Metric Spaces

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To my beloved Evaggelia Leleki for all the good memories.

Definition 1.
A metric space (X,ρ) consists of a set X and a function $\rho : X \times X \to \mathbb{R}$:
1.$\forall$ x,y $\in$ X : ρ (x,y) $\geq$ 0.
2.$\forall$ x,y $\in$ X : ρ (x,y) = ρ (y,x).
3.$\forall$ x,y $\in$ X : (ρ (x,y) = 0 $\iff$ x = y).
4.$\forall$ x,y,z $\in$ X: (ρ (x,y) $\leq$ ρ (x,z) + ρ (z,y)).

Proposition 1.
$\alpha_1, ..., \alpha_n, \beta_1, ...\beta_n \in \mathbb{R}$
$\implies (\sum_{i=1}^{n} \alpha_i^2)(\sum_{i=1}^{n} \beta_i^2) - (\sum_{i=1}^{n} \alpha_i\beta_i)^2 = \sum_{i=1}^{n-1}(\sum_{j=i+1}^{n} \det^2 \begin{pmatrix} \alpha_i & \beta_j \\ \beta_i & \beta_j \end{pmatrix})$

Proof.
Proposition 1 is true for n = 1.

Assume Proposition 1 holds for n > 1.

Let $\alpha = \sum_{i=1}^{n} \alpha_i^2, \beta = \sum_{i=1}^{n} \beta_i^2, \gamma = \sum_{i=1}^{n} \alpha_i\beta_i$

$(\sum_{i=1}^{n+1} \alpha_i^2)(\sum_{i=1}^{n+1} \beta_i^2) - (\sum_{i=1}^{n+1} \alpha_i\beta_i)^2 =$

$(\alpha + \alpha_{n+1}^2)(\beta + \beta_{n+1}^2) - (\gamma + \alpha_{n+1}\beta_{n+1})^2 =$

$\alpha\beta + \alpha\beta_{n+1}^2 + \alpha_{n+1}^2\beta + \alpha_{n+1}^2\beta_{n+1}^2 - \gamma^2 - 2\gamma\alpha_{n+1}\beta_{n+1} - \alpha_{n+1}^2\beta_{n+1}^2 =$

$\sum_{i=1}^{n-1}(\sum_{j=i+1}^{n} \det^2 \begin{pmatrix} \alpha_i & \alpha_j \\ \beta_i & \beta_j \end{pmatrix}) + \alpha\beta_{n+1}^2 + \alpha_{n+1}^2\beta - 2\gamma\alpha_{n+1}\beta_{n+1} =$
$\sum_{i=1}^{n-1}(\sum_{j=i+1}^{n} \det^2 \begin{pmatrix} \alpha_i & \alpha_j \\ \beta_i & \beta_j \end{pmatrix}) + (\sum_{i=1}^{n} \alpha_i^2)\beta_{n+1}^2 + \alpha_{n+1}(\sum_{i=1}^{n} \beta_i^2)$
$- 2\alpha_{n+1}\beta_{n+1}(\sum_{i=1}^{n} \alpha_i\beta_i) =$

$\sum_{i=1}^{n-1}(\sum_{j=i+1}^{n} \det^2 \begin{pmatrix} \alpha_i & \alpha_j \\ \beta_i & \beta_j \end{pmatrix}) + (\sum_{i=1}^{n-1} \det^2 \begin{pmatrix} \alpha_i & \alpha_{n+1} \\ \beta_i & \beta_{n+1} \end{pmatrix})$
$+ (\det^2 \begin{pmatrix} \alpha_n & \alpha_{n+1} \\ \beta_n & \beta_{n+1} \end{pmatrix}) =$
$\sum_{i=1}^{n-1}(\sum_{j=i+1}^{n+1} \det^2 \begin{pmatrix} \alpha_i & \alpha_j \\ \beta_i & \beta_j \end{pmatrix}) + (\det^2 \begin{pmatrix} \alpha_n & \alpha_{n+1} \\ \beta_n & \beta_{n+1} \end{pmatrix}) =$
$\sum_{i=1}^{n}(\sum_{j=i+1}^{n+1} \det^2 \begin{pmatrix} \alpha_i & \alpha_j \\ \beta_i & \beta_j \end{pmatrix})$ □

Theorem 1. $(Cauchy - Schwarz)$
$\alpha_1, ..., \alpha_n, \beta_1, ...\beta_n \in \mathbb{R}$

$\implies (\sum_{i=1}^{n} \alpha_i^2)(\sum_{i=1}^{n} \beta_i^2) \geq (\sum_{i=1}^{n} \alpha_i\beta_i)^2$

Proof.
$(\sum_{i=1}^{n} \alpha_i^2)(\sum_{i=1}^{n} \beta_i^2) - (\sum_{i=1}^{n} \alpha_i\beta_i)^2 = \sum_{i=1}^{n-1}(\sum_{j=i+1}^{n} \det^2 \begin{pmatrix} \alpha_i & \beta_j \\ \beta_i & \beta_j \end{pmatrix}) \geq 0.$

Thus $(\sum_{i=1}^{n} \alpha_i^2)(\sum_{i=1}^{n} \beta_i^2) \geq (\sum_{i=1}^{n} \alpha_i\beta_i)^2$ □

Proposition 2.
$(\alpha, \beta \geq 0) \wedge (p, q \in \mathbb{R}) \wedge (p > 1) \wedge (\frac{1}{p} + \frac{1}{q} = 1) \implies \alpha\beta \leq \frac{\alpha^p}{p} + \frac{\beta^q}{q}$

Proof.
Since log is (strictly) concave function:

$(\lambda \in (0, 1)) \wedge (x, y \in (0, +\infty)) \implies$

$\log(\lambda x + (1 - \lambda)y) \geq \lambda \log(x) + (1 - \lambda) \log(y)$

Let $\lambda = \frac{1}{p}, x = \alpha^p, y = \beta^q$,then

$\log(\frac{\alpha^p}{p} + \frac{\beta^q}{q}) \geq \frac{1}{p} \log(\alpha^p) + \frac{1}{q} \log(\beta^q)$,therefore $\log(\frac{\alpha^p}{p} + \frac{\beta^q}{q}) \geq \log(\alpha\beta)$

Since log is strictly increasing : $\alpha\beta \leq \frac{\alpha^p}{p} + \frac{\beta^q}{q}$ □

Theorem 2. *(Holder)*
$(p, q > 1) \wedge (\frac{1}{p} + \frac{1}{q} = 1) \wedge (\alpha_1, ...\alpha_n \beta_1, ...\beta_n \geq 0) \implies$

$\sum_{i=1}^n \alpha_i \beta_i \leq (\sum_{i=1}^n \alpha_i^p)^{\frac{1}{p}} (\sum_{i=1}^n \beta_i^q)^{\frac{1}{q}}$

Proof.

$(\sum_{i=1}^n \alpha_i^p = 0) \vee (\sum_{i=1}^n \beta_i^q = 0) \implies$

$(i \in \{1, ..., n\} \implies \alpha_i = 0) \vee (i \in \{1, ..., n\} \implies \beta_i = 0) \implies$

$\sum_{i=1}^n \alpha_i \beta_i = 0.$

Now lets assume that $(\sum_{i=1}^n \alpha_i^p \neq 0) \wedge (\sum_{i=1}^n \beta_i^q \neq 0)$.

Let $\gamma_i = \frac{\alpha_i}{(\sum_{i=1}^n \alpha_i^p)^{\frac{1}{p}}}, \delta_i = \frac{\beta_i}{(\sum_{i=1}^n \beta_i^q)^{\frac{1}{q}}}$
$\sum_{i=1}^n \gamma_i \delta_i \leq \sum_{i=1}^n (\frac{\gamma_i^p}{p} + \frac{\delta_i^q}{q}) = 1.$

Therefore $\sum_{i=1}^n \alpha_i \beta_i \leq (\sum_{i=1}^n \alpha_i^p)^{\frac{1}{p}} (\sum_{i=1}^n \beta_i^q)^{\frac{1}{q}}$ □

Theorem 3. *(Minkowski)*
$(p \geq 1) \wedge (\alpha_1, ..., \alpha_n, \beta_1, ...\beta_n \geq 0) \implies$

$(\sum_{i=1}^n (\alpha_i + \beta_i)^p)^{\frac{1}{p}} \leq (\sum_{i=1}^n \alpha_i^p)^{\frac{1}{p}} + (\sum_{i=1}^n \beta_i^p)^{\frac{1}{p}}$

Proof.

