THE WEIGHTED CORE-EP INVERSE AND ITS ASSOCIATED PRE-ORDER

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ABSTRACT. In this paper, we introduce a new pre-order stemming from the w-core-EP inverse in a ring. We characterize this generalized inverse by combing the w-core inverse with nilpotent elements. This characterization allows us to explore a new binary relation among w-core-EP invertible elements, leveraging the w-core pre-order. Using the Pierce matrix for two idempotents as a new tool, we find equivalent conditions for the forward and reverse order laws of w-core-EP invertibility. In addition, we extend the *-DMP property to encompass a broader range of cases within the w-core-EP pre-order.

1. INTRODUCTION

A ring R is called a *-ring if there exists an involution $* : x \to x^*$ satisfying $(x+y)^* = x^* + y^*, (xy)^* = y^*x^*, (x^*)^* = x$. An element a in a *-ring R has core inverse if and only if there exist $x \in R$ such that

$$ax^2 = x, (ax)^* = ax, xa^2 = a.$$

If such x exists, it is unique, and denote it by $a^{\textcircled{\#}}$ (see [34]). An element $a \in \mathcal{A}$ has core-EP inverse (i.e., pseudo core inverse) if there exist $x \in \mathcal{A}$ and $k \in \mathbb{N}$ such that

$$ax^2 = x, (ax)^* = ax, xa^{k+1} = a^k.$$

If such x exists, it is unique, and denote it by $a^{\textcircled{D}}$ (see [10]). The core and core-EP inverses have been extensively studied by many authors from various perspectives, including, for example, [1, 6, 22, 24, 27, 32].

Let $a, w \in R$. Following Zhu et al., an element $a \in \mathcal{A}$ has w-core inverse if there exist $x \in \mathcal{A}$ such that

$$awx^2 = x, (awx)^* = awx, xawa = a.$$

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Many properties of w-core inverse have been investigated in [36, 39]. Recently, Mosić extended the w-core inverse to a new class of weighted generalized inverse. An element $a \in \mathcal{A}$ has w-core-EP inverse if there exist $x \in \mathcal{A}$ such that

$$awx^2 = x, (awx)^* = awx, x(aw)^{k+1}a = (aw)^k a.$$

Such x is unique if it exits, and we denote it by $a_w^{\textcircled{D}}$ (see [25]).

In [28], Rakić and Djordjević, introduced the core pre-order on a ring with involution. That is, if $a \in R^{\textcircled{\#}}, b \in R$, then a is below b under core pre-order (written as $a \leq^{\textcircled{\#}} b$), if $aa^{\textcircled{\#}} = ba^{\textcircled{\#}}$ and $a^{\textcircled{\#}}a = a^{\textcircled{\#}}b$. In [9], Dolinar et al. introduced and studied the core-EP pre-order. Recently, the w-core pre-order, induced by the w-core inverse, has been defined and studied (see [40]). For more papers on various binary relations induced by specific generalized inverses, we refer the reader to [8, 9, 12, 17, 19, 20, 21, 23, 37].

The motivation of this paper is to introduce and study a new pre-order induced by the w-core-EP inverse. Let $a, b, w \in R$.

Definition 1.1. *Let $a \in R_w^{\textcircled{D}}$. We define a binary relation "$\leq_w^{\textcircled{D}}$" on R in the following way: $a \leq_w^{\textcircled{D}} b$ if and only if*

$$awa_w^{\textcircled{D}} = bwa_w^{\textcircled{D}}, a_w^{\textcircled{D}}a = a_w^{\textcircled{D}}b.$$

In Section 2, we characterize the w-core-EP inverse by combining the w-core inverse with nilpotent elements. This characterization establishes a foundation for examining a new binary relation among w-core-EP invertible elements, utilizing the w-core pre-order.

In Section 3, we investigate the pre-order of w-core-EP inverses, which includes certain self-adjoint elements, thereby extending many established results on Hilbert operators to a more comprehensive class of ring elements.

In Section 4, we apply the method used for the Pierce matrix relative to two idempotents to establish equivalent conditions for the forward and reverse order laws of w-core-EP invertibility in a ring setting.

An element a is w-weighted *-DMP if and only if $a \in R_w^{\textcircled{D}}$ and $wawa_w^{\textcircled{D}} = wa_w^{\textcircled{D}}aw$. Finally, in Section 5, we consider w-weighted *-DMP elements, broadening the applicability of the w-core-EP pre-order framework. This extends related results on *-DMP elements to a wider class that includes weighted considerations.

Throughout the paper, all *-rings are associative with an identity. $R^{D,w}, R_w^{\textcircled{\#}}$ and $R_w^{\textcircled{D}}$ denote the sets of all w-Drazin, w-core invertible and w-core-EP invertible elements in R, respectively.

2. WEIGHTED CORE-EP DECOMPOSITION

The objective of this section is to characterize the w-core-EP inverse by combing the w-core inverse and nilpotent within the framework of a *-ring. We begin with

Theorem 2.1. *Let $a, w \in R$. Then the following are equivalent:*

(1) $a \in R_w^{\textcircled{\dagger}}$.
(2) There exists $x \in R$ such that
$$x = awx^2, (awx)^* = awx, x(aw)^{k+1} = (aw)^k$$
for some $k \in \mathbb{N}$.
(3) $a \in R$ has the w-core-EP decomposition, i.e., there exist $x, y \in R$ such that
$$a = x + y, x^*y = ywx = 0, x \in R_w^{\textcircled{\#}}, y \in R_w^{nil}.$$

In this case, $a_w^{\textcircled{\dagger}} = x_w^{\textcircled{\#}}$.

Proof. (1) $\Rightarrow$ (2) By hypothesis, there exists $x \in R$ such that
$$awx^2 = x, x(aw)^{k+1}a = (aw)^k a \text{ and } (awx)^* = awx.$$
Then
$$x(aw)^{k+2} = [x(aw)^{k+1}a]w = [(aw)^k a]w = (aw)^{k+1},$$
as required.

(2) $\Rightarrow$ (3) By hypotheses, there exists $x \in R$ such that
$$x = awx^2, (awx)^* = awx, x(aw)^{k+1} = (aw)^k$$
for some $k \in \mathbb{N}$. Then
$$xawx = [x(aw)^{k+1}]x^{k+1} = (aw)^k x^{k+1} = awx^2 = x.$$
Set $z = awxa$ and $y = a - awxa$. We check that
$$\begin{array}{rcl} ywz & = & (a - awxa)wawxa = awawxa - awx(aw)^2xa \\ & = & awawxa - aw(awx)a = 0, \\ z^*y & = & (awxa)^*y = a^*(awx)y = a^*(awx)(a - awxa) \\ & = & a^*aw(xa - xawxa) = 0. \end{array}$$

We claim that $z \in R_w^{\textcircled{\#}}$ and $z_w^{\textcircled{\#}} = x$.

Claim 1. $x = zwx^2$. We verify that
$$zwx^2 = awx(awx^2) = awx^2 = x.$$

Claim 2. $(zwx)^* = zwx$. Clearly, we have $zwx = aw(xawx) = awx$, and then $(zwx)^* = (awx)^* = awx = zwx$.

Claim 3. $xzwz = z$. One checks that

$$xzwz = (xawx)awawxa = x(aw)^2xa = awxa = z.$$

Therefore $z \in R_w^{\bigoplus}$. Moreover, we see that

$$\begin{array}{rcl}(aw)^n - awx(aw)^n & = & (a - awxa)w(aw)^{n-1} \\ & = & yw(aw)^{n-1} = ywaw(aw)^{n-2} \\ & = & yw(z+y)w(aw)^{n-2} = (yw)^2(aw)^{n-2} \\ & = & \cdots = (yw)^n.\end{array}$$

Therefore

$$\lim_{n\to\infty}(yw)^n = 0,$$

and then $y \in R_w^{nil}$.

(3) $\Rightarrow$ (1) By hypothesis, there exist $z, y \in R$ such that

$$a = z + y, z^*y = ywz = 0, z \in R_w^{\bigoplus}, y \in R_w^{nil}.$$

Set $x = z_w^{\bigoplus}$. Then

$$\begin{array}{rcl}awx & = & (z+y)wz_w^{\bigoplus} = zwz_w^{\bigoplus}, \\ (awx)^* & = & awx, \\ awx^2 & = & (awx)x = zwz_w^{\bigoplus}(z+y) = zwz_w^{\bigoplus}z = x.\end{array}$$

It is easy to verify that

$$\begin{array}{rcl}xawx & = & z_w^{\bigoplus}(zwz_w^{\bigoplus}) = z_w^{\bigoplus} = x, \\ x(aw)^2x & = & (xaw)(awx) = z_w^{\bigoplus}(z+y)wzwz_w^{\bigoplus} = z_w^{\bigoplus}zwzwz_w^{\bigoplus} \\ & = & z_w^{\bigoplus}zwzwz_w^{\bigoplus} = zwz_w^{\bigoplus} = awx.\end{array}$$

Moreover, we have

$$awxa = (awx)a = zwz_w^{\bigoplus}(z+y) = zwz_w^{\bigoplus}z = z,$$

and so

$$a - awxa = a - z = y \in R_w^{nil}.$$

Write $(yw)^n = 0$ for some $n \in \mathbb{N}$. Then we verify that

$$\begin{array}{rcl}(aw)^n - awx(aw)^n & = & (a - awxa)w(aw)^{n-1} \\ & = & yw(aw)^{n-1} = ywaw(aw)^{n-2} \\ & = & yw(z+y)w(aw)^{n-2} = (yw)^2(aw)^{n-2} \\ & = & \cdots = (yw)^n = 0.\end{array}$$

Accordingly,

$$(aw)^n = awx(aw)^n.$$

As $z \in R^{\textcircled{\#}}_w$, there exist $s \in R$ such that

$$zws^2 = s, szwz = z \text{ and } (zws)^* = zws.$$

Then

$$(zw)s^2 = s, s(zw)^2 = zw \text{ and } (zws)^* = zws.$$

Hence, $zw \in R^{\textcircled{D}}$. Therefore $zw \in R^D$. Since $ywz = 0$, it follows by [26, Theorem 3.2] that $aw = zw + yw \in \mathcal{A}^D$.

According, we have

$$awx^2 = x, (awx)^* = awx, (aw)^n = awx(aw)^n = (aw)^n x^n (aw)^n.$$

Let $t = (aw)(aw)^D x$. We claim that $a^{\textcircled{D}}_w = t$. One directly verifies that

$$\begin{array}{rcl} awx - awt & = & [1-(aw)(aw)^D](aw)x = [1-(aw)(aw)^D]^n (aw)^n x^n \\ & = & [(aw)^n - (aw)^{n+1}(aw)^D]x^n = 0, \end{array}$$

Then $awt = awx$, and so $(awt)^* = (awx)^* = awx = awt$.

$$\begin{array}{rcl} t - awt^2 & = & (1-awt)t = (1-awx)(aw)(aw)^d x \\ & = & (1-awx)(aw)^n[(aw)^D]^n x \\ & = & [(aw)^n - (aw)^n x^n (aw)^n][(aw)^D]^n x = 0. \end{array}$$

Hence, $t = awt^2$.

Furthermore, we see that

$$\begin{array}{rl} & (aw)^n - t(aw)^{n+1} \\ = & [(aw)^n - (aw)^{n+1}(aw)^D] + [(aw)^{n+1}(aw)^D - (aw)(aw)^D x(aw)^{n+1}] \\ = & [(aw)^{n+1}(aw)^D - (aw)(aw)^D x(aw)^{n+1}] \\ = & (aw)^{n+1}(aw)^D - (aw)^D[(aw)x(aw)^n](aw) \\ = & (aw)^{n+1}(aw)^D - (aw)^D(aw)^n(aw) = 0. \end{array}$$

Then $(aw)^n = t(aw)^{n+1}$, thus yielding the result. □

Corollary 2.2. *Let $a, b \in R^{\textcircled{D}}_w$. If $a^*b = 0$ and $awb = bwa = 0$, then $a + b \in R^{\textcircled{D}}_w$. In this case,*

$$(a+b)^{\textcircled{D}}_w = a^{\textcircled{D}}_w + b^{\textcircled{D}}_w.$$

Proof. Case 1. $a, b \in R^{\textcircled{\#}}_w$. Set $x = a^{\textcircled{\#}}_w + b^{\textcircled{\#}}_w$. Then we verify that

$$\begin{array}{rcl} (a+b)w(x+y) & = & awx + bwy, \\ \big((a+b)w(x+y)\big)^* & = & (a+b)w(x+y), \\ (a+b)w(x+y)^2 & = & (awx+bwy)(x+y) \\ & = & awx^2 + bwy^2 = x + y, \\ (x+y)(a+b)w(a+b) & = & xawa + ybwb = x + y. \end{array}$$

Hence $a + b \in R^{\textcircled{\#}}_w$ and $(a+b)^{\textcircled{\#}}_w = a^{\textcircled{\#}}_w + b^{\textcircled{\#}}_w$.

Case 2. Since $a, b \in R^{\textcircled{D}}_w$, by virtue of Theorem 2.1, there exist $x, s \in R^{\textcircled{\#}}_w$ and $y, t \in R^{nil}_w$ such that

$$\begin{array}{rcl} a & = & x+y, x^*y=0, ywx=0, x\in R^{\textcircled{\#}}_w, y\in R^{nil}_w; \\ b & = & s+t, s^*t=0, tws=0, s\in R^{\textcircled{\#}}_w, t\in R^{nil}_w. \end{array}$$

As in the proof of Theorem 2.1, $x = awa^{\textcircled{D}}_w a, y = a - awa^{\textcircled{D}}_w a$ and $s = bwb^{\textcircled{D}}_w b, t = b - bwb^{\textcircled{D}}_w b$. Then $a = (x + s) + (y + t)$. Clearly, $(y + t)w = [(a - awa^{\textcircled{D}}_w a) + (b - bwb^{\textcircled{D}}_w b)]w \in R^{nil}$. This implies that $y + b \in R^{nil}_w$. We directly check that

$$\begin{array}{rcl} (x+s)^*(y+t) & = & x^*y+s^*t=0, \\ (y+t)w(x+s) & = & ywx+tws=0. \end{array}$$

By using Theorem 2.1, $a + b \in R^{\textcircled{D}}_w$. In this case,

$$(a+b)^{\textcircled{D}}_w = x^{\textcircled{\#}}_w + y^{\textcircled{\#}}_w = a^{\textcircled{D}}_w + b^{\textcircled{D}}_w.$$

□

Lemma 2.3. *Let $a, w \in R$. Then the following are equivalent:*

(1) $a \in R^{\textcircled{D}}_w$.
(2) $aw \in R^{\textcircled{D}}$.
(3) $a \in R^{D,w}$ and there exists $x \in R$ such that

$$x = awx^2, (awx)^* = awx, (aw)^n = awx(aw)^n.$$

In this case, $a^{\textcircled{D}}_w = x = (aw)^{\textcircled{D}}$.

Proof. (1) ⇔ (2) This is proved in [25, Theorem 2.14].

