The Complete Proof of the Goldbach Binary Conjecture

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Abstract: For $m \in \mathbb{Z}$, let $N = 2m \geq 8$ and $\mathbb{G}_N$ be a set of *goldbach primes*, p, of N defined as $\mathbb{G}_N = \{p : p \leq \frac{N}{2} \text{ and } p \nmid N\}$. By denoting the cardinality of $\mathbb{G}_N$ by $\mid \mathbb{G}_N \mid$ or $g(N)$, we show that $\forall N, \mid \mathbb{G}_N \mid > 0$, and the set of these cardinalities, $\{\mid \mathbb{G}_N \mid\}$, is equal to the set of natural numbers i.e $\{\mid \mathbb{G}_N \mid\} = \mathbb{N}$. We finally prove the famous Goldbach binary conjecture by showing that $\sum_{i=1}^{|\mathbb{G}_N|} \mu(b_i)\Lambda(b_i) < 0$, whenever $b_i = N - p_i$ with $p_i \in \mathbb{G}_N, i \in \mathbb{N}$ and $1 \leq i \leq \mid \mathbb{G}_N \mid$. In particular we show that every N is a sum of two distinct primes.

1. Introduction

The Goldbach Conjecture was first proposed by Christian Goldbach, a Prussian mathematician, in a letter to Leonhard Euler in 1742. Goldbach conjectured that every even integer greater than 2 can be expressed as the sum of two prime numbers. This became known as Goldbach strong(binary) conjecture. Euler, one of the most prolific mathematicians of all time, believed in the truth of the conjecture Goldbach proposed, but he could not provide a formal proof. A weaker form of the conjecture, known as Goldbach's ternary conjecture, states: Every odd integer greater than 5 can be expressed as the sum of three prime numbers. Euler replied that if the strong conjecture were true, it would imply the weak conjecture. He believed the conjecture to be certainly true but was unable to provide a formal proof. While the weak conjecture was eventually proven by Harald Helfgott in 2013, via a preprint made publicly available on arXiv. The strong conjecture remains unproven despite extensive efforts; numerous computational verifications have confirmed the conjecture for even numbers up to 4×10^{18}.

2. Definitions and Notations

Throughout this paper, the following definitions and notations will be used:

- Every even number greater than 6 will be denoted by N.
- The following sets of numbers will be denoted as shown in the table below:

Set	Notation	First Elements of the Set
Natural Numbers	$\mathbb{N}$	$1, 2, 3, 4, 5, 6, \cdots$
Integers	$\mathbb{Z}$	$0, \pm1, \pm2, \pm3, \pm4, \pm5, \cdots$
Even Numbers	$\mathbb{E}$	$0, \pm2, \pm4, \pm6, \pm8, \pm10, \pm12, \cdots$
Prime Numbers	$\mathbb{P}$	$2, 3, 5, 7, 11, 13, 17, 19, 23, 29, \cdots$