The theorem is true for p=1.Lets Assume that p > 1.

Let $q = \frac{p}{p-1}$, *then* $q > 1$ *and* $\frac{1}{p} + \frac{1}{q} = 1$.

Let $A = (\sum_{i=1}^{n}(\alpha_i+\beta_i)^p)^{\frac{1}{p}}$.

$A^p = \sum_{i=1}^{n}(\alpha_i+\beta_i)^p = \sum_{i=1}^{n}(\alpha_i+\beta_i)(\alpha_i+\beta_i)^{p-1} =$

$\sum_{i=1}^{n}\alpha_i(\alpha_i+\beta_i)^{p-1} + \sum_{i=1}^{n}\beta_i(\alpha_i+\beta_i)^{p-1}$

$(Holder)\ \leq (\sum_{i=1}^{n}\alpha_i^p)^{\frac{1}{p}}(\sum_{i=1}^{n}(\alpha_i+\beta_i)^{(p-1)q})^{\frac{1}{q}} +$

$(\sum_{i=1}^{n}\beta_i^p)^{\frac{1}{p}}(\sum_{i=1}^{n}(\alpha_i+\beta_i)^{(p-1)q})^{\frac{1}{q}} =$

$(\sum_{i=1}^{n}\alpha_i^p)^{\frac{1}{p}}(\sum_{i=1}^{n}(\alpha_i+\beta_i)^p)^{\frac{1}{q}} +$

$(\sum_{i=1}^{n}\beta_i^p)^{\frac{1}{p}}(\sum_{i=1}^{n}(\alpha_i+\beta_i)^p)^{\frac{1}{q}} = A^{\frac{p}{q}}((\sum_{i=1}^{n}\alpha_i^p)^{\frac{1}{p}} + (\sum_{i=1}^{n}\beta_i^p)^{\frac{1}{p}})$.

$Therefore$

$A \leq (\sum_{i=1}^{n}\alpha_i^p)^{\frac{1}{p}} + (\sum_{i=1}^{n}\beta_i^p)^{\frac{1}{p}}$ □

Example 1.
$\rho_1 : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$, ρ_1 *(x,y)* $= \sum_{i=1}^{n}|x_i - y_i|$.

Proof.

$x, y, z \in \mathbb{R}^n$

1.

$\forall i \in \{1, ..., n\} : |x_i - y_i| \geq 0,$

$Therefore\ \rho_1(x,y) \geq 0.$

2.

$\forall i \in \{1, ..., n\} : |x_i - y_i| = |y_i - x_i|,$

$Therefore\ \rho_1(x,y) = \rho_1(y,x).$

3.

$\rho_1(x,y) = 0 \implies (\forall i \in \{1, ..., n\} : |x_i - y_i| = 0) \implies x = y$

$x = y \implies (\forall i \in \{1, ..., n\} : |x_i - y_i| = 0) \implies \rho_1(x,y) = 0$

4.

$\forall i \in \{1, ..., n\} : |x_i - y_i| \leq |x_i - z_i| + |z_i - y_i|.$

$Therefore,$

$\sum_{i=1}^{n}|x_i - y_i| \leq \sum_{i=1}^{n}|x_i - z_i| + \sum_{i=1}^{n}|z_i - y_i|$

$Therefore,$

$\rho_1(x,y) \leq \rho_1(x,z) + \rho_1(z,y).$ $\square$

Example 2.
$\rho_2 : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$, ρ_2 $(x,y) = (\sum_{i=1}^n (x_i - y_i)^2)^{\frac{1}{2}}$.

Proof.

$x, y, z \in \mathbb{R}^n$

1.

$(i \in \{1, ..., n\} \implies (x_i - y_i)^2 \geq 0) \implies (\sum_{i=1}^n (x_i - y_i)^2)^{\frac{1}{2}} \geq 0$

$\implies \rho_2(x,y) \geq 0.$

2.

$(i \in \{1, ..., n\} \implies (x_i - y_i)^2 = (y_i - x_i)^2) \implies$

$(\sum_{i=1}^n (x_i - y_i)^2)^{\frac{1}{2}} = (\sum_{i=1}^n (y_i - x_i)^2)^{\frac{1}{2}} \implies \rho_2(x,y) = \rho_2(y,x).$

3.

$\rho_2(x,y) = 0 \iff (i \in \{1, ..., n\} \implies (x_i - y_i)^2 = 0)$

$\iff (i \in \{1, ..., n\} \implies x_i = y_i) \iff x = y.$

4.

$\rho_2^2(x,y) = \sum_{i=1}^n (x_i - y_i)^2 = \sum_{i=1}^n ((x_i - z_i) + (z_i - y_i))^2$

$= \sum_{i=1}^n (x_i - z_i)^2 + 2\sum_{i=1}^n (x_i - z_i)(z_i - y_i) + \sum_{i=1}^n (z_i - y_i)^2 \leq$

$\sum_{i=1}^n (x_i - z_i)^2 + 2\sum_{i=1}^n |x_i - z_i||z_i - y_i| + \sum_{i=1}^n (z_i - y_i)^2 \leq$

$\sum_{i=1}^n (x_i - z_i)^2 + 2(\sum_{i=1}^n (x_i - z_i)^2)^{\frac{1}{2}}(\sum_{i=1}^n (z_i - y_i)^2)^{\frac{1}{2}} + \sum_{i=1}^n (z_i - y_i)^2$

$= ((\sum_{i=1}^n (x_i - z_i)^2)^{\frac{1}{2}} + (\sum_{i=1}^n (z_i - y_i)^2)^{\frac{1}{2}})^2$

$= (\rho_2(x,z) + \rho_2(z,y))^2.$

$Therefore,$

$\rho_2(x,y) = |\rho_2(x,y)| \leq |\rho_2(x,z) + \rho_2(z,y)| = \rho_2(x,z) + \rho_2(z,y).$ $\square$

Example 3.
$\rho_\infty : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$, ρ_∞ $(x,y) = max\ \{|x_i - y_i| : i = 1, ..., n\}$

Proof.

1.

$(i \in \{1, ...n\} \implies |x_i - y_i| \geq 0)$

$\implies max\{|x_i - y_i| : i \in \{1, ...n\}\} \geq 0$

$\implies \rho_\infty(x, y) \geq 0.$

2.

$(i \in \{1, ...n\} \implies |x_i - y_i| = |y_i - x_i|)$

$\implies max\{|x_i - y_i| : i \in \{1, ...n\}\} = max\{|y_i - x_i| : i \in \{1, ...n\}\}$

$\implies \rho_\infty(x, y) = \rho_\infty(y, x)$

3.

$\rho_\infty = 0 \iff (i \in \{1, ...n\} \implies |x_i - y_i| = 0) \iff x = y$

4.

$\exists j \in \{1, ..., n\} : \rho_\infty(x, y) = |x_j - y_j|$

$|x_j - y_j| \leq |x_j - z_j| + |z_j - y_j| \leq$

$max\{|x_i - z_i| : i \in \{1, ..., n\}\} + max\{|z_i - y_i| : i \in \{1, ..., n\}\} =$

$\rho_\infty(x, z) + \rho_\infty(z, y)$

□

Example 4.
$\rho_p : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$, ρ_p *(x,y)* $= (\sum_{i=1}^{n} (x_i - y_i)^p)^{\frac{1}{p}}, p \in [1, \infty)$

Proof.

1.

$(i \in \{1, ..., n\} \implies |x_i - y_i|^p \geq 0)$

$\implies (\sum_{i=1}^{n} |x_i - y_i|^p)^{\frac{1}{p}} \geq 0 \implies \rho_p(x, y) \geq 0.$

2.

