Born's Rule and the L_2 Norm from Pythagorean Geometry: A Deterministic Foundation via Global Phase Dynamics

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Abstract

This paper develops a deterministic foundation for quantum measurement based on the Hidden Deterministic Interpretation (HDI), which avoids probabilistic postulates and wavefunction collapse at the level of individual particles. HDI introduces two principles: the Phase Consistency Criterion, which fixes the global phase of the Heisenberg-picture wavefunction via initial conditions, and the Quantum Hamilton's Principle, which selects the actual trajectory by extremizing accumulated quantum phase. Applied to an ensemble of particles, this framework yields detection statistics that reproduce the Born rule without assuming it. We show that the squared amplitudes associated with measurement outcomes follow directly from orthogonal projections governed by the Pythagorean theorem. As a consequence, the L_2 norm of Hilbert space emerges naturally—not as an imposed structure, but as a mathematically inevitable result of the deterministic phase dynamics. Thus, both Born's rule and the underlying geometry of quantum state space are derived from the first principles, offering a coherent, deterministic, and geometrically grounded alternative to the standard axioms of quantum mechanics.

1 Introduction

Conventional quantum theory, particularly as encapsulated in the Copenhagen interpretation [1–3], has demonstrated remarkable empirical success. At its core, the standard formulation rests on two central assumptions:

- **Postulate 1:** When a quantum system is described as a linear combination of eigenstates corresponding to a particular observable, a measurement causes the system to assume one specific eigenstate from the superposition.
- **Postulate 2:** The likelihood of obtaining a particular outcome during this process is determined by Born's rule, which assigns probabilities based on the squared modulus of the associated amplitude.

The Hidden Deterministic Interpretation (HDI) offers an alternative framework that provides an underlying physical rationale for both of these postulates [4]. Rather than invoking an explicit collapse mechanism—wherein the entire wavefunction discontinuously reduces to a single term—HDI explains wavefunction reduction as an emergent phenomenon rooted in a deterministic ontology. This interpretation is constructed upon two fundamental principles:

1. **Phase Consistency Criterion (PCC):** At the moment of particle emission, the global phase of the Heisenberg state $|\psi\rangle$ is matched with the phase of the position eigenstate $|\mathbf{r}_{i;p}, t_i\rangle$ through the relation:

$$\mathrm{ph}(|\psi\rangle) = \mathrm{ph}(\langle \mathbf{r}_{i;p}, t_i|\psi\rangle), \tag{1}$$

   where $\mathbf{r}_{i;p}$ and t_i denote the particle's initial position and time; the subscript p denotes "particle". As shown in Ref. [4], Eq. (1) resolves the mystery of quantum superposition [5, 6].

2. **Quantum Hamilton's Principle (QHP):** The actual trajectory of the particle is such that the total quantum phase accumulated along the path remains stationary:

$$\delta \left\{ \int_{t_i}^{t_f} dt \; L_c(\mathbf{r}(t), \dot{\mathbf{r}}(t)) \right\} = 0, \tag{2}$$

   where L_c is the classical Lagrangian, and $\mathbf{r}(t)$ and $\dot{\mathbf{r}}(t)$ are the position and velocity eigenvalues at time t; t_f is the final time.

Similar to Bohmian mechanics [7,8], HDI promotes a fully deterministic account of quantum evolution. However, unlike Bohmian theory, it avoids introducing guiding equations or nonlocal hidden variables. Within HDI, the particle evolves deterministically along a trajectory embedded in the Heisenberg-picture wavefunction, its motion governed jointly by the PCC (Eq. 1) and QHP (Eq. 2). This deterministic path is conceptually parallel to a test particle following a geodesic in general relativity's curved spacetime [9].

Upon detection, the prior predictive role of the wavefunction is rendered physically irrelevant, thus rendering it obsolete for subsequent evolution. In this sense, HDI accommodates a notion of wavefunction collapse, not as an imposed or discontinuous process, but as a natural endpoint of the particle's completed trajectory. This framework retains unitary evolution and determinism while effectively reproducing the statistical appearance of measurement collapse. Furthermore, HDI derives Born's rule from first principles, interpreting the observed detection probabilities as limiting frequencies arising from deterministic dynamics—fully consistent with empirical quantum observations.

This geometric derivation leads to a deeper insight. Unlike standard quantum mechanics, which postulates probabilistic outcomes at the level of individual measurements, the Hidden Deterministic Interpretation (HDI) is fundamentally non-probabilistic for single-particle events. Each particle follows a definite trajectory determined by the global phase dynamics, encoded in the Phase Consistency Criterion and Quantum Hamilton's Principle. Statistical behavior—and thus, quantum probabilities—emerge only at the ensemble level, as a macroscopic manifestation of underlying deterministic dynamics.