- $P_n^{\#} = \prod_{k=1}^{n} p_k$ where p_k is the k^{th} prime i.e $p_1 = 2, p_2 = 3, p_3 = 5, p_4 = 7, p_5 = 11, \cdots$
- We use (a, b) to denote the greatest common divisor of two natural numbers a and b.
- We write $a \mid b$ and $a \nmid b$ to denote the assertion that a divides b and a does not divide b respectively.
- $\Rightarrow$ stands for "implies that" ; $\Leftrightarrow$ stands for "if and only if" or simply "iff".
- $\in$ and $\notin$ are statements "is an element of" and "is not an element of" respectively.
- $\forall$ is a statement "for all" and $\exists$ is translated as "there exists"
- $<$ and $\leq$ stand for "less than" and "less than or equal to" respectively.
- $\mathbb{H}_N$ is a set of all primes not exceeding $\frac{N}{2}$ i.e $\mathbb{H}_N = \left\{p : p \leq \frac{N}{2}\right\}$, where p is a prime. The cardinality of a set $\mathbb{H}_N$ will be denoted by $|\mathbb{H}_N|$.
- The Divisor set, $\mathbb{D}_N$ is a set of all positive divisors of N. Hence $\mathbb{D}_N = \{a \in \mathbb{N} : a \mid N\}$.
- **Goldbach prime**: A prime number p such that $p \in \mathbb{H}_N$ and $p \nmid N$ will be called a *Goldbach prime* of N. That is, p is a Goldbach prime if and only if $p \leq \frac{N}{2}$ and $p \nmid N$. The number of Goldbach primes of N will be denoted by $g(N)$.
- $\mathbb{G}_N$ is a set of Goldbach primes of N i.e. for p prime, we have $\mathbb{G}_N = \left\{p : \frac{N}{2} \geq p \nmid N\right\}$. Hence we shall be using $g(N)$ and $|\mathbb{G}_N|$ interchangeably i.e $g(N) = |\mathbb{G}_N|$.
- We define the set of cardinalities: $\{g(N)\} = \{g(N) \ : \ N = 8, 10, 12, 14, 16, 18, 20, \cdots\}$
- Let $a \in \mathbb{H}_N$ be prime, $b \in \mathbb{N}$ and $N \in \mathbb{E}$ with $a + b = N$ where $a \leq b$. We say that a corresponds to a prime iff b is prime; otherwise a corresponds to a composite. Sometimes we shall be writing $a \to b$ if $a + b = N$.

We shall also use the following standard notations for Arithmetic functions:

- $\pi(n)$ is the *prime counting function* i.e the number of primes not exceeding n.
- $\omega(n)$ is the *little prime omega function*, which represents the number of distinct prime factors (divisors) of n.
- $\mu(n)$ is the *Moebius function* of n.
- $\Lambda(n)$ is the *von Mangoldt function* of n.
- We have introduced a new arithmetic function, $g(n)$, the goldbach-prime counting function i.e the number of primes not exceeding $\frac{n}{2}$ and relatively prime to n.

3. Fundamental Concepts:

Lemma 3.1: Let a and b be two natural numbers with $a < b$ such that $a + b = N$.

(i) If $a|N$ then $a|b$.

(ii) If $a \nmid N$ then $a \nmid b$.

(iii) If a is prime and $a \nmid N$ then $(a, b) = (a, N) = (b, N) = 1$.

Proof:

(i) The proof follows from the linearity property of divisibility i.e

$a|N \Rightarrow \exists x \in \mathbb{Z}$ such that $N = ax$

$\Rightarrow b = N - a = ax - a = a(x - 1)$. Hence $a|b$.

(ii) $a \nmid N \Rightarrow N = ax + k$ for some integers x and k with $0 < k < a$.

$\therefore b = N - a = ax + k - a = a(x - 1) + k$. Since $0 < k < a$ it follows that $a \nmid b$.

(iii) It is clear from (ii) that $(a, N) = (a, b) = 1$ since a is prime.

$(b, N) = k \neq 1 \Rightarrow \exists x, y \in \mathbb{Z}$ such that $b = kx$ and $N = ky$.

Hence $a + kx = ky \Rightarrow a = ky - kx = k(y - x)$

Since a is prime then either $k = a$ and $y - x = 1$ or $k = 1$ and $y - x = a$. But $k \neq 1 \Rightarrow k = a$, contradicting that $a \nmid N$. ■

Corollary 3.1(of Lemma 3.1).

(i) Given that $a+b = N$ with $a \in \mathbb{D}_N$ then $a|b$ hence b is composite. This means that all divisors of N correspond to composite numbers, to their multiples in particular(WLOG assume $\frac{N}{2}$ is always composite for all N).

Example. Consider the number 70 being partitioned as shown in the following table.

2	3	5	7	11	13	17	19	23	29	31
68	67	65	63	59	57	53	51	47	41	39

We can observe that all the prime factors of 70 i.e $\{2, 5, 7\}$ correspond to their respective multiples $\{68, 65, 63\}$ and $\mathbb{G}_{70} = \{3, 11, 13, 17, 19, 23, 29, 31\}$.

(ii) If $a \in \mathbb{G}_N$ then $a \nmid b$ hence b is either prime or composite. If b is composite the it must have only the Goldbach prime(s) other than a as its prime factor(s).