$(i \in \{1, ..., n\} \implies |x_i - y_i|^p = |y_i - x_i|^p)$

$\implies (\sum_{i=1}^{n} |x_i - y_i|^p)^{\frac{1}{p}}) = (\sum_{i=1}^{n} |y_i - x_i|^p)^{\frac{1}{p}}$

$\implies \rho_p(x,y) = \rho_p(y,x)$

3.

$\rho_p(x,y) = 0 \iff (i \in \{1,...,n\} \implies |x_i - y_i| = 0) \iff x = y$

4.

$\rho_p(x,y) = (\sum_{i=1}^{n} |x_i - y_i|^p)^{\frac{1}{p}}) = (\sum_{i=1}^{n} |(x_i - z_i) + (z_i - y_i)|^p)^{\frac{1}{p}}$

$\leq (\sum_{i=1}^{n} |x_i - z_i|^p)^{\frac{1}{p}}) + (\sum_{i=1}^{n} |z_i - y_i|^p)^{\frac{1}{p}}) = \rho_p(x,z) + \rho_p(z,y)$ □

Proposition 3.

$Assume\ (X_1, \sigma_1), ..., (X_n, \sigma_n)\ are\ metric\ spaces, and\ X = \prod_{i=1}^{n} X_i.$

$We\ define:$

$1.\rho_1 : X \times X \to \mathbb{R}, \rho_1(x,y) = \sum_{i=1}^{n} \sigma_i(x_i, y_i)$

$2.\rho_2 : X \times X \to \mathbb{R}, \rho_2(x,y) = (\sum_{i=1}^{n} \sigma_i^2(x_i, y_i))^{\frac{1}{2}}$

$3.\rho_\infty : X \times X \to \mathbb{R}, \rho_\infty(x,y) = max\{\sigma_i(x_i, y_i) : i \in \{1, ..., n\}\}$

$4.if\ p \in [1, +\infty)\ we\ define\ \rho_p : X \times X \to \mathbb{R}, \rho_p(x,y) = (\sum_{i=1}^{n} \sigma_i^p(x_i, y_i))^{\frac{1}{p}}$

$(X, \rho_1), (X, \rho_2), (X, \rho_\infty), (X, \rho_p)\ are\ metric\ spaces.$

Example 5.

$p \in [1, +\infty),$

$l^p = \{(x_n) \in \mathbb{R}^{\mathbb{N}} : \sum_{n=1}^{\infty} |x_n|^p < +\infty\}$

$\rho_p(x,y) = (\sum_{n=1}^{\infty} |x_n - y_n|^p)^{\frac{1}{p}}$

$Then\ (l^p, \rho_p)\ is\ a\ metric\ space.$

Proof.

$x, y \in \mathbb{R}^{\mathbb{N}} \wedge n \in \mathbb{N} \implies (\sum_{i=1}^{n} |x_i - y_i|^p)^{\frac{1}{p}} \leq$

$(\sum_{i=1}^{n} |x_i|^p)^{\frac{1}{p}} + (\sum_{i=1}^{n} |y_i|^p)^{\frac{1}{p}} \leq (\sum_{n=1}^{\infty} |x_n|^p)^{\frac{1}{p}} + (\sum_{n=1}^{\infty} |y_n|^p)^{\frac{1}{p}} < +\infty$

$Therefore,$

$\rho_p(x,y) \in [0, +\infty).$

$Let\ x, y, z \in \mathbb{R}^{\mathbb{N}}.$

1.

$(i \in \mathbb{N}) \implies |x_i - y_i|^p \geq 0 \implies$

$(n \in \mathbb{N} \implies \sum_{i=1}^{n} |x_i - y_i|^p \geq 0) \implies$

$\sum_{n=1}^{\infty} |x_i - y_i|^p \geq 0 \implies$

$(\sum_{n=1}^{\infty} |x_i - y_i|^p)^{\frac{1}{p}} \geq 0 \implies$

$\rho_p(x, y) \geq 0.$

2.

$(i \in \mathbb{N}) \implies |x_i - y_i|^p = |y_i - x_i|^p \implies$

$\sum_{i=1}^{n} |x_i - y_i|^p = \sum_{i=1}^{n} |y_i - x_i|^p \implies$

$\sum_{n=1}^{\infty} |x_n - y_n|^p = \sum_{n=1}^{\infty} |y_n - x_n|^p \implies$

$(\sum_{n=1}^{\infty} |x_n - y_n|^p)^{\frac{1}{p}} = (\sum_{n=1}^{\infty} |y_n - x_n|^p)^{\frac{1}{p}} \implies \rho_p(x, y) = \rho_p(y, x).$

3.

$\rho_p(x, y) = 0 \iff \forall i \in \mathbb{N} : |x_i - y_i| = 0 \iff$

$(n \in \mathbb{N} \implies \sum_{i=1}^{n} |x_i - y_i| = 0) \iff \sum_{n=1}^{\infty} |x_n - y_n| = 0 \iff$

$\rho_p(x, y) = 0.$

4.

$(n \in \mathbb{N}) \implies (\sum_{i=1}^{n} |x_i - y_i|^p)^{\frac{1}{p}} \leq (\sum_{i=1}^{n} (|x_i - z_i| + |z_i - y_i|)^p)^{\frac{1}{p}}$

$\leq (\sum_{i=1}^{n} |x_i - z_i|^p)^{\frac{1}{p}} + (\sum_{i=1}^{n} |z_i - y_i|^p)^{\frac{1}{p}}$

$Therefore,$

$(\sum_{n=1}^{\infty} |x_n - y_n|^p)^{\frac{1}{p}} \leq (\sum_{n=1}^{\infty} |x_n - z_n|^p)^{\frac{1}{p}} + (\sum_{n=1}^{\infty} |z_n - y_n|^p)^{\frac{1}{p}}$ □

Example 6.

$l^{\infty} = \{(x_n) \in \mathbb{R}^{\mathbb{N}} : sup\{|x_n| : n \in \mathbb{N}\} < +\infty\},$

$\rho_{\infty}((x_n), (y_n)) = sup\{|x_n - y_n| : n \in \mathbb{N}\},$

Then $(l^{\infty}, \rho_{\infty})$ *is a metric space.*

Proof.

$(x_n),(y_n) \in \mathbb{R}^{\mathbb{N}} \implies (n \in \mathbb{N} \implies |x_n - y_n| \leq |x_n| + |y_n|)$

$\implies sup\{|x_n - y_n| : n \in \mathbb{N}\} \leq sup\{|x_n| + |y_n| : n \in \mathbb{N}\}$

$sup\{|x_n| + |y_n| : n \in \mathbb{N}\} \leq sup\{|x_n| : n \in \mathbb{N}\} + sup\{|y_n| : n \in \mathbb{N}\}$

$Therefore,$

$sup\{|x_n - y_n| : n \in \mathbb{N}\} < +\infty$

$Let (x_n), (y_n), (z_n) \in \mathbb{R}^{\mathbb{N}}.$

1.

$0 \leq |x_1 - y_1| \leq sup\{|x_n - y_n| : n \in \mathbb{N}\}.$

$Therefore,$

$\rho_\infty((x_n),(y_n)) \geq 0.$

2.

$\forall n \in \mathbb{N} : |x_n - y_n| = |y_n - x_n|.$

$Therefore,$

$sup\{|x_n - y_n| : n \in \mathbb{N}\} = sup\{|y_n - x_n| : n \in \mathbb{N}\}$

3.

$\rho_\infty((x_n),(y_n)) = 0 \iff \forall n \in \mathbb{N} : |x_n - y_n| = 0 \iff (x_n) = (y_n).$

4.