(2) ⇒ (3) In view of [25, Theorem 2.4], $a \in R^{D,w}$ and there exists $x \in R$ such that

$$x = awx^2, (awx)^* = awx, (aw)^n = awx(aw)^n.$$

Moreover, we have $(aw)(aw)^D x = a^{\textcircled{D}}_w$. Since $x = awx^2$, by induction, we have $x = (aw)^n x^{n+1}$ for any $n \in \mathbb{N}$. Then

$$\begin{array}{rcl} x-(aw)(aw)^d x & = & x-(aw)(aw)^d x \\ & = & [1-(aw)(aw)^D](aw)^n x^{n+1} \\ & = & (aw)^n-(aw)^D(aw)^{n+1}x. \end{array}$$

As $(aw)^n = (aw)^D(aw)^{n+1}$, we see that

$$x = (aw)(aw)^D x = 0,$$

and therefore $x = (aw)(aw)^D x = a^{\textcircled{D}}_w$, as required.

(3) ⇒ (2) Since $a \in R^{D,w}$, we have $aw \in R^D$. Let m be the Drazin index of aw. Set $k = m + n$. Then $(aw)^k x^k = awx$ and $(aw)^k = (aw)^k x^k (aw)^k$. Hence, $(aw)^k \in R^{(1,3)}$. Therefore $aw \in R^{\textcircled{D}}$ by [10, Theorem 2.3]. □

If a and x satisfy the equations $a = axa$ and $(ax)^* = ax$, then x is called $(1,3)$-inverse of a and is denoted by $a^{(1,3)}$. We use $R^{(1,3)}$ to stand for sets of all $(1,3)$-invertible elements in R. We now derive

Theorem 2.4. *Let $a, w \in R$. Then the following are equivalent:*

(1) $a \in R_w^{\textcircled{D}}$.
(2) $a \in R^{D,w}$ and $a^{D,w} \in R_w^{\textcircled{\#}}$.
(3) $a \in R^{D,w}$ and $a^{D,w} \in R^{(1,3)}$.

In this case, $a_w^{\textcircled{D}} = [(aw)^D]^2[a^{D,w}]_w^{\textcircled{\#}} = a^{D,w}wa^{D,w}(a^{D,w})^{(1,3)}$.

Proof. (1) $\Rightarrow$ (2) In view of Lemma 2.3, $aw \in R^{\textcircled{D}}$ and $x = (aw)^{\textcircled{D}}$. By virtue of [10, Theorem 2.3], $aw \in R^D$. Evidently, $[(aw)^D]^{\textcircled{\#}} = (aw)^2(aw)^{\textcircled{D}}$. Let $x = \big((aw)^D\big)^{\textcircled{\#}}$. Since $(aw)^D = [(aw)^D]^2 aw = a^{D,w}w$, we have $a^{D,w}w \in R^{\textcircled{\#}}$. We direly check that

$$\begin{array}{rcl} a^{D,w}wx^2 &=& x, \\ (a^{D,w}wx)^* &=& a^{D,w}wx, \\ xa^{D,w}wa^{D,w}w &=& a^{D,w}w. \end{array}$$

Hence,

$$\begin{array}{rcl} xa^{D,w}wa^{D,w} &=& xa^{D,w}w[a^{D,w}wawa^{D,w}] \\ &=& [xa^{D,w}wa^{D,w}w]awa^{D,w} = a^{D,w}wawa^{D,w} = a^{D,w}. \end{array}$$

Therefore $a^{D,w} \in R_w^{\textcircled{\#}}$ and $\big(a^{D,w}\big)_w^{\textcircled{\#}} = x$.

Additionally, we have

$$\begin{array}{rcl} a_w^{\textcircled{D}} &=& (aw)^{\textcircled{D}} \\ &=& [(aw)^D]^2[(aw)^D]^{\textcircled{\#}} \\ &=& [(aw)^D]^2[a^{D,w}]_w^{\textcircled{\#}}. \end{array}$$

(2) $\Rightarrow$ (3) Since $a^{D,w} \in R_w^{\textcircled{\#}}$, by virtue of [39, Theorem 2.6], $a^{D,w} \in R^{(1,3)}$, as required.

(3) $\Rightarrow$ (1) Let $x = a^{D,w}wa^{D,w}(a^{D,w})^{(1,3)}$. Then $x = [(aw)^D]^3 a\big(((aw)^D)^2 a\big)^{(1,3)}$. Let $k = i(aw)$. Then $(aw)^D(aw)^{k+1} = (aw)^k$.

Claim 1. $awx(aw)^k a = (aw)^k a$.

We verify that

$$\begin{array}{rcl} awx(aw)^k a &=& aw[(aw)^D]^3 a\big(((aw)^D)^2 a\big)^{(1,3)}(aw)^k a \\ &=& [(aw)^D]^2 a\big(((aw)^D)^2 a\big)^{(1,3)}[(aw)^D]^2 a[w(aw)^{k+1}a] \\ &=& [(aw)^D]^2 a[w(aw)^{k+1}a] \\ &=& [(aw)^D]^2(aw)^2(aw)^k a] = (aw)^k a. \end{array}$$

Step 2. $(aw)^k aR = xR$.

Clearly, $xR \subseteq (aw)^D R \subseteq (aw)^k aR$. Also we see that

$$\begin{array}{rcl} (aw)^k a & = & (aw)^D (aw)^{k+1} a = [((aw)^D)^3 a][w(aw)^{k+2} a] \\ & = & [(aw)^D]^3 a\big(((aw)^D)^2 a\big)^{(1,3)}((aw)^D)^2 a][w(aw)^{k+2} a] \\ & = & x((aw)^D)^2 a][w(aw)^{k+2} a]; \end{array}$$

hence, $(aw)^k a \subseteq xR$. Therefore $(aw)^k aR = xR$.

Step 3. $Rx = R\big((aw)^k a\big)^*$.

We easily verify that

$$\begin{array}{rcl} x & = & [(aw)^D]^3 a\big(((aw)^D)^2 a\big)^{(1,3)} \\ & = & \Big([(aw)^D]^3 a\big(((aw)^D)^2 a\big)^{(1,3)}\Big)^* \\ & = & \Big([(aw)^k aw((aw)^D)^{k+1}][(aw)^D]^3 a\big(((aw)^D)^2 a\big)^{(1,3)}\Big)^* \\ & = & \Big([w((aw)^D)^{k+1}][(aw)^D]^3 a\big(((aw)^D)^2 a\big)^{(1,3)}\Big)^* \big((aw)^k a\big)^*, \end{array}$$

and then, $Rx \subseteq R\big((aw)^k a\big)^*$. Moreover, we have

$$\begin{array}{rcl} (aw)^k a & = & (aw)^D (aw)^{k+1} a = [(aw)^D]^2 aw(aw)^{k+1} a \\ & = & \big([(aw)^D]^2 a([(aw)^D]^2 a)^{(1,3)}[(aw)^D]^2 a\big) w(aw)^{k+1} a, \end{array}$$

and then

$$\begin{array}{rcl} \big((aw)^k a\big)^* & = & \big([(aw)^D]^2 aw(aw)^{k+1} a\big)^* [(aw)^D]^2 a([(aw)^D]^2 a)^{(1,3)} \\ & = & \big([(aw)^D]^2 aw(aw)^{k+1} a\big)^* (aw)[(aw)^D]^3 a([(aw)^D]^2 a)^{(1,3)} \\ & = & \big([(aw)^D]^2 aw(aw)^{k+1} a\big)^* (aw)x. \end{array}$$

Hence $R((aw)^k a)^* \subseteq Rx$. Therefore $Rx = R((aw)^k a)^*$.

Accordingly, $a \in R_w^{\textcircled{D}}$ by [25, Theorem 2.4]. □

Corollary 2.5. *Let $a, w \in R$. Then the following are equivalent:*

1. $a \in R_w^{\textcircled{D}}$.
2. $a \in R^{D,w}$ and $awa^{D,w}w \in R^{(1,3)}$.
3. $a \in R^{D,w}$ and there exists a projection $q \in R$ such that $a^{D,w}R = qR$.

In this case, $a_w^{\textcircled{D}} = a^{D,w}wa^{D,w}(a^{D,w})^{(1,3)} = a^{D,w}wq$.

Proof. (1) ⇒ (3) By virtue of Theorem 2.4, $a^{D,w} \in R^{(1,3)}$ and then

$$a^{D,w} = a^{D,w}(a^{D,w})^{(1,3)}a^{D,w} \text{ and } [a^{D,w}(a^{D,w})^{(1,3)}]^* = a^{D,w}(a^{D,w})^{(1,3)}.$$

Let $q = a^{D,w}(a^{D,w})^{(1,3)}$. Then $a^{D,w}R = qR, q^2 = q = q^*$, as required.

(3) ⇒ (2) Let $x = a^{D,w}wq$. Then $awx = awa^{D,w}wq = aw[(aw)^D]^2 awq = aw(aw)^D q = q$, and so $(awx)^* = q^* = q = awx$. Moreover, we have

$$awx^2 = (awx)x = qa^{D,w}wq = a^{D,w}wq = x.$$

Let n be the Drazin index of aw. Then $(aw)^n = (aw)^D(aw)^{n+1}$. Obviously, $a^{D,w}w(aw) = (aw)a^{D,w}w$, and so

$$\begin{array}{cl} & (aw)^n - x(aw)^{n+1} \\ = & (aw)^n - a^{D,w}wq(aw)^{n+1} \\ = & (aw)^n - (aw)^Dq(aw)^{n+1} = (aw)^n - (aw)^Dq(aw)^D(aw)^{n+2} \\ = & (aw)^n - (aw)^D(aw)^D(aw)^{n+2} = (aw)^n - (aw)^D(aw)^{n+1} = 0. \end{array}$$

Hence $(aw)^n = x(aw)^{n+1}$. Thus $x = a_w^{\textcircled{\dagger}}$. In this case, $a_w^{\textcircled{\dagger}} = x = a^{D,w}wq = a^{D,w}wa^{D,w}(a^{D,w})^{(1,3)}$.

(2) $\Rightarrow$ (1) Let $x = a^{D,w}w(awa^{D,w}w)^{(1,3)}$. Then we verify that

$$\begin{array}{rcl} awx & = & awa^{D,w}w(awa^{D,w}w)^{(1,3)} = aw(aw)^D(aw(aw)^D)^{(1,3)}, \\ (awx)^* & = & awx, \\ awx^2 & = & aw(aw)^D(aw(aw)^D)^{(1,3)}a^{D,w}w(awa^{D,w}w)^{(1,3)} \\ & = & aw(aw)^D(aw(aw)^D)^{(1,3)}aw[(aw)^D]^2(awa^{D,w}w)^{(1,3)} \\ & = & (aw)^D(awa^{D,w}w)^{(1,3)} = x, \end{array}$$

Let n be the Drazin index of aw. Then

$$\begin{array}{rcl} awx(aw)^n & = & aw(aw)^D(aw(aw)^D)^{(1,3)}(aw)^n \\ & = & aw(aw)^D(aw(aw)^D)^{(1,3)}(aw)(aw)^D(aw)^n \\ & = & aw(aw)^D(aw)^n = (aw)^n. \end{array}$$

Hence, $(aw)^n = awx(aw)^n$. Therefore $a \in R_w^{\textcircled{\dagger}}$ by Lemma 2.3. In this case, $a_w^{\textcircled{\dagger}} = a^{D,w}w(awa^{D,w}w)^{(1,3)}$. □

Corollary 2.6. *Let $a \in R$. Then the following are equivalent:*

(1) $a \in R^{\textcircled{\dagger}}$.
(2) $a \in R^D$ and $a^D \in R^{\textcircled{\#}}$.
(3) $a \in R^D$ and $a^D \in R^{(1,3)}$.
(4) $a \in R^D$ and $aa^D \in R^{(1,3)}$.
(5) $a \in R^D$ and there exists a projection $q \in R$ such that $a^DR = qR$.

In this case, $a^{\textcircled{\dagger}} = (a^D)^2(a^{D\textcircled{\#}} = (a^D)^2(a^D)^{(1,3)} = a^Dq$.

Proof. This is obvious by choosing $w = 1$ in Theorem 2.4 and Corollary 2.5. □

3. WEIGHTED CORE-EP ORDERS

Let $a \in R_w^{\textcircled{\#}}, b \in R$. Recall that $a \leq_w^{\textcircled{\#}} b$ if $awa_w^{\textcircled{\#}} = bwa_w^{\textcircled{\#}}$ and $a_w^{\textcircled{\#}}a = a_w^{\textcircled{\#}}b$ (see [40]). By employing the w-core-decomposition as a tool, we now characterize the weighted core-EP inverse through the weight core order.

Theorem 3.1. *Let $a, b \in R_w^{\textcircled{D}}$. If $a = a_1 + a_2, b = b_1 + b_2$ are w-core-EP decompositions of a and b. Then the following are equivalent:*

(1) $a \leq_w^{\textcircled{D}} b$.
(2) $a_1 \leq_w^{\textcircled{\#}} b_1$.
(3) $(aw)^{n+1} = bw(aw)^n$ and $a^*(aw)^n = b^*(aw)^n$ for some $n \in \mathbb{N}$.

Proof. (1) $\Rightarrow$ (2) Since $a \leq_w^{\textcircled{D}} b$, we have $awa_w^{\textcircled{D}} = bwa_w^{\textcircled{D}}$ and $a_w^{\textcircled{D}} a = a_w^{\textcircled{D}} b$. For any $m \in \mathbb{N}$, we derive

$$\begin{array}{rcl} a_1 w(a_1)_w^{\textcircled{\#}} & = & (a_1 + a_2)w(a_1)_w^{\textcircled{\#}} = awa_w^{\textcircled{D}} = bwa_w^{\textcircled{D}} \\ & = & bwaw(a_w^{\textcircled{D}})^2 = bw[awa_w^{\textcircled{D}}]a_w^{\textcircled{D}} = bw[bwa_w^{\textcircled{D}}]a_w^{\textcircled{D}} \\ & = & (bw)^2(a_w^{\textcircled{D}})^2 = \cdots = (bw)^m(a_w^{\textcircled{D}})^m, \\ b_1 w(a_1)_w^{\textcircled{\#}} & = & bwb_w^{\textcircled{D}} bwa_w^{\textcircled{D}} = bwb_w^{\textcircled{D}} bwaw(a_w^{\textcircled{D}})^2 = bwb_w^{\textcircled{D}}(bw)^2(a_w^{\textcircled{D}})^2 \\ & = & \cdots = bwb_w^{\textcircled{D}}(bw)^m(a_w^{\textcircled{D}})^m. \end{array}$$

Thus, we have

$$\begin{array}{rl} & a_1 w(a_1)_w^{\textcircled{\#}} - b_1 w(a_1)_w^{\textcircled{\#}} \\ = & (bw)^m(a_w^{\textcircled{D}})^m - bwb_w^{\textcircled{D}}(bw)^m(a_w^{\textcircled{D}})^m \\ = & [(bw)^m - bwb_w^{\textcircled{D}}(bw)^m](a_w^{\textcircled{D}})^m. \end{array}$$

In view of Lemma 2.3,

$$(bw)^m = bwb_w^{\textcircled{D}}(bw)^m.$$

Hence,

$$a_1 w(a_1)_w^{\textcircled{\#}} = b_1 w(a_1)_w^{\textcircled{\#}}.$$

Since $b_1 = bwb_w^{\textcircled{D}} b$, we verify that

$$awa_w^{\textcircled{D}} = a_1 w(a_1)_w^{\textcircled{\#}} = b_1 w(a_1)_w^{\textcircled{\#}} = bwb_w^{\textcircled{D}} bwa_w^{\textcircled{D}} = bwb_w^{\textcircled{D}} awa_w^{\textcircled{D}}.$$

Thus,

$$[awa_w^{\textcircled{D}}]^* = [bwb_w^{\textcircled{D}} awa_w^{\textcircled{D}}]^*,$$

and so

$$awa_w^{\textcircled{D}} = awa_w^{\textcircled{D}} bwb_w^{\textcircled{D}}.$$

Then we see that

$$\begin{array}{rcl} (a_1)_w^{\textcircled{\#}} a_1 & = & a_w^{\textcircled{D}}(awa_w^{\textcircled{D}} a) = a_w^{\textcircled{D}}(awa_w^{\textcircled{D}})a \\ & = & a_w^{\textcircled{D}}(awa_w^{\textcircled{D}})b \\ & = & a_w^{\textcircled{D}}(awa_w^{\textcircled{D}} bwb_w^{\textcircled{D}})b \\ & = & (a_w^{\textcircled{D}} awa_w^{\textcircled{D}})bwb_w^{\textcircled{D}})b \\ & = & a_w^{\textcircled{D}}(bwb_w^{\textcircled{D}} b) = (a_1)_w^{\textcircled{\#}} b_1. \end{array}$$

Therefore $a_1 \leq_w^{\textcircled{\#}} b_1$.