Definitions: Consider the following Arithmetic functions.

1. The Moebius function $\mu(n)$ defined as:

$$\mu(n) = \begin{cases} 1, & \text{for } n = 1 \\ (-1)^k, & \text{for } n = p_1^{\alpha_1} p_2^{\alpha_2} p_3^{\alpha_3} \cdots p_k^{\alpha_k} \text{ and } \alpha_1 = \alpha_2 = \alpha_3 = \cdots = \alpha_k = 1 \\ 0, & \text{otherwise.} \end{cases} \quad (1)$$

2. The von Mangoldt function $\Lambda(n)$ defined as:

$$\Lambda(n) = \begin{cases} \log p, & \text{for } n = p^m \text{ where } p \text{ is prime and } m \geq 1 \\ 0, & \text{otherwise.} \end{cases} \quad (2)$$

Lemma 3.2: The product $\mu(n)\Lambda(n) \leq 0$, $\forall n \in \mathbb{N}$. In particular,

$$\mu(n)\Lambda(n) < 0, \text{ iff } n \text{ is prime;}$$
$$\mu(n)\Lambda(n) = 0, \text{ otherwise.}$$

Proof: (**Cases:**)

(i) If n is prime then;
$\mu(n) = -1$ and $\Lambda(n) = \log n$.
$\therefore \mu(n)\Lambda(n) = -1 \times \log n < 0$.
It is easy to show that the converse also holds.

(ii) If n is composite then;
Either $n = p_1 p_2 p_3 \cdots p_k$ with $k \geq 2$ and $p_i \neq p_j \, \forall \, 1 \leq i, j \leq k$ i.e n is square free.
$\Rightarrow \mu(n) = 1$ or -1 and $\Lambda(n) = 0$.
$\therefore \mu(n)\Lambda(n) = (1 \text{ or } -1) \times 0 = 0$.

Or $n = p_1^{\alpha_1} p_2^{\alpha_2} p_3^{\alpha_3} \cdots p_k^{\alpha_k}$ with $k \geq 1$ and $\alpha_i \geq 2$ for $i = 1, 2, 3, \cdots, k$.
$\Rightarrow \mu(n) = 0$ and $\Lambda(n) = \log p_i$ for some i in $1 \leq i \leq k$.
$\therefore \mu(n)\Lambda(n) = 0 \times \log p = 0$ ■

Lemma 3.3(Bertrand's postulate): *For every $n > 1$, there is always at least one prime p such that $n < p < 2n$.*

Theorem 3.1 (Existence of Goldbach primes): *The set $\mathbb{G}_N$ of Goldbach primes is non-empty for all even numbers $N > 6$.*

Proof:

We shall consider three cases:

Case(i): $6 < N < 30$.

It can be verified manually that for $6 < N < 30$ the set $\mathbb{G}_N$ is non-empty. The following table summarizes the results.

N	8	10	12	14	16	18	20	22	24	26	28
Elements of $\mathbb{G}_N$	3	3	5	3,5	3,5,7	5,7	3,7	3,5,7	5,7,11	3,5,7,11	3,5,11,13

Case(ii): $N = P_n^{\#}$ for $n \geq 3$.

Here the first value of N is 30, the third primorial, $P_3^{\#}$, and the proof is based on the following result.

Lemma 3.4: *If N is a primorial greater than 6 then $\mathbb{G}_N$ is non-empty.*

Proof:(By Mathematical Induction)

Required to prove that $\forall N = P_{n\geq 3}^{\#}$, $\pi\left(\frac{N}{2}\right) > \omega(N)$.

For $n = 3$, $N = P_3^{\#} = 30$.

$\pi\left(\frac{N}{2}\right) = \pi\left(\frac{30}{2}\right) = \pi(15) = 6$. Also $\omega(N) = \omega(30) = 3$.

Hence $\pi\left(\frac{30}{2}\right) = 6 > 3 = \omega(30)$. Therefore the statement holds for $n = 3$.