$\forall n \in \mathbb{N} : |x_n - y_n| \leq |x_n - z_n| + |z_n - y_n|.$

$Therefore,$

$sup\{|x_n - y_n| : n \in \mathbb{N}\} \leq sup\{|x_n - z_n| + |z_n - y_n| : n \in \mathbb{N}\}$

$sup\{|x_n - z_n| + |z_n - y_n| : n \in \mathbb{N}\} \leq$

$sup\{|x_n - z_n| : n \in \mathbb{N}\} + sup\{|z_n - y_n| : n \in \mathbb{N}\}$

$Therefore,$

$\rho_\infty((x_n),(y_n)) \leq \rho_\infty((x_n),(z_n)) + \rho_\infty((z_n),(y_n)).$ □

Example 7.

$Assume\ that\ (X_1, \sigma_1), (X_2, \sigma_2), ...\ is\ a\ sequence\ of\ metric\ spaces\ such\ that:$

$\forall n \in \mathbb{N} : (\forall x, y \in X_n : (\sigma_n(x, y) \leq 1)).$

$We\ define$

$X = \prod_{i=1}^{\infty} X_n, \rho : X \times X \to \mathbb{R}, \rho(x, y) = \sum_{n=1}^{\infty} \frac{\sigma_n(x_n, y_n)}{2^n}.$

$Then\ (X, \rho)\ is\ a\ metric\ space.$

Proof.

$x, y \in X \implies \sum_{i=1}^{n} \frac{\sigma_i(x_i, y_i)}{2^i} \leq \sum_{i=1}^{n} \frac{1}{2^i} < \sum_{n=1}^{\infty} \frac{1}{2^n} = 1.$

$Therefore,$

$\rho(x, y) \in \mathbb{R}.$

$x, y, z \in X$

1.

$\forall i \in \mathbb{N} : \sigma_i(x_i, y_i) \geq 0 \implies \forall n \in \mathbb{N} : \sum_{i=1}^{n} \frac{\sigma_i(x_i, y_i)}{2^i} \geq 0$

$\implies \sum_{n=1}^{\infty} \frac{\sigma_n(x_n, y_n)}{2^n} \geq 0.$

2.

$\forall i \in \mathbb{N} : \sigma_i(x_i, y_i) = \sigma_i(y_i, x_i)$

$\implies \forall n \in \mathbb{N} : \sum_{i=1}^{n} \frac{\sigma_i(x_i, y_i)}{2^i} = \sum_{i=1}^{n} \frac{\sigma_i(y_i, x_i)}{2^i} \implies \rho(x, y) = \rho(y, x).$

3.

$\rho(x, y) = 0 \iff \forall n \in \mathbb{N} : \sigma_n(x_n, y_n) = 0$

$\iff \forall n \in \mathbb{N} : x_n = y_n \iff x = y.$

4.

$\forall n \in \mathbb{N} : \sum_{i=1}^{n} \frac{\sigma_i(x_i, y_i)}{2^i} \leq \sum_{i=1}^{n} \frac{\sigma_i(x_i, z_i) + \sigma_i(z_i, y_i)}{2^i} =$

$\sum_{i=1}^{n} \frac{\sigma_i(x_i, z_i)}{2^i} + \sum_{i=1}^{n} \frac{\sigma_i(z_i, y_i)}{2^i}$

$Therefore,$

$\sum_{n=1}^{\infty} \frac{\sigma_n(x_n, y_n)}{2^n} \leq \sum_{n=1}^{\infty} \frac{\sigma_n(x_n, z_n)}{2^n} + \sum_{n=1}^{\infty} \frac{\sigma_n(z_n, y_n)}{2^n}$ □

Example 8.

$C([\alpha,\beta]) = \{f : [\alpha,\beta] \to \mathbb{R} : f continuous\}.$

1. $\rho_\infty : C([\alpha,\beta]) \times C([\alpha,\beta]) \to \mathbb{R}, \rho_\infty(f,g) = sup\{|f-g|(x) : x \in [\alpha,\beta]\}.$

2. $\rho_1(f,g) : C([\alpha,\beta]) \times C([\alpha,\beta]) \to \mathbb{R}, \rho_1(f,g) = \int_\alpha^\beta |f-g|$

$(C([\alpha,\beta]),\rho_\infty),(C([\alpha,\beta]),\rho_1)$ *are metric spaces.*

Proof.

1.

$|f-g| : [\alpha,\beta] \to \mathbb{R}$ *is bounded.*

$(Assume\ that\ |f-g|\ is\ not\ bounded.\ Then$

$\forall n \in \mathbb{N} : \exists x_n \in [\alpha,\beta] : |f-g|(x_n) > n.$

(x_n) *contains a convergent subsequence* (x_{k_n}) $(Bolzano-Weirstrass)$

$, such\ that\ x_{k_n} \to x_0 \in [\alpha,\beta]. Since\ |f-g|\ is\ continuous\ f(x_{k_n}) \to f(x_0).$

$Constradiction. Therefore\ |f-g|\ is\ bounded.)$

1.1

$(\forall x \in [\alpha,\beta] : |f-g|(x) \geq 0) \wedge (\exists M > 0 : |f-g|(x) \leq M) \implies$

$0 \leq sup\{|f-g|(x) : x \in [\alpha,\beta]\} < +\infty.$

1.2

$\forall x \in [\alpha,\beta] : |f-g|(x) = |g-f|(x) \implies$

$sup\{|f-g|(x) : x \in [\alpha,\beta]\} = sup\{|g-f|(x) : x \in [\alpha,\beta]\}.$

1.3

$\rho_\infty(f,g) = 0 \iff \forall x \in [\alpha,\beta] : |f-g|(x) = 0 \iff f = g.$

1.4

$\forall x \in [\alpha,\beta] : |f-g|(x) \leq |f-h|(x) + |h-g|(x).$

$Therefore,$

$sup\{|f-g|(x) : x \in [\alpha,\beta]\} \leq sup\{|f-h|(x) + |h-g|(x) : x \in [\alpha,\beta]\}.$

$sup\{|f-h|(x) + |h-g|(x) : x \in [\alpha,\beta]\} \leq sup\{|f-h|(x) : x \in [\alpha,\beta]\}$

$+\, sup\{|h-g|(x) : x \in [\alpha,\beta]\}.$

$Therefore,$

$sup\{|f-g|(x) : x \in [\alpha,\beta]\} \leq sup\{|f-h|(x) : x \in [\alpha,\beta]\} +$

$sup\{|h-g|(x) : x \in [\alpha,\beta]\}.$

2.1

$\forall x \in [\alpha,\beta] : |f-g|(x) \geq 0 \implies \int_\alpha^\beta |f-g| \geq 0.$

2.2

$\forall x \in [\alpha,\beta] : |f-g|(x) = |g-f|(x) \implies \int_\alpha^\beta |f-g| = \int_\alpha^\beta |g-f|.$

2.3

$((f : [\alpha,\beta] \to \mathbb{R}\ continuous) \wedge (\forall x \in [\alpha,\beta] : f(x) \geq 0)$

$\implies \int_\alpha^\beta f = 0 \iff f = 0)$

$Therefore,$

$since\ |f-g|\ is\ continuous\ and\ \forall x \in [\alpha,\beta] : |f-g|(x) \geq 0,$

$\rho_1(f,g) = 0 \iff f = g.$ □

Definition 2. *(Open Sphere)*

$Let\ (X,\rho)\ be\ a\ metric\ space, x \in X, and\ \epsilon > 0.$

$Open\ sphere\ with\ center\ x\ and\ radius\ \epsilon\ is\ the\ set :$

$S(x,\epsilon) = \{x' \in X : \rho(x,x') < \epsilon\}.$

Proposition 4.

$Let\ (X,\rho)\ be\ a\ metric\ space, and\ Y \subset X.$

$\sigma = \rho|Y \times Y \implies (\forall y \in Y : (\forall \epsilon > 0 : (S_\sigma(y,\epsilon) = S_\rho(y,\epsilon) \cap Y)))$

Proof.

$z \in S_\sigma(y,\epsilon) \iff z \in Y \wedge \sigma(y,z) < \epsilon$

$\iff z \in Y \wedge \rho(y,z) < \epsilon \iff z \in Y \cap S_\rho(x,\epsilon).$ □

Proposition 5.