(2) $\Rightarrow$ (1) Obviously, we have

$$awa_w^{\textcircled{D}} = (a_1+a_2)wa_1^{\textcircled{\#}} = a_1wa_1^{\textcircled{\#}} = b_1wa_1^{\textcircled{\#}} = bwb_w^{\textcircled{D}}bwa_w^{\textcircled{D}}.$$

Then

$$a_w^{\textcircled{D}} = aw(a_w^{\textcircled{D}})^2 = bwb_w^{\textcircled{D}}bw(a_w^{\textcircled{D}})^2.$$

Since $(bw)^n = bwb_w^{\textcircled{D}}(bw)^n$ for some $n \in \mathbb{N}$, we derive that

$$bwb_w^{\textcircled{D}}bwa_w^{\textcircled{D}} = bwa_w^{\textcircled{D}}.$$

Then

$$awa_w^{\textcircled{D}} = bwa_w^{\textcircled{D}}.$$

Clearly, $a_w^{\textcircled{D}}a_2 = (a_1)_w^{\textcircled{\#}}a_2 = (a_1)_w^{\textcircled{\#}}a_1w(a_1)_w^{\textcircled{\#}}a_2 = (a_1)_w^{\textcircled{\#}}(a_1w(a_1)_w^{\textcircled{\#}})^*a_2 = (a_1)_w^{\textcircled{\#}}[w(a_1)_w^{\textcircled{\#}}]^*(a_1)^*a_2 = 0$.

Moreover, we have

$$awa_w^{\textcircled{D}} = bwb_w^{\textcircled{D}}bwa_w^{\textcircled{D}} = (bwb_w^{\textcircled{D}})(awa_w^{\textcircled{D}}).$$

Then

$$\begin{array}{rcl} awa_w^{\textcircled{D}} & = & (awa_w^{\textcircled{D}})^* \\ & = & (awa_w^{\textcircled{D}})^*(bwb_w^{\textcircled{D}})^* \\ & = & awa_w^{\textcircled{D}}bwb_w^{\textcircled{D}}. \end{array}$$

Hence, $a_w^{\textcircled{D}} = a_w^{\textcircled{D}}awa_w^{\textcircled{D}} = a_w^{\textcircled{D}}awa_w^{\textcircled{D}}bwb_w^{\textcircled{D}} = a_w^{\textcircled{D}}bwb_w^{\textcircled{D}}$. Accordingly, $a_w^{\textcircled{D}}b = a_w^{\textcircled{D}}bwb_w^{\textcircled{D}}b = (a_1)_w^{\textcircled{D}}b_1 = (a_1)_w^{\textcircled{D}}a_1 = a_w^{\textcircled{D}}(a_1+a_2) = a_w^{\textcircled{D}}a$, as required.

(1) $\Rightarrow$ (3) Since $awa_w^{\textcircled{D}} = bwa_w^{\textcircled{D}}$ and $(aw)^{k+1} = (aw)^k$, we see that $(aw)^{k+1} = (aw)(aw)^k = (bw)(aw)^k$. Also we have $a_w^{\textcircled{D}}a = a_w^{\textcircled{D}}b$. Then $a^*(a_w^{\textcircled{D}})^* = b^*(a_w^{\textcircled{D}})^*$; hence,

$$a^*(a_w^{\textcircled{D}})^*(aw)^*(aw)^k = b^*(a_w^{\textcircled{D}})^*(aw)^*(aw)^k.$$

As $(awa_w^{\textcircled{D}})^* = awa_w$ and $awa_w^{\textcircled{D}}(aw)^{k+1} = (aw)^{k+2}$, we deduce that $a^*(aw)^{k+1} = b^*(aw)^{k+1}$. Choose $n = k+1$. Then $(aw)^{n+1} = bw(aw)^n$ and $a^*(aw)^n = b^*(aw)^n$, as required.

(3) $\Rightarrow$ (1) By hypothesis, $(aw)^{n+1} = bw(aw)^n$ and $a^*(aw)^n = b^*(aw)^n$ for some $n \in \mathbb{N}$. Since $(aw)^{n+1} = bw(aw)^n$, we have $(aw)(aw)^n(a_w^{\textcircled{D}})^{n+1} = (bw)(aw)^n(a_w^{\textcircled{D}})^{n+1}$. As $aw(a_w^{\textcircled{D}})^2 = a_w^{\textcircled{D}}$, we have $(aw)a_w^{\textcircled{D}} = (bw)a_w^{\textcircled{D}}$.

Since $a^*(aw)^n = b^*(aw)^n$, we have $((aw)^n)^*a = ((aw)^n)^*b$. Hence,

$$[(a_w^{\textcircled{D}})^n]^n((aw)^n)^*a = [(a_w^{\textcircled{D}})^n]^n((aw)^n)^*b.$$

This implies that

$$awa_w^{\textcircled{D}}a = awa_w^{\textcircled{D}}b.$$

Therefore

$$\begin{array}{rcl} a_w^{\textcircled{D}}a & = & a_w^{\textcircled{D}}a = a_w^{\textcircled{D}}[awa_w^{\textcircled{D}}a] \\ & = & a_w^{\textcircled{D}}[awa_w^{\textcircled{D}}b] = [aw)^{\textcircled{D}}awa_w^{\textcircled{D}}]b = a_w^{\textcircled{D}}b, \end{array}$$

as required. □

Corollary 3.2. *The relation $\leq_w^{\textcircled{D}}$ for w-core-EP invertible elements is a pre-order on R.*

Proof. Step 1. $a \leq_w^{\textcircled{D}} a$. Let $a = a_1 + a_2$ be the w-core-EP decomposition. In view of [40, Theorem 2.3], $a_1 \leq_w^{\textcircled{\#}} a_1$. By using Theorem 3.1, $a \leq_w^{\textcircled{D}} a$.

Step 2. Assume that $a \leq_w^{\textcircled{D}} b$ and $b \leq_w^{\textcircled{D}} c$. We claim that $a \leq_w^{\textcircled{D}} c$. Let $a = a_1 + a_2, b = b_1 + b_2$ and $c = c_1 + c_2$ be the w-core-EP decompositions of a, b and c, respectively. By virtue of Theorem 3.1, $a_1 \leq_w^{\textcircled{\#}} b_1$ and $b_1 \leq_w^{\textcircled{\#}} c_1$. According to [40, Theorem 2.3], we have $a_1 \leq_w^{\textcircled{\#}} c_1$. By using Theorem 3.1 again, $a \leq_w^{\textcircled{D}} c$.

Therefore the relation $\leq_w^{\textcircled{D}}$ for w-core-EP invertible elements is a pre-order. □

Lemma 3.3. *Let $a, b \in R_w^{\textcircled{D}}$ and $a \leq_w^{\textcircled{D}} b$. Then the following hold:*

(1) $awa_w^{\textcircled{D}} = (awa_w^{\textcircled{D}})(bwb_w^{\textcircled{D}}) = (bwb_w^{\textcircled{D}})(awa_w^{\textcircled{D}})$.
(2) $a_w^{\textcircled{D}} = a_w^{\textcircled{D}}(bwb_w^{\textcircled{D}}) = (bwb_w^{\textcircled{D}})a_w^{\textcircled{D}}$.
(3) $bwa_w^{\textcircled{D}} = awa_w^{\textcircled{D}}awb_w^{\textcircled{D}}$.
(4) $b_w^{\textcircled{D}}a_w^{\textcircled{D}} = (a_w^{\textcircled{D}})^2$.

Proof. In view of Lemma 2.3, $aw, bw \in R^{\textcircled{D}}, (aw)^{\textcircled{D}} = a_w^{\textcircled{D}}$ and $(bw)^{\textcircled{D}} = b_w^{\textcircled{D}}$. Since $a \leq_w^{\textcircled{D}} b$, we have

$$awa_w^{\textcircled{D}} = bwa_w^{\textcircled{D}}, a_w^{\textcircled{D}}a = a_w^{\textcircled{D}}b.$$

Hence, $a_w^{\textcircled{D}}aw = a_w^{\textcircled{D}}bw$. This implies that $aw \leq^{\textcircled{D}} bw$. In view of [7, Lemma 6.2.6], we derive

$$(aw)(aw)^{\textcircled{D}} = [(aw)(aw)^{\textcircled{D}}][(bw)(bw)^{\textcircled{D}}] = [(bw)(bw)^{\textcircled{D}}][(aw)(aw)^{\textcircled{D}}].$$

Therefore

$$awa_w^{\textcircled{D}} = (awa_w^{\textcircled{D}})(bwb_w^{\textcircled{D}}) = (bwb_w^{\textcircled{D}})(awa_w^{\textcircled{D}}).$$

We directly check that

$$\begin{array}{rcl} a_w^{\textcircled{D}} & = & a_w^{\textcircled{D}}[awa_w^{\textcircled{D}}] = [a_w^{\textcircled{D}}awa_w^{\textcircled{D}}](bwb_w^{\textcircled{D}}) \\ & = & a_w^{\textcircled{D}}(bwb_w^{\textcircled{D}}) = [awa_w^{\textcircled{D}}]a_w^{\textcircled{D}} \\ & = & (bw)(bw)^{\textcircled{D}}aw[(aw)^{\textcircled{D}}]^2 = (bwb_w^{\textcircled{D}})a_w^{\textcircled{D}}. \end{array}$$

Analogously, (3) and (4) are proved by using [7, Theorem 6.2.7]. □

Theorem 3.4. *Let $a, b \in R_w^{\textcircled{D}}$. Then the following are equivalent:*

(1) $a \leq_w^{\textcircled{D}} b$.
(2) $a_w^{\textcircled{D}}b = b_w^{\textcircled{D}}awa_w^{\textcircled{D}}a, a_w^{\textcircled{D}} = a_w^{\textcircled{D}}bwa_w^{\textcircled{D}}, bwa_w^{\textcircled{D}} = awa_w^{\textcircled{D}}awb_w^{\textcircled{D}}$.

Proof. (1) ⇒ (2) We claim that

$$\begin{aligned} a_w^{\textcircled{\dagger}}b &= [a_w^{\textcircled{\dagger}}(bwb_w^{\textcircled{\dagger}})]b = a_w^{\textcircled{\dagger}}[bwb_w^{\textcircled{\dagger}}b] \\ &= (a_1)_w^{\oplus}b_1 = (a_1)_w^{\oplus}a_1 = (b_1)_w^{\oplus}a_1 \\ &= b_w^{\textcircled{\dagger}}awa_w^{\textcircled{\dagger}}a. \end{aligned}$$

By virtue of Lemma 3.3, $a_w^{\textcircled{\dagger}} = a_w^{\textcircled{\dagger}}bwa_w^{\textcircled{\dagger}}, bwa_w^{\textcircled{\dagger}} = awa_w^{\textcircled{\dagger}}awb_w^{\textcircled{\dagger}}$.

(2) ⇒ (1) Step 1. By hypothesis, we have

$$(aw)^{\textcircled{\dagger}} = (aw)^{\textcircled{\dagger}}(bw)(aw)^{\textcircled{\dagger}}, (bw)(aw)^{\textcircled{\dagger}} = (aw)(aw)^{\textcircled{\dagger}}(aw)(bw0^{\textcircled{\dagger}}.$$

Since $a_w^{\textcircled{\dagger}}b = b_w^{\textcircled{\dagger}}awa_w^{\textcircled{\dagger}}a$, we get $(aw)^{\textcircled{\dagger}}(bw) = (bw)^{\textcircled{\dagger}}(aw)(aw)^{\textcircled{\dagger}}(aw)$. According to [7, Proposition 6.2.8], $aw \leq^{\textcircled{\dagger}} bw$. Thus, $(aw)(aw)^{\textcircled{\dagger}} = (bw)(aw)^{\textcircled{\dagger}}$; hence, $awa_w^{\textcircled{\dagger}} = bwa_w^{\textcircled{\dagger}}$.

Step 2. We verify that

$$\begin{aligned} a_w^{\textcircled{\dagger}}a &= [a_w^{\textcircled{\dagger}}(bwa_w^{\textcircled{\dagger}})]a = [a_w^{\textcircled{\dagger}}b][wa_w^{\textcircled{\dagger}}a] \\ &= [b_w^{\textcircled{\dagger}}awa_w^{\textcircled{\dagger}}a][wa_w^{\textcircled{\dagger}}a] = b_w^{\textcircled{\dagger}}aw[a_w^{\textcircled{\dagger}}awa_w^{\textcircled{\dagger}}]a \\ &= b_w^{\textcircled{\dagger}}awa_w^{\textcircled{\dagger}}a = a_w^{\textcircled{\dagger}}b. \end{aligned}$$

This completes the proof. □

Employing the technique of Hilbert operator decomposition, many properties of the core-EP pre-order between two Hilbert space operators, grounded in their corresponding self-adjoint operators, were explored in [25]. Through an elementary-wise analysis, we will characterize the pre-order of weighted core-EP inverses, which includes certain self-adjoint elements, thereby extending many established results to a more comprehensive class of ring elements.

Theorem 3.5. *Let $a, b \in R_w^{\textcircled{\dagger}}$. Then the following are equivalent:*

(1) $a \leq_w^{\textcircled{\dagger}} b$.
(2) $bwa_w^{\textcircled{\dagger}}$ is self-adjoint and $a_w^{\textcircled{\dagger}}a = a_w^{\textcircled{\dagger}}b$.
(3) $bwa_w^{\textcircled{\dagger}}$ is self-adjoint and $awa_w^{\textcircled{\dagger}}a = awa_w^{\textcircled{\dagger}}b$.
(4) $bwa_w^{\textcircled{\dagger}}$ is self-adjoint and $a^*a_w^{\textcircled{\dagger}} = b^*a_w^{\textcircled{\dagger}}$.
(5) $bwa_w^{\textcircled{\dagger}}$ is self-adjoint and $a^*(aw)^n = b^*(aw)^n$ for some $n \in \mathbb{N}$.
(6) $bwa_w^{\textcircled{\dagger}}$ is self-adjoint and $a^*a^{D,w} = b^*a^{D,w}$.