Assuming it holds for some $n = k$ i.e $N = P_k^{\#}$, then we have

$$\pi\left(\frac{N}{2}\right) = \pi\left(\frac{P_k^{\#}}{2}\right) = \pi\left(\prod_{i=2}^{k} p_i\right) > \omega(N) = \omega\left(P_k^{\#}\right) = k. \tag{3}$$

For $n = k + 1$, $N = P_{k+1}^{\#}$;

$$\pi\left(\frac{N}{2}\right) = \pi\left(\frac{P_{k+1}^{\#}}{2}\right) = \pi\left(\prod_{i=2}^{k+1} p_i\right) = \pi\left(p_{k+1}\prod_{i=2}^{k} p_i\right) \tag{4}$$

$$\text{It is clear that} \prod_{i=2}^{k} p_i > \pi\left(\prod_{i=2}^{k} p_i\right) \text{and} \, \pi\left(\prod_{i=2}^{k} p_i\right) > k[\text{from (4)}] \text{ implies} \prod_{i=2}^{k} p_i > k. \tag{5}$$

The combination of (4) and (5) gives us $\pi\left(p_{k+1}\prod_{i=2}^{k} p_i\right) \geq \pi\left(k \cdot p_{k+1}\right)$.

But $\pi(p_{k+1}) = k+1$; and since $k \geq 3$, it follows from *Bertrand's postulate(theorem)* that there exists a prime in the open interval $(p_{k+1},\, k \cdot p_{k+1})$.

Hence $\pi(p_{k+1} \cdot k) > k+1$ and therefore $\pi\left(p_{k+1}\prod_{i=2}^{k} p_i\right) > k+1 = \omega(P_{k+1}^{\#}) = \omega(N)$.

$$\therefore \pi\left(\frac{N}{2}\right) = \pi\left(\frac{P_{k+1}^{\#}}{2}\right) = \pi\left(\prod_{i=2}^{k+1} p_i\right) = \pi\left(p_{k+1}\prod_{i=2}^{k} p_i\right) > k+1 = \omega(P_{k+1}^{\#}) = \omega(N).$$

The statement is therefore true for all values of $n \geq 3$.

Case (iii): $P_k^{\#} < N < P_{k+1}^{\#}$.

Since $\omega(P_{k+1}^{\#}) = k+1$, it follows that $\omega(N) \leq k$.

Also $\omega\left(P_k^{\#}\right) = k$ and it is now clear from **Lemma 3.4** that

$$\pi\left(\frac{P_k^{\#}}{2}\right) > \omega(P_k^{\#}) = k \Rightarrow \pi\left(\frac{P_k^{\#}}{2}\right) > \omega(N).$$

$$\text{Now } N > P_k^{\#} \Rightarrow \frac{N}{2} > \frac{P_k^{\#}}{2} \text{ and it follows that } \pi\left(\frac{N}{2}\right) \geq \pi\left(\frac{P_k^{\#}}{2}\right) > \omega(N).$$

$$\therefore \pi\left(\frac{N}{2}\right) > \omega(N).$$

This completes the proof of **Theorem 3.1**. ■

Lemma 3.5: $\{g(N)\} = \{|\mathbb{G}_N|\} = \mathbb{N} = \{1, 2, 3, 4, 5, 6, \cdots\}$.

Proof:

For simplicity, let $N = 2p_n$ where p_n is the n^{th} prime and $n \geq 3$.

It is easy to see that for $1 < i < n$, $p_i \nmid N$ and $|\mathbb{G}_N| = n - 2 = \mathbb{N}$. ■

Corollary 3.2 (of **Theorem 3.1**) Both $\mathbb{G}_N$ and $\mathbb{D}_N$, the proper subsets of $\mathbb{H}_N$ are non-empty.

Axiom 3.1 All prime divisors of N are elements of $\mathbb{H}_N$ and so are the proper prime divisors of any arbitrary composite number $b < N$.

Theorem 3.2 (Goldbach Conjecture): *Every even number greater than 2 is a sum of two primes.*

Proof:

The conjecture is trivial for even numbers 4 and 6 since $4 = 2 + 2$ and $6 = 3 + 3$.

We shall therefore prove, using the principle of mathematical induction, that every even number greater than 6 is a sum of two distinct primes.