$Let\ (X,\rho)\ be\ a\ metric\ space, x \in X\ and\ \epsilon > 0. Then,$

$\forall y \in S(x,\epsilon) : (\exists \delta > 0 : (S(y,\delta) \subset S(x,\epsilon)))$.

Proof.

$Let\ y \in S(x,\epsilon)\ and\ \delta = \epsilon - \rho(x,y).$

$z \in S(y,\delta) \implies \rho(y,z) < \delta.$

$\rho(x,z) \leq \rho(x,y) + \rho(y,z) < \rho(x,y) + \epsilon - \rho(x,y) = \epsilon.$

$Therefore,$

$z \in S(x,\epsilon).$

$Therefore,$

$S(y,\delta) \subset S(x,\epsilon).$ □

Definition 3. *(Open Set)*

$Let\ (X,\rho)\ be\ a\ metric\ space\ and\ G \subset X. G\ is\ an\ open\ set\ if$

$\forall x \in G : (\exists \epsilon > 0 : (S(x,\epsilon) \subset G))$

Corollary 1.

$Let\ (X,\rho)\ be\ a\ metric\ space.$

$x \in X, \epsilon > 0 \implies S(x,\epsilon)\ is\ an\ open\ set.$

Theorem 4.

$Let\ (X,\rho)\ be\ a\ metric\ space. Then,$

1. $\emptyset, X\ are\ open\ sets\ relative\ to\ X.$

2. $(I \subset P(X)) \wedge (\forall c \in I : c\ is\ open\ relative\ to\ X)$

$\implies G = \bigcup_{c \in I} c\ is\ an\ open\ set\ relative\ to\ X.$

3. $(n \in \mathbb{N}) \wedge (G_1, ..., G_n, open\ sets\ relative\ to\ X) \implies$

$G = \bigcap_{i=1}^{n} G_i, is\ an\ open\ set\ relative\ to\ X.$

Proof.

1.

$It\ follows\ from\ the\ definition.$

2.

$x \in G \implies \exists c \in P(X) : x \in c$

$Since\ c\ is\ an\ open\ set\ relative\ to\ X,$

$\exists\ \epsilon > 0 : S(x, \epsilon) \subset c.Therefore,$

$S(x, \epsilon) \subset G.$

3.

$x \in G \implies \forall i \in \{1, ..., n\} : \exists \epsilon_i > 0 : S(x, \epsilon_i) \subset G_i.$

$Let\ \epsilon = min\{\epsilon_i : i \in \{1, ..., n\}\}.Then,$

$\forall i \in \{1, ..., n\} : S(x, \epsilon) \subset G_i.Therefore,$

$S(x, \epsilon) \subset G.$ □

Proposition 6.

$Let\ (X, \rho)\ be\ a\ metric\ space,\ and\ G \subset X.Then,$

$G\ is\ an\ open\ set\ relative\ to\ X \iff$

$(\exists I \subset P(X) : (\forall c \in I : c\ is\ an\ open\ set\ relative\ to\ X) \wedge (G = \bigcup_{c \in I} c)).$

Proof.

$\longrightarrow (G\ is\ open\ relative\ to\ X)$

$\forall x \in G : \exists \epsilon_x > 0 : S(x, \epsilon_x) \subset X.$

$Let\ I = \{S(x, \epsilon_x) : x \in G\}.Then,$

$G = \bigcap_{c \in I} c.$

$\longleftarrow$

$It\ follows\ from\ Theorem 4.$ □

Proposition 7.

$Let\ (X, \rho)\ be\ a\ metric\ space, Y \subset X, \sigma = \rho|Y \times Y, and\ G \subset Y.Then,$

$G\ is\ open\ relative\ to\ Y \iff \exists A \subset X : ((A\ open\ relative\ to\ X) \wedge (G = A \cap Y))$

Proof.

$\longrightarrow$

$\forall x \in G : (\exists \epsilon_x > 0 : (S_\sigma(x, \epsilon_x) \subset G))$

$S_\sigma(x, \epsilon_x) = S_\rho(x, \epsilon_x) \cap Y$

$Let\ A = \bigcup_{x \in G} S_\rho(x, \epsilon_x).$

$A\ is\ open\ relative\ to\ X\ and$

$G = \bigcup_{x \in G} S_\sigma(x, \epsilon_x) = \bigcup_{x \in G}(S_\rho(x, \epsilon_x) \cap Y) = (\bigcup_{x \in G}(S_\rho(x, \epsilon_x)) \cap Y = A \cap Y$

$\longleftarrow$

$Let\ x \in A \cap Y.$

$\exists \epsilon_x > 0 : (S_\rho(x, \epsilon_x) \subset A)$

$S_\sigma(x, \epsilon_x) = S_\rho(x, \epsilon_x) \cap Y \subset A \cap Y.$

$Therefore,$

$G = A \cap Y\ is\ open\ relative\ to\ Y.$ □

Definition 4. *(Interior Set)*

$Let\ (X, \rho)\ be\ a\ metric\ space\ and\ A \subset X. The\ interior\ of\ A\ is\ the\ set$

$A^{\circ} = \bigcup\{G \subset X : (G\ open\ relative\ to\ X) \wedge (G \subset A)\}$

Proposition 8.

$Let\ (X, \rho)\ be\ a\ metric\ space. Then :$

$1.(G \subset A) \wedge (G\ is\ open\ relative\ to\ X) \implies G \subset A^{\circ}$

$2.A\ is\ open\ relative\ to\ X \iff A = A^{\circ}$

Proposition 9.

$Let\ (X, \rho)\ be\ a\ metric\ space\ and\ A \subset X. Then,$

$A^{\circ} = \{x \in A : \exists \epsilon > 0 : S(x, \epsilon) \subset A\}$

Definition 5.

$Let\ (X, \rho)\ be\ a\ metric\ space\ and\ F \subset X.$

$F\ is\ closed\ relative\ to\ X\ if\ F'\ is\ open\ relative\ to\ X.$

Theorem 5.

$Let\ (X, \rho)\ be\ a\ metric\ space. Then,$

$1.\emptyset$ *and* X *are closed relative to* X.

$2.(n \in \mathbb{N}) \wedge (i = 1, ...n) \wedge (F_i\ closed\ \subset X) \implies F = \bigcup_{i=1}^{n} F_i\ is\ closed\ \subset X.$

$3.F = \bigcap\{F \subset X : F\ is\ closed\} \implies F\ is\ closed.$

Proposition 10.

$Let\ (X, \rho)\ be\ a\ metric\ space, Y \subset X, \sigma = \rho|Y \times Y\ and\ F \subset Y. Then,$

$F\ is\ closed\ relative\ to\ Y \iff \exists K\ closed\ \subset X : F = K \cap Y.$

Proof.

$F\ is\ closed\ relative\ to\ Y \iff$

$Y \setminus F\ open\ relative\ to\ Y \iff$

$\exists A\ open \subset X : Y \setminus F = A \cap Y \iff$

$\exists A\ open \subset X : F = Y \setminus (A \cap Y) = Y \cap (A \cap Y)^c = Y \cap (A^c \cup Y^c) = A^c \cap Y.$ □

Definition 6.

$Let\ (X, \rho)\ be\ a\ metric\ space\ and\ A \subset X.$

$The\ closure\ of\ A\ relative\ to\ X is\ the\ set :$

$\tilde{A} = \bigcap\{F \subset X : F\ closed\ and\ A \subset F\}$

Proposition 11.

$Let\ (X, \rho)\ be\ a\ metric\ space\ and\ A \subset X.$

$Then,$

$1.K \subset X, K\ is\ a\ close\ set\ , A \subset K \implies \tilde{A} \subset K.$

$2.A\ is\ close \iff A = \tilde{A}$

Proposition 12.

$Let\ (X, \rho)\ be\ a\ metric\ space\ and\ A \subset X. Then,$

$\tilde{A} = \{x \in X : \forall \epsilon > 0 : S(x, \epsilon) \cap A \neq \emptyset\}.$

Proof.