Proof. (1) Since $a \leq_w^{\textcircled{\dagger}} b$, we have $awa_w^{\textcircled{\dagger}} = bwa_w^{\textcircled{\dagger}}b$ and $a_w^{\textcircled{\dagger}}a = a_w^{\textcircled{\dagger}}b$. Since $(awa_w^{\textcircled{\dagger}})^* = awa_w^{\textcircled{\dagger}}$, we see that $(bwa_w^{\textcircled{\dagger}})^* = bwa_w^{\textcircled{\dagger}}$. That is, $bwa_w^{\textcircled{\dagger}}$ is self-adjoint, as required.

(2) ⇒ (3) This is obvious.

(3) $\Rightarrow$ (1) By hypothesis, $(bwa_w^{\textcircled{D}})^*=bwa_w^{\textcircled{D}}$. In view of —
cite[Theorem 2.4]MZ, $a_w^{\textcircled{D}}=a_w^{\textcircled{D}}awa_w^{\textcircled{D}}$ and $(awa_w^{\textcircled{D}})^*=awa_w^{\textcircled{D}}$. Then

$$\begin{array}{lll} bwa_w^{\textcircled{D}} & = & [bw(a_w^{\textcircled{D}}awa_w^{\textcircled{D}})]^*\\ & = & (awa_w^{\textcircled{D}})^*(bwa_w^{\textcircled{D}})^*\\ & = & (awa_w^{\textcircled{D}})(bwa_w^{\textcircled{D}})\\ & = & [awa_w^{\textcircled{D}}b]wa_w^{\textcircled{D}}\\ & = & [awa_w^{\textcircled{D}}a]wa_w^{\textcircled{D}}\\ & = & aw[a_w^{\textcircled{D}}awa_w^{\textcircled{D}}]\\ & = & awa_w^{\textcircled{D}}, \end{array}$$

as desired.

(1) $\Rightarrow$ (4) Obviously, $bwa_w^{\textcircled{D}}$ is self-adjoint and $a_w^{\textcircled{D}}a=a_w^{\textcircled{D}}b$. Hence, $awa_w^{\textcircled{D}}a=awa_w^{\textcircled{D}}b$. This implies that $(awa_w^{\textcircled{D}})^*a=(awa_w^{\textcircled{D}})^*b$. Hence, $a^*(awa_w^{\textcircled{D}})=b^*(awa_w^{\textcircled{D}})$. Therefore $a^*a_w^{\textcircled{D}}=[a^*awa_w^{\textcircled{D}}]a_w^{\textcircled{D}}=[b^*awa_w^{\textcircled{D}}]a_w^{\textcircled{D}}=b^*a_w^{\textcircled{D}}$, as required.

(4) $\Rightarrow$ (5) Since $a_w^{\textcircled{D}}a^{n+1}=a^n$ for some $n\in\mathbb{N}$, we have

$$a^*(aw)^n=[a^*a_w^{\textcircled{D}}]a^{n+1}[b^*a_w^{\textcircled{D}}]a^{n+1}=b^*(aw)^n,$$

as desired.

(5) $\Rightarrow$ (6) As $a^*(aw)^n=b^*(aw)^n$, we have $a^*(aw)^D=a^*(aw)^n[(aw)^D]^{n+1}=b^*(aw)^n[(aw)^D]^{n+1}=b^*(aw)^D$. Therefore $a^*a^{D,w}=[a^*(aw)^D][(aw)^Da]=[b^*(aw)^D][(aw)^Da]=b^*a^{D,w}$.

(6) $\Rightarrow$ (3) Since $aw(aw)^D=a^{D,w}waw$, we have $a^*aw(aw)^D=b^*aw(aw)^D$. In view of [10, Theorem 2.3], $(aw)^{\textcircled{D}}\in(aw)^DR$, and then $a^*aw(aw)^{\textcircled{D}}=b^*aw(aw)^{\textcircled{D}}$. Hence $a^*awa_w^{\textcircled{D}}=b^*awa_w^{\textcircled{D}}$. As $(awa_w^{\textcircled{D}})^*=awa_w^{\textcircled{D}}$, we deduce that $awa_w^{\textcircled{D}}a=awa_w^{\textcircled{D}}b$. Therefore

$$a_w^{\textcircled{D}}a=a_w^{\textcircled{D}}[awa_w^{\textcircled{D}}a]=a_w^{\textcircled{D}}[awa_w^{\textcircled{D}}b]=a_w^{\textcircled{D}}a,$$

as asserted. $\square$

Theorem 3.6. *Let $a,b\in R_w^{\textcircled{D}}$. Then the following are equivalent:*

(1) $a\leq_w^{\textcircled{D}}b$.
(2) $(awa_w^{\textcircled{D}}a)^*b$ is self-adjoint and $awa_w^{\textcircled{D}}=bwa_w^{\textcircled{D}}$.
(3) $(awa_w^{\textcircled{D}}a)^*b$ is self-adjoint and $awa_w^{\textcircled{D}}a=bwa_w^{\textcircled{D}}a$.
(4) $(awa_w^{\textcircled{D}}a)^*b$ is self-adjoint and $a_w^{\textcircled{D}}(aw)^*=a_w^{\textcircled{D}}(bw)^*$.

Proof. (1) $\Rightarrow$ (2) By hypothesis, we have

$$awa_w^{\textcircled{D}}=bwa_w^{\textcircled{D}}, a_w^{\textcircled{D}}a=a_w^{\textcircled{D}}b.$$

Hence,

$$(awa_w^{\textcircled{D}}a)^*b=a^*(awa_w^{\textcircled{D}})^*b=a^*aw[a_w^{\textcircled{D}}b]=a^*aw[a_w^{\textcircled{D}}a]=a^*[awa_w^{\textcircled{D}}]a.$$

This implies that $(awa_w^{\textcircled{D}}a)^*b$ is self-adjoint.

(2) $\Rightarrow$ (3) This is obvious.

(3) $\Rightarrow$ (1) Since $awa_w^{\textcircled{D}}a = bwa_w^{\textcircled{D}}a$, we see that

$$\begin{array}{rcl} awa_w^{\textcircled{D}} & = & aw[a_w^{\textcircled{D}}awa_w^{\textcircled{D}}] \\ & = & [awa_w^{\textcircled{D}}a]wa_w^{\textcircled{D}} \\ & = & [bwa_w^{\textcircled{D}}a]wa_w^{\textcircled{D}} \\ & = & bwa_w^{\textcircled{D}}. \end{array}$$

Since $(awa_w^{\textcircled{D}}a)^*b$ is self-adjoint, we verify that

$$\begin{array}{rcl} a_w^{\textcircled{D}}b & = & a_w^{\textcircled{D}}(awa_w^{\textcircled{D}})^*awa_w^{\textcircled{D}}b \\ & = & a_w^{\textcircled{D}}(wa_w^{\textcircled{D}})^*a^*(awa_w^{\textcircled{D}})^*b \\ & = & a_w^{\textcircled{D}}(wa_w^{\textcircled{D}})^*(awa_w^{\textcircled{D}}a)^*b \\ & = & a_w^{\textcircled{D}}(wa_w^{\textcircled{D}})^*\big((awa_w^{\textcircled{D}}a)^*b\big)^* \\ & = & a_w^{\textcircled{D}}(awa_w^{\textcircled{D}})^*(wa_w^{\textcircled{D}})^*\big((awa_w^{\textcircled{D}}a)^*b\big)^* \\ & = & a_w^{\textcircled{D}}(wa_w^{\textcircled{D}})^*a^*(wa_w^{\textcircled{D}})^*\big((awa_w^{\textcircled{D}}a)^*b\big)^* \\ & = & a_w^{\textcircled{D}}(wa_w^{\textcircled{D}})^*(wa_w^{\textcircled{D}}a)^*\big((awa_w^{\textcircled{D}}a)^*b\big)^* \\ & = & a_w^{\textcircled{D}}(wa_w^{\textcircled{D}})^*\big((awa_w^{\textcircled{D}}a)^*[bwa_w^{\textcircled{D}}a]\big)^* \\ & = & a_w^{\textcircled{D}}(wa_w^{\textcircled{D}})^*\big((awa_w^{\textcircled{D}}a)^*[awa_w^{\textcircled{D}}a]\big)^* \\ & = & a_w^{\textcircled{D}}(wa_w^{\textcircled{D}})^*(awa_w^{\textcircled{D}}a)^*[awa_w^{\textcircled{D}}a] \\ & = & a_w^{\textcircled{D}}(wa_w^{\textcircled{D}})^*a^*(awa_w^{\textcircled{D}})^*[awa_w^{\textcircled{D}}a] \\ & = & a_w^{\textcircled{D}}(awa_w^{\textcircled{D}})^*(awa_w^{\textcircled{D}})^*[awa_w^{\textcircled{D}}a] \\ & = & a_w^{\textcircled{D}}(awa_w^{\textcircled{D}})(awa_w^{\textcircled{D}})(awa_w^{\textcircled{D}})a \\ & = & a_w^{\textcircled{D}}a. \end{array}$$

Therefore $a \leq_w^{\textcircled{D}} b$.

(1) $\Rightarrow$ (4) Obviously, $(awa_w^{\textcircled{D}}a)^*b$ is self-adjoint. Clearly, $awa_w^{\textcircled{D}}(aw)^2a_w^{\textcircled{D}} = bwa_w^{\textcircled{D}}(aw)^2a_w^{\textcircled{D}}$. Since $a_w^{\textcircled{D}}(aw)^2a_w^{\textcircled{D}} = (aw)a_w^{\textcircled{D}}$, we deduce that $aw(awa_w^{\textcircled{D}}) = bw(awa_w^{\textcircled{D}})$. As $(awa_w^{\textcircled{D}})^* = awa_w^{\textcircled{D}}$, we have $(awa_w^{\textcircled{D}})(aw)^* = (awa_w^{\textcircled{D}})(bw)^*$. Since $a_w^{\textcircled{D}}(awa_w^{\textcircled{D}}) = a_w^{\textcircled{D}}$, we get $a_w^{\textcircled{D}}(aw)^* = a_w^{\textcircled{D}}(bw)^*$, as required.

(4) $\Rightarrow$ (2) Since $a_w^{\textcircled{D}}(aw)^* = a_w^{\textcircled{D}}(bw)^*$, we have $[awa_w^{\textcircled{D}}](aw)^* = [awa_w^{\textcircled{D}}](bw)^*$. As $[awa_w^{\textcircled{D}}]^* = awa_w^{\textcircled{D}}$, we deduce that $aw[awa_w^{\textcircled{D}}] = bw[awa_w^{\textcircled{D}}]$; and then $awa_w^{\textcircled{D}} = aw[aw(a_w^{\textcircled{D}})^2] = [(aw)^2a_w^{\textcircled{D}}]a_w^{\textcircled{D}} = [bwawa_w^{\textcircled{D}}]a_w^{\textcircled{D}} = bw[aw(a_w^{\textcircled{D}})^2] = bwa_w^{\textcircled{D}}$, as desired. □

Theorem 3.7. *Let $a, b \in R_w^{\textcircled{D}}$. Then the following are equivalent:*

(1) $a \leq_w^{\textcircled{D}} b$.
(2) $awa_w^{\textcircled{D}}a = bwa_w^{\textcircled{D}}b$ and $awa_w^{\textcircled{D}} = bwa_w^{\textcircled{D}}$.
(3) $awa_w^{\textcircled{D}}(a - bwa_w^{\textcircled{D}}b) = 0$ and $awa_w^{\textcircled{D}} = bwa_w^{\textcircled{D}}$.
(4) $a_w^{\textcircled{D}}(a - bwa_w^{\textcircled{D}}b) = 0$ and $awa_w^{\textcircled{D}} = bwa_w^{\textcircled{D}}$.

Proof. (1) ⇒ (2) Since $a \leq^{\textcircled{D}}_w b$, we have $a^{\textcircled{D}}_w a = a^{\textcircled{D}}_w b$ and $awa^{\textcircled{D}}_w = bwa^{\textcircled{D}}_w$. Then $awa^{\textcircled{D}}_w a = bwa^{\textcircled{D}}_w a = bwa^{\textcircled{D}}_w b$, as required.

(2) ⇒ (3) We directly check that

$$awa^{\textcircled{D}}_w(a - bwa^{\textcircled{D}}_w b) = awa^{\textcircled{D}}_w(a - awa^{\textcircled{D}}_w a) = awa^{\textcircled{D}}_w a - aw[a^{\textcircled{D}}_w awa^{\textcircled{D}}_w]a = 0,$$

as desired.

(3) ⇒ (4) Since $a^{\textcircled{D}}_w = a^{\textcircled{D}}_w awa^{\textcircled{D}}_w$, we conclude that $a^{\textcircled{D}}_w(a - bwa^{\textcircled{D}}_w b) = a^{\textcircled{D}}_w[awa^{\textcircled{D}}_w(a - bwa^{\textcircled{D}}_w b)] = 0$.

(4) ⇒ (1) By hypothesis, we have

$$a^{\textcircled{D}}_w a = a^{\textcircled{D}}_w(bwa^{\textcircled{D}}_w)b = a^{\textcircled{D}}_w(awa^{\textcircled{D}}_w)b = [a^{\textcircled{D}}_w awa^{\textcircled{D}}_w]b = a^{\textcircled{D}}_w b.$$

This completes the proof. □

We are ready to prove:

Theorem 3.8. *Let $a, b \in R^{\textcircled{D}}_w$. Then the following are equivalent:*

1. $a \leq^{\textcircled{D}}_w b$.
2. $(1 - awa^{\textcircled{D}}_w)bwa^{\textcircled{D}}_w = 0$ and $a^{\textcircled{D}}_w a = a^{\textcircled{D}}_w b$.
3. $(1 - awa^{\textcircled{D}}_w)bwa^{\textcircled{D}}_w a = 0$ and $awa^{\textcircled{D}}_w a = awa^{\textcircled{D}}_w b$.
4. $(1 - awa^{\textcircled{D}}_w)bwa^{\textcircled{D}}_w = 0$ and $a^*(aw)^n = b^*(aw)^n$ for some $n \in \mathbb{N}$.
5. $(1 - awa^{\textcircled{D}}_w)bwa^{\textcircled{D}}_w = 0, a^{\textcircled{D}}_w a = a^{\textcircled{D}}_w bwa^{\textcircled{D}}_w a$ and $a^{\textcircled{D}}_w a(1 - wawa^{\textcircled{D}}_w) = a^{\textcircled{D}}_w b(1 - wawa^{\textcircled{D}}_w)$.
6. $(1 - awa^{\textcircled{D}}_w)bwa^{\textcircled{D}}_w a = 0, a^{\textcircled{D}}_w = a^{\textcircled{D}}_w bwa^{\textcircled{D}}_w$ and $awa^{\textcircled{D}}_w a(1 - wawa^{\textcircled{D}}_w) = awa^{\textcircled{D}}_w b(1 - wawa^{\textcircled{D}}_w)$.

Proof. (1) ⇒ (2) By hypothesis, we have $awa^{\textcircled{D}}_w = bwa^{\textcircled{D}}_w$ and $a^{\textcircled{D}}_w a = a^{\textcircled{D}}_w b$. Then $(1 - awa^{\textcircled{D}}_w)bwa^{\textcircled{D}}_w a = (1 - awa^{\textcircled{D}}_w)awa^{\textcircled{D}}_w a = 0$, as required.

(2) ⇒ (3) This is trivial.