Within this framework, the L_2 norm arises not as an imposed structure but as an unavoidable consequence of the geometric projection of quantum states in Hilbert space. Specifically, the squared amplitudes corresponding to detection outcomes obey the Pythagorean theorem, yielding a direct derivation of Born's rule. This derivation inherently requires and uniquely selects the L_2 norm, as it is the only norm consistent with orthogonal decompositions and ensemble frequencies. Thus, HDI not only eliminates the need to postulate probabilities and collapse but also provides a first-principles justification for the Hilbert space structure of quantum mechanics—deriving rather than assuming both the Born rule and the L_2 norm.

2 Deriving Born's Rule and the L_2 Norm from Global Phase Geometry

This section presents the central result of this paper: the derivation of Born's rule and the emergence of the L_2 norm from deterministic phase dynamics. Within the HDI framework, the global phase of each particle's Heisenberg-picture state vector governs its individual trajectory, while ensemble-level projections yield statistical detection patterns. By analyzing the geometric relation between the initial and detected states, we demonstrate that the probability rule of quantum mechanics arises naturally from a Pythagorean structure in Hilbert space, requiring no probabilistic postulates or collapse axioms. This provides a first-principles physical explanation for both foundational principles of orthodox quantum mechanics.

As shown in figure 1, the detector D detects the particle at $\mathbf{r}_{f;p}$:

$$|\psi\rangle = \int_{\mathbb{R}^3} d^3\mathbf{r} |\mathbf{r}; t_f\rangle\langle\mathbf{r}; t_f|\psi\rangle \xrightarrow[\text{at D}]{\text{detection}} (\Delta V)|\mathbf{r}_{f;p}, t_f\rangle\langle\mathbf{r}_{f;p}, t_f|\psi\rangle, \tag{3}$$

where ΔV denotes an infinitesimal volume around $\mathbf{r}_{f;p}$. Equation (3) arises because, apart from the position eigenstate $|\mathbf{r}_{f;p}, t_f\rangle\langle\mathbf{r}_{f;p}, t_f|\psi\rangle$, all others remain unoccupied [4]. This provides the physical basis for the phenomenon commonly referred to as "wavefunction collapse" or "state vector reduction" [1–3], thereby resolving **Principle 1**. In the limit $\Delta V \to 0$, (3) explains why quantum experiments always yield measurement outcomes as discrete, point-like events. Various earlier interpretations have predicted the same using different mechanisms [7, 8, 10, 11].

Consider N particles represented by identical state vectors differing only in global phase, emerging from the source S. These can be described by the ensemble:

$$S_N = \{|\psi\rangle \mid \mathrm{ph}(|\psi\rangle) \in [0, 2\pi]\}. \tag{4}$$

The tips of all vectors in S_N lie on a circle of radius $R_N = |\sqrt{\langle\psi|\psi\rangle}|$, as shown in figure 1. The areal number density σ_N is:

$$\sigma_N = \frac{N}{\pi\langle\psi|\psi\rangle}. \tag{5}$$

According to (3), the denominator reduces upon detection:

$$\pi\langle\psi|\psi\rangle \xrightarrow[\text{at D}]{\text{detection}} \pi(\Delta V)|\langle\mathbf{r}_{f;p}, t_f|\psi\rangle|^2. \tag{6}$$

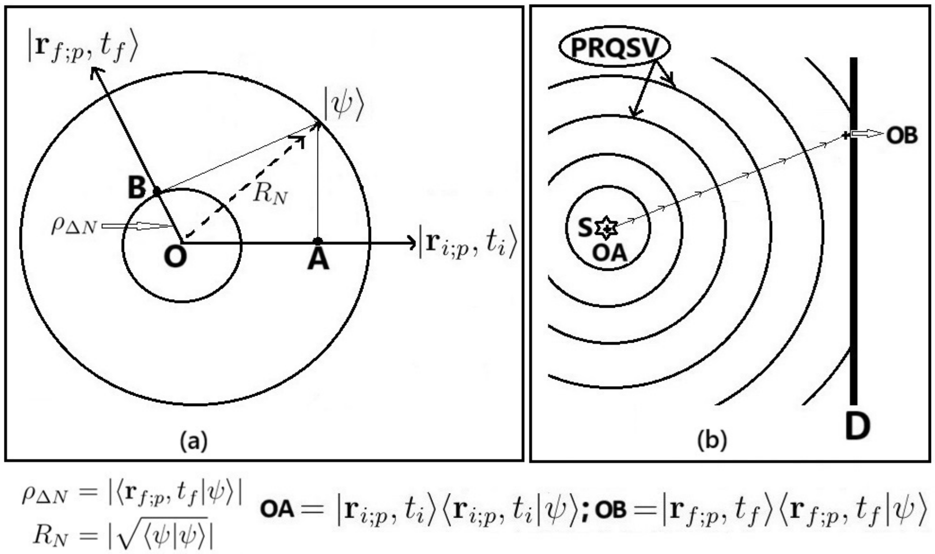

Figure 1: **Schematic Diagram of the Phase Consistency Criterion**: At the initial time t_i, the phase consistency criterion is given by $\text{ph}(|\psi\rangle) = \text{ph}(\langle \mathbf{r}_{i;p}, t_i|\psi\rangle)$. (a) Particle in the Hilbert space: OA and OB represent the position eigenstates, where the particle is initially at $\mathbf{r}_{i;p}$ at time t_i and finally at $\mathbf{r}_{f;p}$ at time t_f, respectively. $|\psi\rangle$ is the state vector in the Heisenberg picture. $\rho_{\Delta N}$ and R_N are the radii of the inner and outer circles, respectively. (b) Particle in the position representation of the quantum state vector (PRQSV): S is the single-particle source, and D is the detector screen. The position eigenvalues of the particle state trace a specific path (a straight-line path is shown for a free particle) within the PRQSV.