$We\ assume\ that\ \exists \epsilon > 0 : S(x, \epsilon) \cap A = \emptyset.$

$Let\ K = S^c(x,\epsilon).$

$K\ is\ closed\ and\ A \subset K, therefore,$

$\tilde{A} \subset K. Since\ x \notin K, x \notin \tilde{A}.$

$Therefore,$

$x \in \tilde{A} \implies \forall \epsilon > 0 : S(x,\epsilon) \cap A \neq \emptyset.$

$Lets\ assume\ that\ \forall \epsilon > 0 : S(x,\epsilon) \cap A \neq \emptyset.$

$x \in (\tilde{A})^c \implies \exists \epsilon > 0 : S(x,\epsilon) \subset (\tilde{A})^c$

$\implies \exists \epsilon > 0 : S(x,\epsilon) \cap A \subset (\tilde{A})^c \cap A = \emptyset.$

$Therefore,$

$x \in \tilde{A}.$

$Therefore,$

$x \in \tilde{A} \iff \forall \epsilon > 0 : S(x,\epsilon) \cap A \neq \emptyset.$ □

Definition 7.

$Let\ (X,\rho)\ be\ a\ metric\ space, \emptyset \neq A \subset X, x \in X.$

$\rho(x,A) = inf\{\rho(x,y) : y \in A\}.$

Proposition 13.

$Let\ (X,\rho)\ be\ a\ metric\ space. Then,$

$\emptyset \neq A \subset X, x \in X \implies (x \in \tilde{A} \iff \rho(x,A) = 0)$

Proof.

$(x \in \tilde{A} \iff \forall \epsilon > 0 : S(x,\epsilon) \cap A \neq \emptyset.)$

$Lets\ assume\ that\ x \in \tilde{A}.$

$Let\ (\epsilon_n) : \mathbb{N} \to \mathbb{R}, \epsilon_n = \frac{1}{n}. \forall n \in N : \exists x_n \in A : \rho(x,x_n) < \epsilon_n = \frac{1}{n}$

$0 \leq \rho(x,A) \leq \rho(x,x_n) = \frac{1}{n}.$

$Therefore,$

$\rho(x,A) = 0.$

$Lets\ assume\ that\ \rho(x, A) = 0.$

$Then\ \forall \epsilon > 0 : \exists y \in A : \rho(x, y) < \epsilon.$

$Therefore,$

$\forall \epsilon > 0 : S(x, \epsilon) \cap A \neq \emptyset.$

$Therefore,$

$x \in \tilde{A}.$ ☐

Proposition 14.

$Let\ (X, \rho)\ be\ a\ metric\ space, A \subset X. Then,$

$1.(\tilde{A^c}) = (A^o)^c$

$2.(A^c)^o = (\tilde{A})^c$

Proof.

1.

$x \in (\tilde{A^c}) \iff \forall \epsilon > 0 : S(x, \epsilon) \cap A^c \neq \emptyset.$

$\iff \forall \epsilon > 0 : S(x, \epsilon) \not\subset A \iff x \notin A^o \iff x \in (A^o)^c.$

2.

$x \in (A^c)^o \iff \exists \epsilon > 0 : S(x, \epsilon) \subset A^c$

$\iff \exists \epsilon > 0 : S(x, \epsilon) \cap A = \emptyset \iff x \notin \tilde{A} \iff x \in (\tilde{A})^c$ ☐

Definition 8.

$Let\ (X, \rho)\ be\ a\ metric\ space\ and\ A \subset X.$

$1.A\ member\ x \in X\ is\ called\ accumulation\ point\ of\ A\ if :$

$\forall \epsilon > 0 : (S(x, \epsilon) \backslash \{x\}) \cap A \neq \emptyset.$

$A' = \{x \in X : x\ is\ an\ accumulation\ point\ of\ A\}$

$2.A\ member\ x \in X\ is\ called\ isolated\ point\ of\ A\ if :$

$\exists \epsilon > 0 : S(x, \epsilon) \cap A = \{x\}.$

Proposition 15.

$Let\ (X, \rho)\ be\ a\ metric\ space, and\ A \subset X.Then,$

$\tilde{A} = A' \cup A.$

Definition 9.

$Let\ (X, \rho)\ be\ a\ metric\ space, and\ D \subset X.D\ is\ dence\ (in\ X)\ if\ \tilde{D} = X.$

Definition 10.

$A\ metric\ space\ is\ called\ separatable\ if\ it\ contains\ a\ countable\ dence\ subset.$

Proposition 16.

$Let\ (X, \rho)\ be\ a\ metric\ space\ and\ D \subset X.Then,$

$D\ is\ dence\ (in\ X) \iff \forall x \in X : \forall \epsilon > 0 : D \cap S(x, \epsilon) \neq \emptyset.$

Proof.

$Assume\ that\ x \in X, and\ \epsilon > 0.$

$x \in \tilde{D}, therefore\ S(x, \epsilon) \cap D \neq \emptyset.$ □

Definition 11.

$Let\ (X, \rho)\ be\ a\ metric\ space, (x_n) : \mathbb{N} \to X.$

$(x_n)\ converges\ to\ x \in X \iff \forall \epsilon > 0 : \exists n_0 \in \mathbb{N} : \forall n \geq n_0 : \rho(x_n, x) < \epsilon$

Proposition 17.

$Let\ (X, \rho)\ be\ a\ metric\ space\ and\ (x_n) : \mathbb{N} \to X.Then,$

$x_n \to x, x_n \to y \implies x = y.$

Proof.

$\rho(x, y) \leq \rho(x, x_n) + \rho(x_n, y) \to 0.$

$Therefore,$

$\rho(x, y) = 0,$

$Therefore,$

$x = y.$

□

Proposition 18.

Let (X_n, ρ_n) *be a sequence of metric spaces such that :*

$\forall n \in \mathbb{N} : \forall x, y \in X_n : \rho_n(x, y) \leq 1.$

$X = \prod_{n=1}^{\infty} X_n, \rho : X \times X \to R, \rho(x, y) = \sum_{i=1}^{\infty} \frac{\rho(x_n, y_n)}{2^n}.$

$((X, \rho)$ *is a metric space(example*$7))$

$(x_n) : \mathbb{N} \to X, y \in X \implies (x_n \to y \iff \forall m \in \mathbb{N} : \lim_{n\to\infty} x_{n_m} \to y_m)$

Proof.

Lets assume that $x_n \to y$ *and* $m \in \mathbb{N}$*.Then,*

$\forall n \in \mathbb{N} : \rho(x_{n_m}, y_m) \leq 2^m \rho(x_n, y) \to 0.$

$Therefore,$

$x_n \to y \implies \forall m \in \mathbb{N} : \lim_{n\to\infty} x_{n_m} \to y_m$

Lets assume that $\forall m \in \mathbb{N} : \lim_{n\to\infty} x_{n_m} \to y_m,$ *and* $\epsilon > 0.$

$\exists n_0 \in \mathbb{N} : \sum_{n=n_0}^{\infty} \frac{1}{2^n} < \frac{\epsilon}{2}.$

$\lim_{n\to\infty} \sum_{m=1}^{n_0-1} \frac{\rho_m(x_{n_m}, y_m)}{2^m} = 0.$

$Therefore,$

$\exists n_1 \in \mathbb{N} : \forall n \geq n_1 : \sum_{m=1}^{n_0-1} \frac{\rho_m(x_{n_m}, y_m)}{2^m} < \frac{\epsilon}{2}.$

Assume that $n \geq n_1.$

$\rho(x_n, y) = \sum_{m=1}^{\infty} \frac{\rho_m(x_{n_m}, y_m)}{2^m} = \sum_{m=1}^{n_0-1} \frac{\rho_m(x_{n_m}, y_m)}{2^m} + \sum_{m=n_0}^{\infty} \frac{\rho_m(x_{n_m}, y_m)}{2^m}$

$\leq \frac{\epsilon}{2} + \sum_{m=n_0}^{\infty} \frac{1}{2^m} < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$

$Therefore,$

$\forall m \in \mathbb{N} : \lim_{n\to\infty} x_{n_m} \to y_m \implies x_n \to y$ □

Proposition 19.