(3) ⇒ (1) By hypothesis, we have $bwa^{\textcircled{D}}_w a = awa^{\textcircled{D}}_w bwa^{\textcircled{D}}_w a$ and $awa^{\textcircled{D}}_w a = awa^{\textcircled{D}}_w b$. Then $bwa^{\textcircled{D}}_w a = [awa^{\textcircled{D}}_w b]wa^{\textcircled{D}}_w a = [awa^{\textcircled{D}}_w a]wa^{\textcircled{D}}_w a = aw[a^{\textcircled{D}}_w awa^{\textcircled{D}}_w]a = awa^{\textcircled{D}}_w a$. Hence, $bwa^{\textcircled{D}}_w = [bwa^{\textcircled{D}}_w a]wa^{\textcircled{D}}_w = [awa^{\textcircled{D}}_w a]wa^{\textcircled{D}}_w = awa^{\textcircled{D}}_w$. Since $awa^{\textcircled{D}}_w a = awa^{\textcircled{D}}_w b$, we deduce that $a^{\textcircled{D}}_w a = a^{\textcircled{D}}_w[awa^{\textcircled{D}}_w a] = a^{\textcircled{D}}_w[awa^{\textcircled{D}}_w b] = a^{\textcircled{D}}_w b$, as desired.

(1) ⇒ (4) By the argument above, we have $(1 - awa^{\textcircled{D}}_w)bwa^{\textcircled{D}}_w = 0$. In view of Theorem 3.1, $a^*(aw)^n = b^*(aw)^n$ for some $n \in \mathbb{N}$.

(4) ⇒ (1) By hypothesis, we verify that

$$\begin{array}{rcl} \big(bwa^{\textcircled{\tiny D}}_w\big)^* & = & \big(awa^{\textcircled{\tiny D}}_w bwa^{\textcircled{\tiny D}}_w\big)^* \\ & = & (wa^{\textcircled{\tiny D}}_w)^* b^* (awa^{\textcircled{\tiny D}}_w)^* \\ & = & (wa^{\textcircled{\tiny D}}_w)^* b^* awa^{\textcircled{\tiny D}}_w \\ & = & (wa^{\textcircled{\tiny D}}_w)^* b^* (aw)^n (a^{\textcircled{\tiny D}}_w)^n \\ & = & (wa^{\textcircled{\tiny D}}_w)^* a^* (aw)^n (a^{\textcircled{\tiny D}}_w)^n \\ & = & (awa^{\textcircled{\tiny D}}_w)^* awa^{\textcircled{\tiny D}}_w \\ & = & aw[a^{\textcircled{\tiny D}}_w awa^{\textcircled{\tiny D}}_w] \\ & = & awa^{\textcircled{\tiny D}}_w. \end{array}$$

Therefore $bwa^{\textcircled{\tiny D}}_w = (awa^{\textcircled{\tiny D}}_w)^* = awa^{\textcircled{\tiny D}}_w$. Obviously, we can find some $n \in \mathbb{N}$ such that $a^*(aw)^n = b^*(aw)^n$ and $a^{\textcircled{\tiny D}}_w(aw)^{n+1} = (aw)^n$; hence, $bwa^{\textcircled{\tiny D}}_w(aw)^{n+1} = (awa^{\textcircled{\tiny D}}_w)^* = awa^{\textcircled{\tiny D}}_w(aw)^{n+1}$. This implies that $bw(aw)^n = awa^{\textcircled{\tiny D}}_w(aw)^n$. Accordingly, $a \leq^{\textcircled{\tiny D}}_w b$ by Theorem 3.1.

(1) ⇒ (5) Since $awa^{\textcircled{\tiny D}}_w = bwa^{\textcircled{\tiny D}}_w$ and $a^{\textcircled{\tiny D}}_w a = a^{\textcircled{\tiny D}}_w b$. We verify that

$$a^{\textcircled{\tiny D}}_w a = [a^{\textcircled{\tiny D}}_w awa^{\textcircled{\tiny D}}_w]a = a^{\textcircled{\tiny D}}_w bwa^{\textcircled{\tiny D}}_w a,$$

as required.

(5) ⇒ (6) Obviously, $a^{\textcircled{\tiny D}}_w = [a^{\textcircled{\tiny D}}_w a]wa^{\textcircled{\tiny D}}_w = [a^{\textcircled{\tiny D}}_w bwa^{\textcircled{\tiny D}}_w a]wa^{\textcircled{\tiny D}}_w = a^{\textcircled{\tiny D}}_w bw[a^{\textcircled{\tiny D}}_w awa^{\textcircled{\tiny D}}_w] = a^{\textcircled{\tiny D}}_w bwa^{\textcircled{\tiny D}}_w$, as desired.

(6) ⇒ (3) By hypothesis, we have

$$\begin{array}{rcl} awa^{\textcircled{\tiny D}}_w bwawa^{\textcircled{\tiny D}}_w & = & awa^{\textcircled{\tiny D}}_w bw[a^{\textcircled{\tiny D}}_w (aw)^2]a^{\textcircled{\tiny D}}_w \\ & = & aw[a^{\textcircled{\tiny D}}_w bwa^{\textcircled{\tiny D}}_w](aw)^2 a^{\textcircled{\tiny D}}_w \\ & = & awa^{\textcircled{\tiny D}}_w (aw)^2 a^{\textcircled{\tiny D}}_w. \end{array}$$

Therefore $awa^{\textcircled{\tiny D}}_w a = awa^{\textcircled{\tiny D}}_w b$, as asserted. □

4. Weighted core-EP inverse of product and difference

Let $p, q \in \mathcal{A}$ be idempotents. Then for any $x \in \mathcal{A}$, we have $x = pxq + px(1-p) + (1-p)xq + (1-p)x(1-q)$. Thus x can be represented in the matrix form $x = \begin{pmatrix} pxp & px(1-p) \\ (1-p)xp & (1-p)x(1-q) \end{pmatrix}_{(p,q)}$. With respect to the orthogonal sum of a Hilbert space, Stanimirović and Mosić provided conditions for the equivalence between the forward order law and the reverse order law for the core-EP inverse of Hilbert space operators. We will utilize the preceding matrix, in relation to idempotents, to extend the main results in [25] to a broader class of ring elements. The following theorem is crucial.

Theorem 4.1. *Let $a \in \mathcal{A}^{\textcircled{\tiny D}}_w$. Then the following are equivalent:*

(1) $a \leq_w^{\textcircled{D}} b$.
(2) a, w and b are represented as

$$a = \begin{pmatrix} a_1 & a_{12} \\ 0 & a_2 \end{pmatrix}_{p\times q}, w = \begin{pmatrix} w_1 & w_{12} \\ 0 & w_2 \end{pmatrix}_{q\times p}, b = \begin{pmatrix} a_1 & a_{12} \\ 0 & b_2 \end{pmatrix}_{p\times q},$$

where

$$\begin{gathered} p = awa_w^{\textcircled{D}}, q = wa_w^{\textcircled{D}}a, \\ a_1w_1 \in (p\mathcal{A}p)^{-1}, w_1a_1 \in (q\mathcal{A}q)^{-1}, \\ a_2w_2 \in ((1-p)\mathcal{A}(1-p))^{qnil}, \\ w_2a_2 \in ((1-q)\mathcal{A}(1-q))^{qnil}, \\ w_1a_{12} + w_{12}a_2 = w_1b_{12} + w_{12}b_2. \end{gathered}$$

Proof. (1) $\Rightarrow$ (2) Let $p = awa_w^{\textcircled{D}}, q = wa_w^{\textcircled{D}}a$. Then we verify that

$$\begin{array}{rcl} (1-p)aq & = & [1 - awa_w^{\textcircled{D}}]awa_w^{\textcircled{D}}a \\ & = & awa_w^{\textcircled{D}}a - aw[a_w^{\textcircled{D}}awa_w^{\textcircled{D}}]a \\ & = & 0; \\ (1-q)wp & = & [1 - wa_w^{\textcircled{D}}a]wawa_w^{\textcircled{D}} \\ & = & wawa_w^{\textcircled{D}} - wa_w^{\textcircled{D}}(aw)^2a_w^{\textcircled{D}} \\ & = & 0. \end{array}$$

Moreover, we verify that

$$\begin{array}{rcl} a_1w_1 & = & aw[a_w^{\textcircled{D}}awa_w^{\textcircled{D}}](aw)^2a_w^{\textcircled{D}} \\ & = & awa_w^{\textcircled{D}}(aw)^2a_w^{\textcircled{D}} \\ & = & (aw)^2a_w^{\textcircled{D}} \\ & \in & (p\mathcal{A}p)^{-1}; \\ w_1a_1 & = & wawa_w^{\textcircled{D}}a \\ & \in & (q\mathcal{A}q)^{-1}; \\ a_2w_2 & = & [1 - awa_w^{\textcircled{D}}]a[1 - wa_w^{\textcircled{D}}a]w[1 - awa_w^{\textcircled{D}}] \\ & = & aw - awa_w^{\textcircled{D}}aw \\ & \in & [(1-p)\mathcal{A}(1-p)]^{qnil}; \\ w_2a_2 & = & [1 - wa_w^{\textcircled{D}}a]w[1 - awa_w^{\textcircled{D}}]a[1 - wa_w^{\textcircled{D}}a] \\ & = & wa - wawa_w^{\textcircled{D}}a \\ & \in & [(1-q)\mathcal{A}(1-q)]^{qnil}. \end{array}$$

Write $b = \begin{pmatrix} b_1 & b_{12} \\ b_{21} & b_2 \end{pmatrix}_{p\times q}$. Clearly, we have

$$\begin{array}{rcl} a_w^{\textcircled{D}}(1-p) & = & a_w^{\textcircled{D}}(1 - awa_w^{\textcircled{D}}) = 0, \\ (1-p)a_w^{\textcircled{D}} & = & (1 - awa_w^{\textcircled{D}})a_w^{\textcircled{D}} = 0. \end{array}$$

Then

$$a_w^{\textcircled{\tiny D}} = \begin{pmatrix} a_w^{\textcircled{\tiny D}} & 0 \\ 0 & 0 \end{pmatrix}_{p\times p}.$$

Since $a \leq_w^{\textcircled{\tiny D}} b$, we have

$$\begin{aligned}
& \begin{pmatrix} a_1 w_1 a_w^{\textcircled{\tiny D}} & 0 \\ 0 & 0 \end{pmatrix}_{p\times q} \\
= & \begin{pmatrix} a_1 & a_{12} \\ 0 & a_2 \end{pmatrix}_{p\times q} \begin{pmatrix} w_1 & w_{12} \\ 0 & w_2 \end{pmatrix}_{q\times p} \begin{pmatrix} a_w^{\textcircled{\tiny D}} & 0 \\ 0 & 0 \end{pmatrix}_{p\times q} \\
= & (aw)a^{\textcircled{d},w} = (bw)a^{\textcircled{d},w} \\
= & \begin{pmatrix} b_1 & b_{12} \\ b_{21} & b_2 \end{pmatrix}_{p\times q} \begin{pmatrix} w_1 & w_{12} \\ 0 & w_2 \end{pmatrix}_{q\times p} \begin{pmatrix} a_w^{\textcircled{\tiny D}} & 0 \\ 0 & 0 \end{pmatrix}_{p\times p} \\
= & \begin{pmatrix} b_1 w_1 a_w^{\textcircled{\tiny D}} & 0 \\ b_{21} w_1 a_w^{\textcircled{\tiny D}} & 0 \end{pmatrix}_{p\times q}.
\end{aligned}$$

Then $a_1 w_1 a_w^{\textcircled{\tiny D}} = b_1 w_1 a_w^{\textcircled{\tiny D}}, b_{21} w_1 a_w^{\textcircled{\tiny D}} = 0$. Hence, $a_1 w_1 = b_1 w_1$, and then $a_1 = (a_1 w_1)(a_1 w_1) a_w^{\textcircled{\tiny D}} = (b_1 w_1)(a_1 w_1) a_w^{\textcircled{\tiny D}} = b_1$. Also we have $b_{21} w_1 = 0$, and so $b_{21} = (b_{21} w_1) a_1 w_1 a_w^{\textcircled{\tiny D}} = 0$. Moreover, we have

$$\begin{aligned}
& \begin{pmatrix} a_w^{\textcircled{\tiny D}} w_1 a_1 & a_w^{\textcircled{\tiny D}} w_1 a_{12} + a_w^{\textcircled{\tiny D}} w_{12} a_2 \\ 0 & 0 \end{pmatrix}_{p\times q} \\
= & \begin{pmatrix} a_w^{\textcircled{\tiny D}} & 0 \\ 0 & 0 \end{pmatrix}_{p\times q} \begin{pmatrix} w_1 & w_{12} \\ 0 & w_2 \end{pmatrix}_{q\times p} \begin{pmatrix} a_1 & a_{12} \\ 0 & a_2 \end{pmatrix}_{p\times q} \\
= & a_w^{\textcircled{\tiny D}}(wa) = a_w^{\textcircled{\tiny D}}(wb) \\
= & \begin{pmatrix} a_w^{\textcircled{\tiny D}} & 0 \\ 0 & 0 \end{pmatrix}_{p\times q} \begin{pmatrix} w_1 & w_{12} \\ 0 & w_2 \end{pmatrix}_{q\times p} \begin{pmatrix} a_1 & b_{12} \\ 0 & b_2 \end{pmatrix}_{p\times q} \\
= & \begin{pmatrix} a_w^{\textcircled{\tiny D}} w_1 a_1 & a_w^{\textcircled{\tiny D}} w_1 b_{12} + a_w^{\textcircled{\tiny D}} w_{12} b_2 \\ 0 & 0 \end{pmatrix}_{p\times q}.
\end{aligned}$$

Thus, we have

$$a_w^{\textcircled{\tiny D}} w_1 a_{12} + a_w^{\textcircled{\tiny D}} w_{12} a_2 = a_w^{\textcircled{\tiny D}} w_1 b_{12} + a_w^{\textcircled{\tiny D}} w_{12} b_2.$$

Thus $w_1 a_{12} + w_{12} a_2 = w_1 b_{12} + w_{12} b_2$.

Further, we see that

$$\begin{array}{rl} & \begin{pmatrix} a_w^{\textcircled{D}}a_1 & a_w^{\textcircled{D}}a_{12} \\ 0 & 0 \end{pmatrix}_{p\times q} \\ = & \begin{pmatrix} a_w^{\textcircled{D}} & 0 \\ 0 & 0 \end{pmatrix}_{p\times p} \begin{pmatrix} a_1 & a_{12} \\ 0 & a_2 \end{pmatrix}_{p\times q} \\ = & a_w^{\textcircled{D}}a = a_w^{\textcircled{D}}b \\ = & \begin{pmatrix} a_w^{\textcircled{D}} & 0 \\ 0 & 0 \end{pmatrix}_{p\times p} \begin{pmatrix} a_1 & b_{12} \\ 0 & b_2 \end{pmatrix}_{p\times q} \\ = & \begin{pmatrix} a_w^{\textcircled{D}}a_1 & a_w^{\textcircled{D}}b_{12} \\ 0 & 0 \end{pmatrix}_{p\times q}. \end{array}$$

Hence $a_w^{\textcircled{D}}a_{12} = a_w^{\textcircled{D}}b_{12}$. Therefore $a_{12} = b_{12}$, as required.

