1}{(n-2)(n-1)} \left(g_{ac} g_{bd} - g_{ad} g_{bc} \right) T \end{aligned} \tag{48}$$

First I examine covariant derivative connections as their are simplest part that comes out of those equations, to make notation simpler I will use this notation:

$$\begin{aligned} \tilde{R}_{abcd} = \frac{1}{n-2} \left(R_{ac} g_{bd} - R_{ad} g_{bc} + R_{bd} g_{ac} - R_{bc} g_{ad} \right) \\ - \frac{n}{2(n-2)(n-1)} \left(g_{ac} g_{bd} - g_{ad} g_{bc} \right) R \end{aligned} \tag{49}$$

$$\begin{aligned} \tilde{T}_{abcd} = \frac{1}{n-2} \left(T_{ac} g_{bd} - T_{ad} g_{bc} + T_{bd} g_{ac} - T_{bc} g_{ad} \right) \\ - \frac{1}{(n-2)(n-1)} \left(g_{ac} g_{bd} - g_{ad} g_{bc} \right) T \end{aligned} \tag{50}$$

There is cross relation between covariant derivatives:

$$\nabla^a \tilde{R}_{abcd} = \frac{1}{n-3} \nabla^a \hat{T}_{abcd} \tag{51}$$

$$\nabla^a \tilde{T}_{abcd} = \frac{1}{n-3} \nabla^a C_{abcd} \tag{52}$$

That suggest strongly that there is relation between opposites in those equations. Matter field affects gravity or more precise Weyl tensor [1] and opposite vacuum energy affects Ricci tensor part. Still in general sense this equation is conserved only if those relations hold. And I can add new equations to them with minus sign as if I switch objects in those equations will get minus sign:

$$\nabla^a \tilde{R}_{abcd} = -\frac{1}{n-3} \nabla^a C_{abcd} \tag{53}$$

$$\nabla^a \tilde{T}_{abcd} = -\frac{1}{n-3} \nabla^a \hat{T}_{abcd} \tag{54}$$

That is just normal need in field equations for covariant derivative to vanish. Where they do differ by a constant that in four dimensions is just equal to one.

7 Matter field can't exist without gravity field

From field equations there are two kinds of solutions taking out trivial ones where I arrive just at flat spacetime. Those trivial ones are just saying four index Einstein tensor is equal to zero so that $G_{abcd} = 0$, but skipping those I have two class of equations left. First are vacuum solutions where Ricci tensor does vanish and Im left only with Weyl tensor [1]. This can be written as:

$$C_{abcd} = \kappa \hat{T}_{abcd} \tag{55}$$

So vacuum solutions give zero covariant derivative on both sides as Ricci tensor goes to zero so:

$$\nabla^a C_{abcd} = \nabla^a \hat{T}_{abcd} = 0 \tag{56}$$

They are correct solutions from point of view of those equations. But there is second type of solutions, where matter field does not vanish. Those type of solutions are more complex, as to make covariant derivative vanish both Weyl and Ricci tensor must be non zero. It means that in those extensions of General Relativity there is not pure Ricci solutions, there are either Ricci flat or or whole curvature is not equal to zero, both Weyl and Ricci. It gives physical consequences, matter field can't exist without a vacuum field that is connected to it, or saying it more simply, both shapes and volumes of matter field change when energy is present. This statement can be written as:

$$\nabla^a G_{abcd} = \nabla^a T_{abcd} = 0 \tag{57}$$

$$\nabla^a \tilde{R}_{abcd} + \frac{1}{n-3} \nabla^a C_{abcd} = 0 \tag{58}$$

$$\nabla^a \tilde{T}_{abcd} + \frac{1}{n-3} \nabla^a \hat{T}_{abcd} = 0 \tag{59}$$

So in order to both side vanish when taking covariant derivative there is need for both Weyl tensor [1] and Ricci tensor to not vanish. It means that from point of view of this model standard cosmological solutions are not correct as there Weyl tensor [1] does vanish. It means that there has to be some kind of gravity propagation in universe solution for this model to hold. This affects other stuff like gravitational collapse where both matter and gravity will be acting on both Ricci and Weyl tensor [1]. Still its not easy task to derive complete model or solutions to those equations. As biggest problem is that there is one tensor that is not fully defined. Only properties of that tensor are defined I mean vacuum energy tensor is defined by two equations:

$$g^{ab} \hat{T}_{abcd} = g^{ac} \hat{T}_{abcd} = g^{ad} \hat{T}_{abcd} = g^{bc} \hat{T}_{abcd} = g^{bd} \hat{T}_{abcd} = g^{cd} \hat{T}_{abcd} = 0 \tag{60}$$

$$\nabla^a \hat{T}_{abcd} = -\nabla^a C_{abcd} \tag{61}$$

And for vacuum solutions there is third equation:

$$\kappa \hat{T}_{abcd} = C_{abcd} \tag{62}$$

That is all information about this tensor , it's lacking direct form and it's not easy task to create an equation for this tensor that is well defined. Still it has to fulfill all those three equations.

References

[1] A. García, F. W. Hehl, C. Heinicke, A. Macías, *The Cotton tensor in Riemannian spacetimes*, arXiv:gr-qc/0309008, 2004.

[2] J. O. Weatherall, J. B. Manchak, *Conformal Tensors and the Bianchi Identities*, arXiv:2302.07128, 2023.