Let (X, ρ) *be a metric space* $\emptyset \neq A \subset X$ *and* $x \in X$*.Then,*

$x \in \tilde{A} \iff \exists (x_n) : \mathbb{N} \to A : x_n \to x.$

Proof.

$(x \in \tilde{A})$

$Let\ \epsilon_n = \frac{1}{n}.Since\ x \in \tilde{A}, \forall n \in \mathbb{N} : S(x, \epsilon_n) \cap A \neq \emptyset.$

$Therefore,$

$\forall n \in \mathbb{N} : \exists x_n \in A : \rho(x, x_n) < \epsilon_n = \frac{1}{n}.$

$Since\ \rho(x, x_n) < \frac{1}{n}, x_n \to x.$

$Therefore,$

$x \in \tilde{A} \implies \exists (x_n) : \mathbb{N} \to A : x_n \to x.$

$(\exists (x_n) : \mathbb{N} \to A : x_n \to x)$

$Let\ \epsilon > 0.Since\ x_n \to x, \exists n_0 \in \mathbb{N} : \forall n \geq n_0 : \rho(x, x_n) < \epsilon.$

$Therefore,$

$S(x, \epsilon) \cap A \neq \emptyset.$

$Therefore,$

$x \in \tilde{A}$

$Therefore,$

$\exists (x_n) : \mathbb{N} \to A : x_n \to x \implies x \in \tilde{A}$ □

Proposition 20.

$Let\ (X, \rho)\ be\ a\ metric\ space\ and\ A \subset X.Then,$

$A\ is\ closed \iff ((x_n) : \mathbb{N} \to A, x_n \to x \implies x \in A)$

Proof.

$(A\ is\ closed)$

$Let\ (x_n) : \mathbb{N} \to A, x_n \to x.$

$x \in A^c \implies \exists \epsilon > 0 : S(x, \epsilon) \subset A.$

$Since\ x_n \to x, \exists n_0 \in \mathbb{N} : \forall n \geq n_0 : x_n \in S(x, \epsilon).Contradiction.Therefore,$

$x \in A.$

$Therefore,$

$(A\ is\ closed) \implies ((x_n):\mathbb{N}\to A, x_n \to x \implies x\in A)$

$((x_n):\mathbb{N}\to A, x_n \to x \implies x\in A)$

$We\ assume\ that\ A^c\ isnt\ open. Then,$

$\exists x\in A^c : \forall \epsilon>0 : S(x,\epsilon)\cap A \neq \emptyset.$

$Let\ \epsilon_n = \frac{1}{n}. Then,$

$\forall n\in\mathbb{N} : \exists x_n : \rho(x,x_n) < \epsilon_n = \frac{1}{n}. x_n \to x, therefore\ x\in A. Contradiction.$

$Therefore,$

$A\ is\ closed.$

$Therefore,$

$((x_n):\mathbb{N}\to A, x_n \to x \implies x\in A) \implies A\ is\ closed$ □

Definition 12. *(Subsequence)*

$Let\ X\ be\ a\ set\ , (x_n):\mathbb{N}\to X, (n_k):\mathbb{N}\to\mathbb{N}\ strictly\ increesing.$

$Then\ x_{n_k}:\mathbb{N}\to X, is\ a\ subsequence\ of\ (x_n).$

Proposition 21.

$Let\ (X,\rho)\ be\ a\ metric\ space\ and\ (x_n):\mathbb{N}\to X. Then,$

$x_n\to x \iff (y_n\ is\ a\ subsequence\ of x_n \implies y_n\to x)$

Proof.

$(x_n\to x)$

$Let\ (y_n)\ be\ a\ subsequence\ of\ (x_n).$

$Let\ \epsilon>0. \exists n_0\in\mathbb{N} : \forall n\geq n_0 : \rho(x_n,x)<\epsilon. Then,$

$\forall n\geq n_0 : \rho(y_n,x)<\epsilon.$

$Therefore,$

$y_n\to x.$

$Therefore,$

$x_n\to x \implies (y_n\ is\ a\ subsequence\ of x_n \implies y_n\to x)$

$((y_n) \; is \; a \; subsequence \; of x_n \implies y_n \to x))$

$(x_n) \; is \; a \; subsequence \; of \; (x_n)$

$Therefore,$

$x_n \to x.$

$Therefore,$

$(y_n \; is \; a \; subsequence \; of x_n \implies y_n \to x) \implies x_n \to x.$

□

Corollary 2.

Let (X, ρ) *be a metric space and* $(x_n) : \mathbb{N} \to X.Then$

$x_n \not\to x \iff \exists (y_n) : \mathbb{N} \to X : (y_n) \; subsequence \; of \; (x_n) \; and \; y_n \not\to x.$

Definition 13.

Let $(X, \rho), (Y, \sigma)$ *be metric spaces,* $f : X \to Y,$ *and* $x_0 \in X.$

$Then, f \; is \; continuous \; at \; x_0 \; if :$

$\forall \epsilon > 0 : (\exists \delta > 0 : (x \in X, \rho(x_0, x) < \delta \implies \sigma(f(x_0), f(x)))) < \epsilon.$

Theorem 6.

Let $(X, \rho), (Y, \sigma)$ *be metric spaces and* $f : X \to Y.Then,$

$f \; is \; continuous \iff$

$G \subset Y, G \; is \; open \implies f^{-1}(G) \; is \; open \subset X \iff$

$F \subset Y, F \; is \; close \implies f^{-1}(F) \; is \; close \subset X \iff$

$\forall A \subset X : f(\tilde{A}) \subset f(\tilde{A})$

Proof.

1.

$(f \; is \; continuous \implies (G \subset Y, G \; is \; open \implies f^{-1}(G) \; is \; open \subset X)$

$Assume \; that \; G \subset Y \; is \; open, and \; x \in f^{-1}(G).Then \; f(x) \in G.$

$\exists \epsilon > 0 : S(f(x), \epsilon) \subset G.Since \; f \; is \; continuous$

$\exists\delta > 0 : x' \in X, \rho(x, x') < \delta \implies \sigma(f(x), f(x')) < \epsilon$

$Therefore,$

$S(x, \delta) \subset f^{-1}(G).$

2.

$((G \subset Y, G \textit{ is open } \implies f^{-1}(G) \textit{ is open } \subset X) \implies$

$(F \subset Y, F \textit{ is close } \implies f^{-1}(F) \textit{ is close } \subset X))$

$Assume\ that\ F \subset Y\ is\ close. F^c\ is\ open, therefore\ f^{-1}(F^c)\ is\ open.$

$f^{-1}(F^c) = (f^{-1}(F))^c\ is\ open. Therefore,$

$((f^{-1}(F))^c)^c = f^{-1}(F)\ is\ close.$

3.

$((F \subset Y, F \textit{ is close } \implies f^{-1}(F) \textit{ is close } \subset X) \implies$

$(\forall A \subset X : f(\tilde{A}) \subset \tilde{f(A)}))$

$A \subset f^{-1}(f(A)) \subset f^{-1}(\tilde{f(A)}). Since\ \tilde{f(A)}\ is\ closed,\ f^{-1}(\tilde{f(A)})\ is\ closed.$

$Therefore,$

$\tilde{A} \subset f^{-1}(\tilde{f(A)})$

$Therefore,$

$f(\tilde{A}) \subset f(f^{-1}(\tilde{f(A)})) = \tilde{f(A)}$

4.