(2) $\Rightarrow$ (1) By hypothesis, we have

$$a_w^{\textcircled{D}} = \begin{pmatrix} a_w^{\textcircled{D}} & 0 \\ 0 & 0 \end{pmatrix}_{p\times q}.$$

Moreover, we derive

$$aw = \begin{pmatrix} a_1w_1 & a_1w_{12} + a_{12}w_2 \\ 0 & a_2w_2 \end{pmatrix}_{p\times p}, bw = \begin{pmatrix} a_1w_1 & a_1w_{12} + b_{12}w_2 \\ 0 & b_2w_2 \end{pmatrix}_{p\times p}.$$

Then

$$awa^{\textcircled{d},w} = \begin{pmatrix} a_1w_1a_w^{\textcircled{D}} & 0 \\ 0 & 0 \end{pmatrix}_{p\times p} = bwa_w^{\textcircled{D}}.$$

Moreover, we have

$$\begin{array}{rcl} a_w^{\textcircled{D}}a & = & \begin{pmatrix} a_w^{\textcircled{D}} & 0 \\ 0 & 0 \end{pmatrix}_{p\times q} \begin{pmatrix} a_1 & a_{12} \\ 0 & a_2 \end{pmatrix}_{p\times q} \\ & = & \begin{pmatrix} a_w^{\textcircled{D}} & 0 \\ 0 & 0 \end{pmatrix}_{p\times q} \begin{pmatrix} a_1 & a_{12} \\ 0 & b_2 \end{pmatrix}_{p\times q} \\ & = & a_w^{\textcircled{D}}b. \end{array}$$

Therefore $a \leq_w^{\textcircled{D}} b$, as asserted. □

Corollary 4.2. *Let $a, b \in \mathcal{A}^{\textcircled{D}}$. Then the following are equivalent:*

(1) $a \leq_w^{\textcircled{D}} b$.
(2) a and b are represented as

$$a = \begin{pmatrix} a_1 & a_{12} \\ 0 & a_2 \end{pmatrix}_{p\times p}, b = \begin{pmatrix} a_1 & a_{12} \\ 0 & b_2 \end{pmatrix}_{p\times p},$$

where $p = aa^{\textcircled{D}}, a_1 \in (p\mathcal{A}p)^{-1}, b_1 \in (p\mathcal{A}p)^{qnil}, a_2 \in ((1-p)\mathcal{A}(1-p))^{-1}$ and $b_2 \in ((1-p)\mathcal{A}(1-p))^{qnil}$.

Proof. This is evident by setting $w = 1$ in Theorem 4.1. □

We now derive the equivalent conditions for the forward order law of the weighted core-EP inverse.

Theorem 4.3. *Let $a, b, awb \in R_w^{\textcircled{D}}$ and $a \leq_w^{\textcircled{D}} b$. Then the following are equivalent:*

(1) $(awb)_w^{\textcircled{D}} = a_w^{\textcircled{D}} b_w^{\textcircled{D}}$.
(2) $(awb)_w^{\textcircled{D}}(1 - awa_w^{\textcircled{D}}) = a_w^{\textcircled{D}} b_w^{\textcircled{D}}(1 - awa_w^{\textcircled{D}}) = 0$.
(3) $(awb)_w^{\textcircled{D}}(1 - awa_w^{\textcircled{D}}) = awa_w^{\textcircled{D}} b_w^{\textcircled{D}}(1 - awa_w^{\textcircled{D}}) = 0$.
(4) $(awb)_w^{\textcircled{D}} = b_w^{\textcircled{D}} a_w^{\textcircled{D}}$ and $awa_w^{\textcircled{D}} b_w^{\textcircled{D}}(1 - awa_w^{\textcircled{D}}) = 0$.
(5) $(awb)_w^{\textcircled{D}} = b_w^{\textcircled{D}} a_w^{\textcircled{D}}$ and $a_w^{\textcircled{D}} b_w^{\textcircled{D}}(1 - awa_w^{\textcircled{D}}) = 0$.

Proof. In view of Theorem 4.1, a, w and b are represented as

$$a = \begin{pmatrix} a_1 & a_{12} \\ 0 & a_2 \end{pmatrix}_{p\times q}, w = \begin{pmatrix} w_1 & w_{12} \\ 0 & w_2 \end{pmatrix}_{q\times p}, b = \begin{pmatrix} a_1 & a_{12} \\ 0 & b_2 \end{pmatrix}_{p\times q},$$

where

$$\begin{gathered} p = awa_w^{\textcircled{D}}, q = wa_w^{\textcircled{D}}a, \\ a_1w_1 \in (p\mathcal{A}p)^{-1}, w_1a_1 \in (q\mathcal{A}q)^{-1}, \\ a_2w_2 \in ((1-p)\mathcal{A}(1-p))^{qnil}, \\ w_2a_2 \in ((1-q)\mathcal{A}(1-q))^{qnil}, \\ w_1a_{12} + w_{12}a_2 = w_1b_{12} + w_{12}b_2. \end{gathered}$$

Then

$$awb = \begin{pmatrix} a_1w_1a_1 & (a_1w_1)a_{12} + (a_1w_{12} + a_{12}w_2)b_2 \\ 0 & a_2w_2b_2 \end{pmatrix}_{p\times q}.$$

$$a_w^{\textcircled{D}} = \begin{pmatrix} a_w^{\textcircled{D}} & 0 \\ 0 & 0 \end{pmatrix}_{p\times p}, b_w^{\textcircled{D}} = \begin{pmatrix} a_w^{\textcircled{D}} & -a_w^{\textcircled{D}}a_{12}b_w^{\textcircled{D}}(1-p) \\ 0 & (1-p)b_w^{\textcircled{D}}(1-p) \end{pmatrix}_{(p,p)},$$

$$(awb)_w^{\textcircled{D}} = \begin{pmatrix} (a_w^{\textcircled{D}})^2 & -(a_w^{\textcircled{D}})^2[(a_1w_1)a_{12} + (a_1w_{12} + a_{12}w_2)b_2](awb)_w^{\textcircled{D}}(1-p) \\ 0 & (1-p)(awb)_w^{\textcircled{D}}(1-p) \end{pmatrix}_{(p,p)}.$$

$$a_w^{\textcircled{D}}b_w^{\textcircled{D}} = \begin{pmatrix} (a_w^{\textcircled{D}})^2 & -(a_w^{\textcircled{D}})^2a_{12}b_w^{\textcircled{D}}(1-p) \\ 0 & 0 \end{pmatrix}_{(p,p)}.$$

(1) ⇒ (2) Since $(awb)_w^{\textcircled{D}} = a_w^{\textcircled{D}} b_w^{\textcircled{D}}$, we see that $a_w^{\textcircled{D}} b_w^{\textcircled{D}}(1 - awa_w^{\textcircled{D}}) = 0$. Moreover, we have

$$p(awb)_w^{\textcircled{D}}p = (1-p)(awb)_w^{\textcircled{D}}(1-p),$$

and then $(awb)_w^{\textcircled{D}}(1-awa_w^{\textcircled{D}})=0$.

(2) $\Rightarrow$ (3) This is trivial.

(3) $\Rightarrow$ (1) Since $awa_w^{\textcircled{D}}b_w^{\textcircled{D}}(1-awa_w^{\textcircled{D}})=0$, we see that

$$b_w^{\textcircled{D}}=\begin{pmatrix} a_w^{\textcircled{D}} & 0 \\ 0 & (1-p)b_w^{\textcircled{D}}(1-p) \end{pmatrix}_{(p,p)}.$$

Hence,

$$a_w^{\textcircled{D}}b_w^{\textcircled{D}}=\begin{pmatrix} (a_w^{\textcircled{D}})^2 & 0 \\ 0 & 0 \end{pmatrix}_{(p,p)}.$$

Therefore $a_w^{\textcircled{D}}b_w^{\textcircled{D}}=(awb)_w^{\textcircled{D}}$, as required.

(3) $\Leftrightarrow$ (4) $\Leftrightarrow$ (5) By the preceding argument, we have

$$a_w^{\textcircled{D}}=\begin{pmatrix} a_w^{\textcircled{D}} & 0 \\ 0 & 0 \end{pmatrix}_{p\times p}, b_w^{\textcircled{D}}=\begin{pmatrix} a_w^{\textcircled{D}} & -a_w^{\textcircled{D}}a_{12}b_w^{\textcircled{D}}(1-p) \\ 0 & (1-p)b_w^{\textcircled{D}}(1-p) \end{pmatrix}_{(p,p)},$$

$$(awb)_w^{\textcircled{D}}=\begin{pmatrix} (a_w^{\textcircled{D}})^2 & -(a_w^{\textcircled{D}})^2[(a_1w_1)a_{12}+(a_1w_{12}+a_{12}w_2)b_2](awb)_w^{\textcircled{D}}(1-p) \\ 0 & (1-p)(awb)_w^{\textcircled{D}}(1-p) \end{pmatrix}_{(p,p)}.$$

Then

$$b_w^{\textcircled{D}}a_w^{\textcircled{D}}=\begin{pmatrix} a_w^{\textcircled{D}} & -a_w^{\textcircled{D}}a_{12}b_w^{\textcircled{D}}(1-p) \\ 0 & (1-p)b_w^{\textcircled{D}}(1-p) \end{pmatrix}_{(p,p)}\begin{pmatrix} a_w^{\textcircled{D}} & 0 \\ 0 & 0 \end{pmatrix}_{p\times p}=\begin{pmatrix} (a_w^{\textcircled{D}})^2 & 0 \\ 0 & 0 \end{pmatrix}_{p\times p}.$$

Then $(awb)_w^{\textcircled{D}}=b_w^{\textcircled{D}}a_w^{\textcircled{D}}$ if and only if $(awb)_w^{\textcircled{D}}(1-awa_w^{\textcircled{D}})=0$, as required. □

As an immediate consequence of Theorem 4.3, we extend [29, Theorem 3.2] from the core-EP pre-order for Hilbert space operators to that for elements of a ring with involution.

Corollary 4.4. *Let $a,b,ab\in R^{\textcircled{D}}$ and $a\leq^{\textcircled{D}} b$. Then the following are equivalent:*

(1) $(ab)^{\textcircled{D}}=a^{\textcircled{D}}b^{\textcircled{D}}$.
(2) $(ab)^{\textcircled{D}}(1-aa^{\textcircled{D}})=a^{\textcircled{D}}b^{\textcircled{D}}(1-aa^{\textcircled{D}})=0$.
(3) $(ab)^{\textcircled{D}}(1-aa^{\textcircled{D}})=aa^{\textcircled{D}}b^{\textcircled{D}}(1-aa^{\textcircled{D}})=0$.
(4) $(ab)^{\textcircled{D}}=b^{\textcircled{D}}a^{\textcircled{D}}$ and $aa^{\textcircled{D}}b^{\textcircled{D}}(1-aa^{\textcircled{D}})=0$.
(5) $(ab)^{\textcircled{D}}=b^{\textcircled{D}}a^{\textcircled{D}}$ and $a^{\textcircled{D}}b^{\textcircled{D}}(1-aa^{\textcircled{D}})=0$.

Lemma 4.5. *Let $a,b\in R_w^{\textcircled{D}}$ and $a\leq_w^{\textcircled{D}} b$. Then the following are equivalent:*

(1) $a_w^{\textcircled{D}}\leq^{\textcircled{D}} b_w^{\textcircled{D}}$.
(2) $a_w^{\textcircled{D}}b_w^{\textcircled{D}}=b_w^{\textcircled{D}}a_w^{\textcircled{D}}$.
(3) $a_w^{\textcircled{D}}b_w^{\textcircled{D}}=(a_w^{\textcircled{D}})^2$.

Proof. By virtue of Theorem 4.1, a, w and b are represented as

$$a = \begin{pmatrix} a_1 & a_{12} \\ 0 & a_2 \end{pmatrix}_{p\times q}, w = \begin{pmatrix} w_1 & w_{12} \\ 0 & w_2 \end{pmatrix}_{q\times p}, b = \begin{pmatrix} a_1 & a_{12} \\ 0 & b_2 \end{pmatrix}_{p\times q},$$

where $p = awa_w^{\textcircled{D}}, q = wa_w^{\textcircled{D}}a$.

As in the proof of Theorem 4.3, we have

$$a_w^{\textcircled{D}} = \begin{pmatrix} a_w^{\textcircled{D}} & 0 \\ 0 & 0 \end{pmatrix}_{p\times p}, b_w^{\textcircled{D}} = \begin{pmatrix} a_w^{\textcircled{D}} & -a_w^{\textcircled{D}}a_{12}b_w^{\textcircled{D}}(1-p) \\ 0 & (1-p)b_w^{\textcircled{D}}(1-p) \end{pmatrix}_{(p,p)}.$$

Then

$$\begin{aligned} a_w^{\textcircled{D}}b_w^{\textcircled{D}} &= \begin{pmatrix} a_w^{\textcircled{D}} & 0 \\ 0 & 0 \end{pmatrix}_{p\times p}\begin{pmatrix} a_w^{\textcircled{D}} & -a_w^{\textcircled{D}}a_{12}b_w^{\textcircled{D}}(1-p) \\ 0 & (1-p)b_w^{\textcircled{D}}(1-p) \end{pmatrix}_{(p,p)} \\ &= \begin{pmatrix} (a_w^{\textcircled{D}})^2 & -(a_w^{\textcircled{D}})^2a_{12}b_w^{\textcircled{D}}(1-p) \\ 0 & 0 \end{pmatrix}_{p\times p}, \\ b_w^{\textcircled{D}}a_w^{\textcircled{D}} &= \begin{pmatrix} a_w^{\textcircled{D}} & -a_w^{\textcircled{D}}a_{12}b_w^{\textcircled{D}}(1-p) \\ 0 & (1-p)b_w^{\textcircled{D}}(1-p) \end{pmatrix}_{(p,p)}\begin{pmatrix} a_w^{\textcircled{D}} & 0 \\ 0 & 0 \end{pmatrix}_{p\times p} \\ &= \begin{pmatrix} (a_w^{\textcircled{D}})^2 & 0 \\ 0 & 0 \end{pmatrix}_{p\times p}. \end{aligned}$$

(1) $\Leftrightarrow$ (2) Obviously,

$$[a_w^{\textcircled{D}}]^{\textcircled{D}} = (aw)^2a_w^{\textcircled{D}} = \begin{pmatrix} (aw)^2a_w^{\textcircled{D}} & 0 \\ 0 & 0 \end{pmatrix}_{p\times p}.$$

Then $a_w^{\textcircled{D}} \leq^{\textcircled{D}} b_w^{\textcircled{D}}$ if and only if the following hold:

$$\begin{aligned} a_w^{\textcircled{D}}[a_w^{\textcircled{D}}]^{\textcircled{D}} &= \begin{pmatrix} (aw)^2a_w^{\textcircled{D}} & 0 \\ 0 & 0 \end{pmatrix}_{p\times p} = b_w^{\textcircled{D}}[a_w^{\textcircled{D}}]^{\textcircled{D}}, \\ \left[a_w^{\textcircled{D}}\right]^{\textcircled{D}}a_w^{\textcircled{D}} &= \begin{pmatrix} awa_w^{\textcircled{D}} & 0 \\ 0 & 0 \end{pmatrix}_{p\times p} \\ &= \begin{pmatrix} awa_w^{\textcircled{D}} & -pa_{12}b_w^{\textcircled{D}}(1-p) \\ 0 & 0 \end{pmatrix}_{p\times p} \\ &= \left[a_w^{\textcircled{D}}\right]^{\textcircled{D}}b_w^{\textcircled{D}}. \end{aligned},$$

i.e., $pa_{12}b_w^{\textcircled{D}}(1-p) = 0$.