Let ΔN be the number of particles detected at $\mathbf{r}_{f;p}$, described by the subset:

$$S_{\Delta N} = \{|\mathbf{r}_{f;p}, t_f\rangle\langle \mathbf{r}_{f;p}, t_f|\psi\rangle \mid \text{ph}(\langle \mathbf{r}_{f;p}, t_f|\psi\rangle) \in [0, 2\pi], t_f \in T_a\}, \tag{7}$$

where T_a is the set of arrival times. These detected states correspond to a circle of radius $|\langle \mathbf{r}_{f;p}, t_f|\psi\rangle|\sqrt{\Delta V}$. Their areal number density is:

$$\sigma_{\Delta N} = \frac{1}{\pi|\langle \mathbf{r}_{f;p}, t_f|\psi\rangle|^2}\frac{\Delta N}{\Delta V}, \tag{8}$$

with constant inner radius $\rho_{\Delta N} \equiv |\langle \mathbf{r}_{f;p}, t_f|\psi\rangle|$.

Equating $\sigma_N = \sigma_{\Delta N}$ yields a key geometric identity, a manifestation of the Pythagoras theorem (which, in infinite-dimensional Hilbert space, corresponds to Parseval's identity):

$$\frac{N}{\pi\langle\psi|\psi\rangle} = \frac{1}{\pi|\langle \mathbf{r}_{f;p}, t_f|\psi\rangle|^2}\frac{\Delta N}{\Delta V}. \tag{9}$$

This equation shows that quantum probabilities arise from projection geometry, validating ensemble statistics regardless of whether particles are measured collectively or individually. The equality reflects a conservation law reminiscent of incompressible fluid flow. Rearranging gives:

$$\left(\frac{1}{N}\frac{\Delta N}{\Delta V}\right)_{\text{Experiment}} = \left(\frac{|\langle \mathbf{r}_{f;p}, t_f|\psi\rangle|^2}{\langle\psi|\psi\rangle}\right)_{\text{Theory}} = \left(\frac{\pi\rho_{\Delta N}^2}{\pi R_N^2}\right)_{\text{Figure 1}}. \tag{10}$$

The left-hand side is the experimentally observed relative frequency density at $\mathbf{r}_{f;p}$, which in the continuum limit matches the Born probability density — thereby resolving **Principle 2**. The rightmost expression relates this probability to geometric projection areas in Hilbert space, anchored by the Pythagorean identity.

By contrast, the wavefunction in (3) represents a single particle localized in a position eigenstate, making "wavefunction collapse" a natural and observer-independent consequence of trajectory completion. Through the PCC and QHP, HDI renders the resolution of the measurement problem [12–16].

In conclusion, this derivation establishes that Born's rule and the L_2 norm both follow from deterministic global phase dynamics and geometric projection. No probabilistic or statistical axioms are invoked. This positions HDI as a rigorous and physically motivated foundation for the formal structure of quantum mechanics.

3 Conclusions

In this work, we have developed a deterministic foundation for quantum mechanics based on global phase dynamics, encapsulated by the Hidden Deterministic Interpretation (HDI). By introducing the Phase Consistency Criterion (PCC) and Quantum Hamilton's Principle (QHP), we formulated a framework in which each quantum particle follows a definite, physically meaningful trajectory without requiring stochasticity or discontinuous collapse. Measurement outcomes emerge not through postulates, but as statistical patterns across ensembles of such deterministic events.

We have shown that Born's rule, which governs the probability of quantum outcomes, naturally arises from the geometric structure of Hilbert space. Specifically, the squared amplitudes associated with measurement outcomes emerge from orthogonal projections governed by the Pythagorean theorem. This leads to a deeper insight: the use of the L_2 norm, central to the Hilbert space formalism of quantum mechanics, is not an assumed mathematical convenience but a necessary consequence of the underlying phase-based dynamics. The L_2 structure arises unavoidably from the geometry of ensemble-level projections, reinforcing the internal consistency and completeness of the HDI framework.

Thus, HDI offers more than an interpretational alternative—it provides a coherent physical derivation of both Born's rule and the structure of quantum state space from deterministic first principles. This positions HDI as a viable foundational framework capable of resolving the measurement problem without invoking additional postulates and opens a path toward a deeper understanding of quantum theory grounded in deterministic geometry.

Future work will explore the implications of this framework for multi-particle entanglement, quantum field theory, and quantum gravity.

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