$((\forall A \subset X : f(\tilde{A}) \subset \tilde{f(A)}) \implies (f\ is\ continuous))$

$Assume\ that\ f\ isn't\ continuous\ at\ x \in X. Then,$

$\exists\epsilon > 0 : (\forall\delta > 0 : (\exists x' \in X : (\rho(x, x') < \delta, \sigma(f(x), f(x')) > \epsilon))))$

$Let\ \delta_n : \mathbb{N} \to \mathbb{R}, \delta_n = \frac{1}{n}. Then,$

$\forall n \in \mathbb{N} : (\exists x_n : (\rho(x_n, x) < \delta_n, \sigma(f(x), f(x_n) > \epsilon)))$

$Let\ A = \{x_n \in X : n \in \mathbb{N}\}. Then\ x \notin A, x \in \tilde{A}, therefore$

$f(x) \in f(\tilde{A}).$

$Since\ S(f(x),\epsilon)\cap f(A)=\emptyset, f(x)\notin \tilde{f(A)}.$

$Therefore,$

$f(\tilde{A})\not\subset \tilde{f(A)}$

$\square$

Proposition 22.

Let $(X,\rho),(Y,\sigma)$ *be metric spaces and* $f:X\to Y.Then,$

$f\ is\ continuous \iff (x_n):\mathbb{N}\to\mathbb{X}:x_n\to x \implies f(x_n)\to f(x).$

Proof.

$(f\ is\ continuous \implies (x_n):\mathbb{N}\to\mathbb{X}:x_n\to x \implies f(x_n)\to f(x))$

$Let\ x\in X\ and\ \epsilon>0.$

$\exists\delta>0: x'\in X,\rho(x,x')<\delta \implies \sigma(f(x),f(x'))<\epsilon.$

$\exists n_0\in\mathbb{N}:\forall n\geq n_0:\rho(x,x_n)<\delta.$

$Therefore,$

$\forall n\geq n_0:\sigma(f(x),f(x_n))<\epsilon.$

$(((x_n):\mathbb{N}\to\mathbb{X}:x_n\to x \implies f(x_n)\to f(x)) \implies f\ is\ continuous)$

$Assume\ that\ f\ isn't\ continuous\ at\ x\in X.Then,$

$\exists\epsilon>0:\forall\delta>0:\exists x'\in X:\rho(x,x')<\delta,\sigma(f(x),f(x'))>\epsilon.$

$Let\ (\delta_n):\mathbb{N}\to X,\delta_n=\frac{1}{n}.Then$

$\forall n\in\mathbb{N}:\exists x_n\in X:\rho(x,x_n)<\frac{1}{n},\sigma(f(x),f(x_n))>\epsilon.$

$x_n\to x,$

$Therefore,$

$f(x_n)\to f(x).Contradiction.$

$Therefore,$

$f\ is\ continuoss.$ $\square$

Definition 14.

Let $(X,\rho),(Y,\sigma)$ *be metric spaces and* $f : X \to Y$.

$f\ is\ uniformly\ continuous\ if:$

$\forall \epsilon > 0 : \exists \delta > 0 : x, x' \in X, \rho(x, x') < \delta \implies \sigma(f(x), f(x')) < \epsilon.$

Proposition 23.

Let $(X,\rho),(Y,\sigma)$ *be metric spaces and* $f : X \to Y$.

$\exists k > 0 : \forall x, x' \in X : \sigma(f(x), f(x')) < k\rho(x, x') \implies f\ is\ uniformly\ continuous.$

Proposition 24.

Let (X,ρ) *be a metric space,* $\sigma : X \times X \to \mathbb{R}, \sigma(x,y) = \rho(x_1,y_1) + \rho(x_2,y_2)$.

Then $\rho : (X \times X, \sigma) \to \mathbb{R}$ *is uniformly continuous.*

Proof.

Let $x, y \in X \times X$.

$\rho(x_1,x_2) \leq \rho(x_1,y_1) + \rho(y_1,y_2) + \rho(y_2,x_2),$

$\rho(y_1,y_2) \leq \rho(y_1,x_1) + \rho(x_1,x_2) + \rho(x_2,y_2)$

Therefore,

$|\rho(x_1,x_2) - \rho(y_1,y_2)| \leq \sigma(x,y).$ □

Proposition 25.

Let (X,ρ) *be a metric space* $\emptyset \neq A \subset X, f : X \to \mathbb{R}, f(x) = \rho(x, A)$. *Then,*

f *is uniformly continuous.*

Proof.

Let $x, y \in X$ *and* $\epsilon > 0$.

$\exists x' \in A : \rho(x,x') < \rho(x,A) + \epsilon,$

$\rho(y,A) \leq \rho(y,x') \leq \rho(y,x) + \rho(x,x') < \rho(x,y) + \rho(x,A) + \epsilon.$

Therefore,

$\rho(y,A) - \rho(x,A) \leq \rho(x,y).$

$...\rho(x, A) - \rho(y, A) \leq \rho(x, y).$

$Therefore,$

$|\rho(x, A) - \rho(y, A)| \leq \rho(x, y).$

$\square$

Definition 15.

Let $(X, \rho), (Y, \sigma)$ *be metric spaces and* $f : X \to Y$.

f *is called an isometry if :*

$\forall x, y \in X : \rho(x, y) = \sigma((f(x), f(y)).$

Definition 16.

Let $(X, \rho), (X, \sigma)$ *be metric spaces.*

ρ, σ *are equivalent if :*

$\forall A \subset X : (A \ is \ \rho - open \iff A \ is \ \sigma - open)$

Proposition 26.

Let $(X, \rho), (Y, \sigma)$ *be metric spaces.Then,*

ρ, σ *are equivalent* $\iff$

$\forall x \in X : (\forall \epsilon > 0 : (\exists \delta_1, \delta_2 : (S_\sigma(x, \delta_1) \subset S_\rho(x, \epsilon), S_\rho(x, \delta_2) \subset S_\sigma(x, \epsilon)))) \iff$

$(x_n) : \mathbb{N} \to X \implies (x_n \to_\rho x \iff x_n \to_\sigma x) \iff$

$id : (X, \rho) \to (X, \sigma), id^{-1} : (X, \sigma) \to (X, \rho)$ *are continuous.*

Proof.

1.

$(\rho, \sigma$ *are equivalent* $\implies$

$(\forall x \in X : (\forall \epsilon > 0 : (\exists \delta_1, \delta_2 : (S_\sigma(x, \delta_1) \subset S_\rho(x, \epsilon), S_\rho(x, \delta_2) \subset S_\sigma(x, \epsilon))))))$

Let $x \in X$ *and* $\epsilon > 0$.

$S_\rho(x,\epsilon)\ is\ \rho-open\, therefore\ is\ \sigma-open, therefore,$

$\exists\delta_1>0 : S_\sigma(x,\delta_1)\subset S_\rho(x,\epsilon).$

$S_\sigma(x,\epsilon)\ is\ \sigma-open\, therefore\ is\ \rho-open, therefore,$

$\exists\delta_2>0 : S_\rho(x,\delta_2)\subset S_\sigma(x,\epsilon).$

2.

$(\forall x\in X : (\forall\epsilon>0 : (\exists\delta_1,\delta_2 : (S_\sigma(x,\delta_1)\subset S_\rho(x,\epsilon), S_\rho(x,\delta_2)\subset S_\sigma(x,\epsilon)))) \implies$

$(x_n):\mathbb{N}\to X \implies (x_n\to_\rho x \iff x_n\to_\sigma x))$

$Let\ (x_n):\mathbb{N}\to X, x_n\to_\rho x.$

$Let\ \epsilon>0. Then,$

$\exists\delta>0 : S_\rho(x,\delta)\subset S_\sigma(x,\epsilon).$

$\exists n_0\in\mathbb{N} : \forall n\geq n_0 : x_n\in S_\rho(x,\delta).$

$Therefore,$

$\forall n\geq n_0 : x_n\in S_\sigma(x,\epsilon).$

3.

$((x_n):\mathbb{N}\to X \implies (x_n\to_\rho x \iff x_n\to_\sigma x) \implies$

$id:(X,\rho)\to(X,\sigma), id^{-1}:(X,\sigma)\to(X,\rho)\ are\ continuous)$

$It\ follows\ from\ proposition\ 24.$

4.

$(id:(X,\rho)\to(X,\sigma), id^{-1}:(X,\sigma)\to(X,\rho)\ are\ continuous \implies$

$\rho,\sigma\ are\ equivalent)$

$It\ follows\ from\ theorem\ 6.$ □