On the other hand, $a_w^{\textcircled{D}}b_w^{\textcircled{D}} = b_w^{\textcircled{D}}a_w^{\textcircled{D}}$ if and only if $-a_w^{\textcircled{D}}a_{12}b_w^{\textcircled{D}}(1-p) = 0$. Clearly, $a_w^{\textcircled{D}} = a_w^{\textcircled{D}}p$. Then $pa_{12}b_w^{\textcircled{D}}(1-p) = 0$ if and only if $-a_w^{\textcircled{D}}a_{12}b_w^{\textcircled{D}}(1-p) = 0$, as required.

(1) $\Leftrightarrow$ (3) Since $(a_w^{\textcircled{D}})^2 = \begin{pmatrix} (a_w^{\textcircled{D}})^2 & 0 \\ 0 & 0 \end{pmatrix}_{p\times p}$, we see that $a_w^{\textcircled{D}} b_w^{\textcircled{D}} = (a_w^{\textcircled{D}})^2$ if and only if $-(a_w^{\textcircled{D}})^2 a_{12} b_w^{\textcircled{D}}(1-p) = 0$.

Since $p = aw(a_w^{\textcircled{D}})^2$ and $(a_w^{\textcircled{D}})^2 = a_w^{\textcircled{D}} p$, we have $-(a_w^{\textcircled{D}})^2 a_{12} b_w^{\textcircled{D}}(1-p) = 0$ if and only if $pa_{12}b_w^{\textcircled{D}}(1-p) = 0$. By the argument above, we complete the proof. $\square$

Theorem 4.6. *Let $a, b, awb \in R_w^{\textcircled{D}}$ and $a \leq_w^{\textcircled{D}} b$. Then $(awb)_w^{\textcircled{D}} = a_w^{\textcircled{D}} b_w^{\textcircled{D}}$ if and only if*

(1) $a_w^{\textcircled{D}} \leq^{\textcircled{D}} b_w^{\textcircled{D}}$;
(2) $(awb)_w^{\textcircled{D}}(1 - awa_w^{\textcircled{D}}) = 0$.

Proof. $\Longrightarrow$ Since $a \leq_w^{\textcircled{D}} b$, by using Theorem 4.3 and Lemma 4.5, $a_w^{\textcircled{D}} \leq^{\textcircled{D}} b_w^{\textcircled{D}}$. According to Theorem 4.3, $(awb)_w^{\textcircled{D}}(1 - awa_w^{\textcircled{D}}) = 0$, as required.

$\Longleftarrow$ By virtue of Lemma 4.5, $a_w^{\textcircled{D}} b_w^{\textcircled{D}} = b_w^{\textcircled{D}} a_w^{\textcircled{D}}$. This completes the proof by Theorem 4.3. $\square$

Corollary 4.7. *Let $a, b, ab \in R^{\textcircled{D}}$ and $a \leq^{\textcircled{D}} b$. Then $(ab)^{\textcircled{D}} = a^{\textcircled{D}} b^{\textcircled{D}}$ if and only if*

(1) $a^{\textcircled{D}} \leq^{\textcircled{D}} b^{\textcircled{D}}$;
(2) $(ab)^{\textcircled{D}}(1 - aa^{\textcircled{D}}) = 0$.

Proof. This is obvious by choosing $w = 1$ in Theorem 4.6. $\square$

Dually, we derive the equivalent conditions for the reverse order law of the weighted core-EP inverse.

Theorem 4.8. *Let $a, b, bwa \in R_w^{\textcircled{D}}$ and $a \leq_w^{\textcircled{D}} b$. Then the following are equivalent:*

(1) $(bwa)_w^{\textcircled{D}} = b_w^{\textcircled{D}} a_w^{\textcircled{D}}$.
(2) $(bwa)_w^{\textcircled{D}}(1 - awa_w^{\textcircled{D}}) = awa_w^{\textcircled{D}} b_w^{\textcircled{D}}(1 - awa_w^{\textcircled{D}}) = 0$.
(3) $(bwa)_w^{\textcircled{D}}(1 - awa_w^{\textcircled{D}}) = a_w^{\textcircled{D}} b_w^{\textcircled{D}}(1 - awa_w^{\textcircled{D}}) = 0$.
(4) $(bwa)_w^{\textcircled{D}} = a_w^{\textcircled{D}} b_w^{\textcircled{D}}$ and $awa_w^{\textcircled{D}} b_w^{\textcircled{D}}(1 - awa_w^{\textcircled{D}}) = 0$.
(5) $(bwa)_w^{\textcircled{D}} = a_w^{\textcircled{D}} b_w^{\textcircled{D}}$ and $a_w^{\textcircled{D}} b_w^{\textcircled{D}}(1 - awa_w^{\textcircled{D}}) = 0$.

Proof. By virtue of Theorem 4.1, a, w and b are represented as

$$a = \begin{pmatrix} a_1 & a_{12} \\ 0 & a_2 \end{pmatrix}_{p\times q}, w = \begin{pmatrix} w_1 & w_{12} \\ 0 & w_2 \end{pmatrix}_{q\times p}, b = \begin{pmatrix} a_1 & a_{12} \\ 0 & b_2 \end{pmatrix}_{p\times q},$$

where $p = awa_w^{\textcircled{\dagger}}, q = wa_w^{\textcircled{\dagger}}a$. Then $a_w^{\textcircled{\dagger}}$ and $b_w^{\textcircled{\dagger}}$ can be written in the matrix forms as in the proof of Theorem 4.3. Moreover, we check that

$$b_w^{\textcircled{\dagger}}a_w^{\textcircled{\dagger}} = \begin{pmatrix} (a_w^{\textcircled{\dagger}})^2 & 0 \\ 0 & 0 \end{pmatrix}_{(p,p)}.$$

(1) $\Rightarrow$ (2) Since $(bwa)_w^{\textcircled{\dagger}} = b_w^{\textcircled{\dagger}}a_w^{\textcircled{\dagger}}$, we have $(bwa)_w^{\textcircled{\dagger}}(1 - awa_w^{\textcircled{\dagger}}) = b_w^{\textcircled{\dagger}}a_w^{\textcircled{\dagger}}(1 - awa_w^{\textcircled{\dagger}}) = 0$. Moreover, we have

$$p(awb)_w^{\textcircled{\dagger}}p = (1-p)(awb)_w^{\textcircled{\dagger}}(1-p),$$

and then $(awb)_w^{\textcircled{\dagger}}(1 - awa_w^{\textcircled{\dagger}}) = 0$.

(2) $\Rightarrow$ (3) This is trivial.

(3) $\Rightarrow$ (1) Since $awa_w^{\textcircled{\dagger}}b_w^{\textcircled{\dagger}}(1 - awa_w^{\textcircled{\dagger}}) = 0$, we see that

$$b_w^{\textcircled{\dagger}} = \begin{pmatrix} a_w^{\textcircled{\dagger}} & 0 \\ 0 & (1-p)b_w^{\textcircled{\dagger}}(1-p) \end{pmatrix}_{(p,p)}.$$

Hence,

$$a_w^{\textcircled{\dagger}}b_w^{\textcircled{\dagger}} = \begin{pmatrix} (a_w^{\textcircled{\dagger}})^2 & 0 \\ 0 & 0 \end{pmatrix}_{(p,p)}.$$

Therefore $a_w^{\textcircled{\dagger}}b_w^{\textcircled{\dagger}} = (awb)_w^{\textcircled{\dagger}}$, as required.

(3) $\Leftrightarrow$ (4) $\Leftrightarrow$ (5) By the preceding argument, we have

$$a_w^{\textcircled{\dagger}} = \begin{pmatrix} a_w^{\textcircled{\dagger}} & 0 \\ 0 & 0 \end{pmatrix}_{p\times p}, b_w^{\textcircled{\dagger}} = \begin{pmatrix} a_w^{\textcircled{\dagger}} & -a_w^{\textcircled{\dagger}}a_{12}b_w^{\textcircled{\dagger}}(1-p) \\ 0 & (1-p)b_w^{\textcircled{\dagger}}(1-p) \end{pmatrix}_{(p,p)},$$

$$(awb)_w^{\textcircled{\dagger}} = \begin{pmatrix} (a_w^{\textcircled{\dagger}})^2 & -(a_w^{\textcircled{\dagger}})^2[(a_1w_1)a_{12} + (a_1w_{12} + a_{12}w_2)b_2](awb)_w^{\textcircled{\dagger}}(1-p) \\ 0 & (1-p)(awb)_w^{\textcircled{\dagger}}(1-p) \end{pmatrix}_{(p,p)}.$$

Then

$$b_w^{\textcircled{\dagger}}a_w^{\textcircled{\dagger}} = \begin{pmatrix} a_w^{\textcircled{\dagger}} & -a_w^{\textcircled{\dagger}}a_{12}b_w^{\textcircled{\dagger}}(1-p) \\ 0 & (1-p)b_w^{\textcircled{\dagger}}(1-p) \end{pmatrix}_{(p,p)} \begin{pmatrix} a_w^{\textcircled{\dagger}} & 0 \\ 0 & 0 \end{pmatrix}_{p\times p} = \begin{pmatrix} (a_w^{\textcircled{\dagger}})^2 & 0 \\ 0 & 0 \end{pmatrix}_{p\times p}.$$

Then $(awb)_w^{\textcircled{\dagger}} = b_w^{\textcircled{\dagger}}a_w^{\textcircled{\dagger}}$ if and only if $(awb)_w^{\textcircled{\dagger}}(1 - awa_w^{\textcircled{\dagger}}) = 0$, as required. □

We now explore the equivalent conditions for $(b-a)_w^{\textcircled{\dagger}} = b_w^{\textcircled{\dagger}} - a_w^{\textcircled{\dagger}}$.

Theorem 4.9. *Let $a, b, b-a \in R_w^{\textcircled{\dagger}}$ and $a \leq_w^{\textcircled{\dagger}} b$. Then the following are equivalent:*

(1) $(b-a)_w^{\textcircled{\dagger}} = b_w^{\textcircled{\dagger}} - a_w^{\textcircled{\dagger}}$.
(2) $(b-a)_w^{\textcircled{\dagger}} = b_w^{\textcircled{\dagger}}(1 - awa_w^{\textcircled{\dagger}})$.
(3) $(b-a)_w^{\textcircled{\dagger}} = (1 - awa_w^{\textcircled{\dagger}})b_w^{\textcircled{\dagger}}$ and $a_w^{\textcircled{\dagger}}b_w^{\textcircled{\dagger}} = (a_w^{\textcircled{\dagger}})^2$.

(4) $(b-a)^{\textcircled{D}}_w=(1-awa^{\textcircled{D}}_w)b^{\textcircled{D}}_w$ and $a^{\textcircled{D}}_w b^{\textcircled{D}}_w(1-awa^{\textcircled{D}}_w)=0$.

Proof. By virtue of Theorem 4.1, we have

$$a=\begin{pmatrix} a_1 & a_{12}\\ 0 & a_2\end{pmatrix}_{p\times q}, w=\begin{pmatrix} w_1 & w_{12}\\ 0 & w_2\end{pmatrix}_{q\times p}, b=\begin{pmatrix} a_1 & a_{12}\\ 0 & b_2\end{pmatrix}_{p\times q},$$

where $p=awa^{\textcircled{D}}_w, q=wa^{\textcircled{D}}_w a.$. Moreover, we compute that

$$a^{\textcircled{D}}_w=\begin{pmatrix} a^{\textcircled{D}}_w & 0\\ 0 & 0\end{pmatrix}_{p\times p}, b^{\textcircled{D}}_w=\begin{pmatrix} a^{\textcircled{D}}_w & -a^{\textcircled{D}}_w a_{12}b^{\textcircled{D}}_w(1-p)\\ 0 & (1-p)b^{\textcircled{D}}_w(1-p)\end{pmatrix}_{(p,p)},$$

$$(b-a)^{\textcircled{D}}_w=\begin{pmatrix} 0 & 0\\ 0 & (1-p)(b-a)^{\textcircled{D}}_w(1-p)\end{pmatrix}_{(p,p)}.$$

(1) $\Leftrightarrow$ (2) Obviously, $-a^{\textcircled{D}}_w a_{12}b^{\textcircled{D}}_w(1-p)=pb^{\textcircled{D}}_w(1-p)$. Then $(b-a)^{\textcircled{D}}_w=b^{\textcircled{D}}_w-a^{\textcircled{D}}_w$ if and only if $(1-p)(b-a)^{\textcircled{D}}_w(1-p)=(1-p)b^{\textcircled{D}}_w(1-p), pb^{\textcircled{D}}_w(1-p)=0$, i.e., $(b-a)^{\textcircled{D}}_w=(1-p)(b-a)^{\textcircled{D}}_w(1-p)=b^{\textcircled{D}}_w(1-awa^{\textcircled{D}}_w)$.

(2) $\Rightarrow$ (3) By the argument above, we have

$$a^{\textcircled{D}}_w=\begin{pmatrix} a^{\textcircled{D}}_w & 0\\ 0 & 0\end{pmatrix}_{p\times p}, b^{\textcircled{D}}_w=\begin{pmatrix} a^{\textcircled{D}}_w & 0\\ 0 & (1-p)b^{\textcircled{D}}_w(1-p)\end{pmatrix}_{(p,p)}.$$

Then $a^{\textcircled{D}}_w b^{\textcircled{D}}_w=\begin{pmatrix} (a^{\textcircled{D}}_w)^2 & 0\\ 0 & 0\end{pmatrix}_{(p,p)}=(a^{\textcircled{D}}_w)^2$. Accordingly, $(b-a)^{\textcircled{D}}_w=b^{\textcircled{D}}_w-a^{\textcircled{D}}_w=b^{\textcircled{D}}_w-aw(a^{\textcircled{D}}_w)^2=b^{\textcircled{D}}_w-awa^{\textcircled{D}}_w b^{\textcircled{D}}_w=(1-awa^{\textcircled{D}}_w)b^{\textcircled{D}}_w$.

(3) $\Rightarrow$ (4) This is obvious as $a^{\textcircled{D}}_w b^{\textcircled{D}}_w(1-awa^{\textcircled{D}}_w)=(a^{\textcircled{D}}_w)^2(1-awa^{\textcircled{D}}_w)=0$.