Mathematically, we have;

$$p + b = N; \text{ where } p \in \mathbb{G}_N,\ N \in \mathbb{E}_{>6} \text{ and } b \text{ is a prime.}$$

We have already shown (from **Theorem 3.1**) that every even number $N > 6$ has Goldbach prime(s). From **Corollary 3.2 (ii)** we said that if $p \in \mathbb{G}_N$ then b is either prime or composite. It is now left to prove that for all $N \in \mathbb{E}_{>6}$, there exists a prime b such that $p + b = N$ for some $p \in \mathbb{G}_N$.

Therefore the Goldbach binary conjecture is equivalent to the statement:

$$\boxed{\sum_{i=1}^{g(N)} \mu(b_i)\Lambda(b_i) < 0, \text{ whenever } p_i + b_i = N \text{ with } p_i \in \mathbb{G}_N. \qquad (\star)}$$

This means that there exists $b_i \in \mathbb{P}$ whenever $p_i + b_i = N$ for $1 \leq i \leq g(N)$ and $p_i \in \mathbb{G}_N$.

Lemma 3.5 enables us to use mathematical induction to prove the boxed inequality $(\star)$.

– For $g(N) = |\mathbb{G}_N| = 1$; we have $p_1 + b_1 = N$.

$p_1 \in \mathbb{G}_N$, $\Rightarrow p_1 \nmid N$ and from **Lemma 3.1** (ii), $p_1 \nmid b_1$. Also for some $q \in \mathbb{D}_N$, $q \nmid b_1$, because if $q|b_1$ then $(b_1, N) = q$, a contradiction to **Lemma 3.1** (iii). Hence b_1 has no prime factors in set $\mathbb{H}_N$, thus b_1 is prime (according to **Axiom 3.1**).

Alternatively,

If some $q \in \mathbb{D}_N$ divides b_1 then $\exists x, y \in \mathbb{Z}$ such that $b_1 = xq$ and $N = yq$. Now $p_1 = N - b_1 = yq - xq = q(y - x)$.

Now if $y - x \leq 0$; $p_1 \leq 0$, this is unthinkable since p_1 is prime and from definition of a prime number, $p_1 > 1$. If $y - x = 1$; $p_1 = q$ which contradicts **Theorem 3.1** (existence of *Goldbach primes*). If $y - x > 1$; p_1 becomes composite, a contradiction.

Hence b_1 must be prime whenever $|\mathbb{G}_N| = 1$.

Since b_1 is prime for $|\mathbb{G}_N| = 1$, it follows from **Lemma 3.2** that

$$\sum_{i=1}^{1} \mu(b_i)\Lambda(b_i) = \mu(b_1)\Lambda(b_1) < 0\,.$$

Hence the conjecture holds for $g(N) = 1$.

– Assuming $(\star)$ is true for $g(N) = k$ then;

$$\sum_{i=1}^{k} \mu(b_i)\Lambda(b_i) < 0.$$

– For $n = k + 1$; $\quad \displaystyle\sum_{i=1}^{k+1} \mu(b_i)\Lambda(b_i) = \sum_{i=1}^{k} \mu(b_i)\Lambda(b_i) + \mu(b_{k+1})\Lambda(b_{k+1}).$

$$\text{But} \quad \sum_{i=1}^{k} \mu(b_i)\Lambda(b_i) < 0 \quad \text{and} \quad \mu(b_{k+1})\Lambda(b_{k+1}) \leq 0.$$

$$\text{Hence} \quad \sum_{i=1}^{k+1} \mu(b_i)\Lambda(b_i) < 0.$$

Therefore for all values of $g(N)$ there exists a pair of primes (a, b) whose sum is an even number greater than 6; completing the proof of the famous Goldbach conjecture. ■

References:

1. I. Niven, H.S. Zuckerman, *An Introduction to the Theory of Numbers, Fifth edition.*
2. W. Sierpinski(1988), *Elementary Theory of numbers.* Volume 31.
3. T.M. Apostol (1976), *Introduction to Analytic Number Theory.* Springer-Verlag(New York).