(4) $\Rightarrow$ (1) Since $a\leq^{\textcircled{D}}_w b$, we have $b^{\textcircled{D}}_w a=a^{\textcircled{D}}_w a$. By hypothesis, $a^{\textcircled{D}}_w b^{\textcircled{D}}_w=a^{\textcircled{D}}_w b^{\textcircled{D}}_w awa^{\textcircled{D}}_w$, and then

$$\begin{array}{rcl}(b-a)^{\textcircled{D}}_w &=& b^{\textcircled{D}}_w-aw(a^{\textcircled{D}}_w b^{\textcircled{D}}_w)\\ &=& b^{\textcircled{D}}_w-aw(a^{\textcircled{D}}_w b^{\textcircled{D}}_w awa^{\textcircled{D}}_w)\\ &=& b^{\textcircled{D}}_w-awa^{\textcircled{D}}_w(b^{\textcircled{D}}_w a)wa^{\textcircled{D}}_w\\ &=& b^{\textcircled{D}}_w-awa^{\textcircled{D}}_w(a^{\textcircled{D}}_w a)wa^{\textcircled{D}}_w\\ &=& b^{\textcircled{D}}_w-[aw(a^{\textcircled{D}}_w)^2]awa^{\textcircled{D}}_w\\ &=& b^{\textcircled{D}}_w-a^{\textcircled{D}}_w awa^{\textcircled{D}}_w\\ &=& b^{\textcircled{D}}_w-a^{\textcircled{D}}_w,\end{array}$$

as asserted. □

5. WEIGHTED *-DMP ELEMENTS

The aim of this section is to characterize the weighted *-DMP element using a specific weighted core-EP decomposition. Subsequently, we will investigate the weighted core-EP order involving the weighted *-DMP element in a ring. An element a is w-weighted EP if and only if $a \in R^{\oplus}_w$ and $wawa^{\oplus}_w = wa^{\oplus}_w aw$. We now express the weighted core-EP element by combining the weighted EP element with a nilpotent element in a ring.

Theorem 5.1. *Let $a \in R$. Then the following are equivalent:*

(1) a is w-weighted *-DMP.
(2) There exist $x, y \in R$ such that

$$a = x + y, x^*y = ywx = 0, x \in R \text{ is } w\text{-weighted EP}, y \in R^{nil}_w.$$

Proof. (1) ⇒ (2) Since a is a w-weighted *-DMP element, $a \in R^{\textcircled{D}}_w$. By virtue of Theorem 2.1, There exist $x, y \in R$ such that

$$a = x + y, x^*y = ywx = 0, x \in R^{\oplus}_w, y \in R^{nil}_w.$$

Evidently, $a^{\textcircled{D}}_w = x^{\oplus}_w$. Since $wawa^{\textcircled{D}}_w = wa^{\textcircled{D}}_w aw$, we check that

$$\begin{aligned} wx^{\oplus}_w xw &= wx^{\oplus}_w xw + wx^{\oplus}_w [wx^{\oplus}_w]^*(x^*y)w \\ &= wx^{\oplus}_w xw + wx^{\oplus}_w [xwx^{\oplus}_w]yw = wx^{\oplus}_w xw + w[x^{\oplus}_w xwx^{\oplus}_w]yw \\ &= wx^{\oplus}_w xw + wx^{\oplus}_w yw = wx^{\oplus}_w (x+y)w = wa^{\textcircled{D}}_w aw \\ &= awa^{\textcircled{D}}_w = w(x+y)wx^{\oplus}_w = wxwx^{\oplus}_w. \end{aligned}$$

Therefore x is a w-EP element, as required.

(2) ⇒ (1) By virtue of Theorem 2.1, $a \in R^{\textcircled{D}}_w$. Since x is a w-EP element, we see that $wx^{\oplus}_w xw = wxwx^{\oplus}_w$. Hence, we have

$$\begin{aligned} wa^{\textcircled{D}}_w aw &= wx^{\oplus}_w xw + wx^{\oplus}_w yw \\ &= wx^{\oplus}_w xw + w[x^{\oplus}_w xwx^{\oplus}_w]yw \\ &= wx^{\oplus}_w xw + w[x^{\oplus}_w (wx^{\oplus}_w)^*](x^*y)w \\ &= wx^{\oplus}_w xw = w(x+y)wx^{\oplus}_w = wawa^{\oplus}_w. \end{aligned}$$

Therefore a is w-weighted *-DMP, as asserted. □

Corollary 5.2. *Let $a \in R$. Then the following are equivalent:*

(1) a is *-DMP.
(2) There exist $x, y \in R$ such that

$$a = x + y, x^*y = ywx = 0, x \in R \text{ is EP}, y \in R^{nil}_w.$$

Proof. This is obtained by choosing $w = 1$ in Theorem 5.1. □

Theorem 5.3. *Let $a \in R$.Then the following are equivalent:*

(1) $a \in R$ is w-weighted *-DMP.
(2) $(aw)^D \in R^{\oplus}$ and $(aw)^D[(aw)^D]^{\oplus}a$ is w-weighted EP.

Proof. (1) $\Rightarrow$ (2) Obviously, $a \in R^{\textcircled{D}}_w$; and so $aw \in R^{\textcircled{D}}$. In light of [4, Theorem 4.1], $(aw)^D \in R^{\textcircled{\#}}$. By virtue of Theorem 5.1, there exist $x, y \in R$ such that

$$a = x + y, x^*y = ywx = 0, x \in R \text{ is } w\text{-weighted EP}, y \in R^{nil}_w.$$

Evidently, $x = awa^{\textcircled{D}}_w a$. By using Theorem 2.4, we have $x = awa^{\textcircled{D}}_w a = aw[(aw)^D]^2[(aw)^D]^{\textcircled{\#}}a = (aw)^D[(aw)^D]^{\textcircled{\#}}a$. This implies that $(aw)^D[(aw)^D]^{\textcircled{\#}}a$ is w-weighted EP.

(2) $\Rightarrow$ (1) Let $x = (aw)^D[(aw)^D]^{\textcircled{\#}}a$ and $y = a - (aw)^D[(aw)^D]^{\textcircled{\#}}a$. Then $a = x + y$. Let n be the Deazin index of aw. Then $(aw)^D(aw)^{n+1} = (aw)^n$. Obviously, we have

$$\begin{array}{rl} & aw - [(aw)^D]^n[(aw)^D]^{\textcircled{\#}}(aw)^n \\ = & aw - [(aw)^D]^n[(aw)^D]^{\textcircled{\#}}(aw)^D(aw)^{n+1} \\ = & aw - [(aw)^D]^{n-1}\big((aw)^D[(aw)^D]^{\textcircled{\#}}(aw)^D\big)(aw)^{n+1} \\ = & aw - [(aw)^D]^{n-1}(aw)^D(aw)^{n+1} \\ = & aw - [(aw)^D]^{n-1}(aw)^n = aw - (aw)^D(aw)^2. \end{array}$$

Hence, $aw - [(aw)^D]^n[(aw)^D]^{\textcircled{\#}}(aw)(aw)^{n-1}$ is nilpotent. Since $yw = aw - (aw)^D[(aw)^D]^{\textcircled{\#}}aw = aw - (aw)^{n-1}[(aw)^D]^n[(aw)^D]^{\textcircled{\#}}aw$, we show that $yw \in R$ is nilpotent. One easily checks that

$$\begin{array}{rcl} x^*y & = & \big((aw)^D[(aw)^D]^{\textcircled{\#}}a\big)^*[a - (aw)^D[(aw)^D]^{\textcircled{\#}}a] \\ & = & a^*\big((aw)^D[(aw)^D]^{\textcircled{\#}}\big)^*[1 - (aw)^D[(aw)^D]^{\textcircled{\#}}]a \\ & = & a^*(aw)^D[(aw)^D]^{\textcircled{\#}}[1 - (aw)^D[(aw)^D]^{\textcircled{\#}}]a = 0, \\ ywx & = & [a - (aw)^D[(aw)^D]^{\textcircled{\#}}a]w(aw)^D[(aw)^D]^{\textcircled{\#}}a \\ & = & [1 - (aw)^D[(aw)^D]^{\textcircled{\#}}]aw(aw)^D[(aw)^D]^{\textcircled{\#}}a \\ & = & [1 - (aw)^D[(aw)^D]^{\textcircled{\#}}][(aw)^D]^{\textcircled{\#}}a \\ & = & 0. \end{array}$$

Therefore $a \in R$ is w-weighted *-DMP by Theorem 2.1. □

As an immediate consequence of Theorem 5.3, we derive

Corollary 5.4. *Let $a \in R$. Then the following are equivalent:*

(1) $a \in R$ is *-DMP.
(2) $a^D \in R^{\textcircled{\#}}$ and $a^D(a^D)^{\textcircled{\#}}a$ is EP.

Lemma 5.5. *Let $a \in R$ be w-weighted *-DMP. If $a \leq^{\textcircled{D}}_w b$, then $(awa^{\textcircled{D}}_w)bw = bw(awa^{\textcircled{D}}_w)$ and $a^{\textcircled{D}}_w b^{\textcircled{D}}_w = b^{\textcircled{D}}_w a^{\textcircled{D}}_w$.*

Proof. Since a is w-weighted *-DMP, we have $w(awa_w^{\textcircled{D}}) = w(a_w^{\textcircled{D}}aw)$. Assume that $a \leq_w^{\textcircled{D}} b$. Then $awa_w^{\textcircled{D}} = bwa_w^{\textcircled{D}}$ and $a_w^{\textcircled{D}}a = a_w^{\textcircled{D}}b$. We check that

$$\begin{aligned}(awa_w^{\textcircled{D}})bw &= aw(a_w^{\textcircled{D}}b)w = aw(a_w^{\textcircled{D}}a)w\\ &= (awa_w^{\textcircled{D}})aw = (bwa_w^{\textcircled{D}})aw\\ &= bw(a_w^{\textcircled{D}}aw) = bw(awa_w^{\textcircled{D}}).\end{aligned}$$

Since $(awa_w^{\textcircled{D}})^* = (awa_w^{\textcircled{D}})$, we deduce that $(awa_w^{\textcircled{D}})(bw)^* = (bw)^*(awa_w^{\textcircled{D}})$. In view of [3, Theorem 15.2.12], $(awa_w^{\textcircled{D}})(bw)^{\textcircled{D}} = (bw)^{\textcircled{D}}(awa_w^{\textcircled{D}})$. Thus we have

$$\begin{aligned}(awa_w^{\textcircled{D}})b_w^{\textcircled{D}} &= b_w^{\textcircled{D}}(awa_w^{\textcircled{D}}) = b_w^{\textcircled{D}}(bwa_w^{\textcircled{D}})\\ &= b_w^{\textcircled{D}}(bw)(awa_w^{\textcircled{D}})a_w^{\textcircled{D}} = b_w^{\textcircled{D}}(bw)^2(a_w^{\textcircled{D}})^2\\ &\ \vdots\\ &= b_w^{\textcircled{D}}(bw)^{k+1}(a_w^{\textcircled{D}})^{k+1} = (bw)^k(a_w^{\textcircled{D}})^{k+1} = a_w^{\textcircled{D}}.\end{aligned}$$

Therefore

$$\begin{aligned}a_w^{\textcircled{D}}b_w^{\textcircled{D}} &= [a_w^{\textcircled{D}}awa_w^{\textcircled{D}}]b_w^{\textcircled{D}} = a_w^{\textcircled{D}}[(awa_w^{\textcircled{D}})b_w^{\textcircled{D}}]\\ &= (a_w^{\textcircled{D}})^2 = [b_w^{\textcircled{D}}(awa_w^{\textcircled{D}})]a_w^{\textcircled{D}} = b_w^{\textcircled{D}}[aw(a_w^{\textcircled{D}})^2] = b_w^{\textcircled{D}}a_w^{\textcircled{D}},\end{aligned}$$

as desired. □

We are ready to prove:

Theorem 5.6. *Let $a \in R$ be w-weighted *-DMP. If $a \leq_w^{\textcircled{D}} b$, then $bw(1 - awa_w^{\textcircled{D}}) \in \mathcal{R}^{\textcircled{D}}$.*

Proof. Since $a \leq_w^{\textcircled{D}} b$, we have

$$awa_w^{\textcircled{D}} = bwa_w^{\textcircled{D}}, a_w^{\textcircled{D}}a = a_w^{\textcircled{D}}b.$$

Since a is w-weighted *-DMP, by virtue of Lemma 5.5, we have

$$(awa_w^{\textcircled{D}})bw = bw(awa_w^{\textcircled{D}}).$$

We claim that $[bw(1 - awa_w^{\textcircled{D}})]^{\textcircled{D}} = b_w^{\textcircled{D}} - a_w^{\textcircled{D}}$. Since b is w-weighted *-DMP, we have $wbwb_w^{\textcircled{D}} = wb_w^{\textcircled{D}}bw$. In light of Lemma 4.5 and Lemma 5.5, we have $a_w^{\textcircled{D}}b_w^{\textcircled{D}} = (a_w^{\textcircled{D}})^2$. Then

$$\begin{aligned}&[bw(1 - awa_w^{\textcircled{D}})][b_w^{\textcircled{D}} - a_w^{\textcircled{D}}]\\ =\ &[b(w - wawa_w^{\textcircled{D}})]b_w^{\textcircled{D}} - [b(w - wawa_w^{\textcircled{D}})]a_w^{\textcircled{D}}\\ =\ &bwb_w^{\textcircled{D}} - bwaw[a_w^{\textcircled{D}}b_w^{\textcircled{D}}] - bwa_w^{\textcircled{D}} + bw[aw(a_w^{\textcircled{D}})^2]\\ =\ &bwb_w^{\textcircled{D}} - bw[aw(a_w^{\textcircled{D}})^2] - bwa_w^{\textcircled{D}} + bw[aw(a_w^{\textcircled{D}})^2]\\ =\ &bwb_w^{\textcircled{D}} - awa_w^{\textcircled{D}};\end{aligned}$$

Since $(bwb_w^{\textcircled{D}})^* = bwb_w^{\textcircled{D}}$ and $(awa_w^{\textcircled{D}})^* = awa_w^{\textcircled{D}}$, we have $\big([bw(1-awa_w^{\textcircled{D}})][b_w^{\textcircled{D}} - a_w^{\textcircled{D}}]\big)^* = [bw(1-awa_w^{\textcircled{D}})][b_w^{\textcircled{D}} - a_w^{\textcircled{D}}]$. Moreover, we check that

$$\begin{array}{rl}
& [bw(1-awa_w^{\textcircled{D}})][b_w^{\textcircled{D}} - a_w^{\textcircled{D}}]^2 \\
= & [bwb_w^{\textcircled{D}} - awa_w^{\textcircled{D}}][b_w^{\textcircled{D}} - a_w^{\textcircled{D}}] \\
= & bw(b_w^{\textcircled{D}})^2 - bwb_w^{\textcircled{D}}a_w^{\textcircled{D}} - awa_w^{\textcircled{D}}b_w^{\textcircled{D}} + aw(a_w^{\textcircled{D}})^2 \\
= & b_w^{\textcircled{D}} - [bwa_w^{\textcircled{D}}]b_w^{\textcircled{D}} = b_w^{\textcircled{D}} - [awa_w^{\textcircled{D}}]b_w^{\textcircled{D}} \\
= & b_w^{\textcircled{D}} - aw(a_w^{\textcircled{D}})^2 = b_w^{\textcircled{D}} - a_w^{\textcircled{D}}; \\
& [b_w^{\textcircled{D}} - a_w^{\textcircled{D}}][bw(1-awa_w^{\textcircled{D}})]^{k+1} \\
= & [b_w^{\textcircled{D}} - a_w^{\textcircled{D}}](bw)^{k+1}(1-awa_w^{\textcircled{D}}) = b_w^{\textcircled{D}}((bw)^{k+1}(1-awa_w^{\textcircled{D}}) \\
= & (bw)^k(1-awa_w^{\textcircled{D}}) = [bw(1-awa_w^{\textcircled{D}})]^k.
\end{array}$$

This completes the proof. □

Conflict of interest

The authors declare there is no conflicts of interest

Data Availability Statement

The data used to support the findings of this study are included within the article.

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