The Energy-Restitution Connection: A First-Principles Derivation for Rigid Body Collisions

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Abstract

The coefficient of restitution is a dimensionless parameter commonly used to classify the elasticity of collisions. In this paper, we derive a direct relationship between the restitution coefficient and the change in kinetic energy of a two-body system during a one-dimensional collision. This formulation enhances the interpretability of energy loss in terms of velocity and mass, providing an analytic tool for physical and computational modeling.

1 Introduction

The restitution coefficient ε is typically introduced as a descriptive ratio between relative velocities. However, its deeper connection to energy conservation principles is often overlooked. By expressing ε in terms of the system's kinetic energy loss T, we establish a physically grounded perspective suitable for theoretical modeling.

2 Definitions and Setup

Consider two point masses m_a and m_b undergoing a head-on collision. Let v_a and v_b be their initial velocities, and v'_a and v'_b their velocities after the collision. Define:

- **Restitution coefficient:**

$$\varepsilon = \frac{v'_b - v'_a}{v_a - v_b}$$

- **Kinetic energy change:**

$$\Delta T = \frac{1}{2} m_a {v'_a}^2 + \frac{1}{2} m_b {v'_b}^2 - \left(\frac{1}{2} m_a v_a^2 + \frac{1}{2} m_b v_b^2 \right)$$

3 Main Result

Lemma 1 (Restitution Coefficient from Energy Loss)**.** *Let m_a, m_b be the masses of two colliding bodies, with initial velocities v_a, v_b and final velocities v'_a, v'_b. If momentum is conserved, then the restitution coefficient ε can be written in terms of the kinetic energy loss ΔT as:*

$$\varepsilon = \sqrt{1 - \frac{2\Delta T}{\mu (v_a - v_b)^2}}\ ,\ \mu = \frac{m_a m_b}{m_a + m_b}$$

Proof. From conservation of momentum:

$$m_a v_a + m_b v_b = m_a v'_a + m_b v'_b$$

Substitute the restitution expression:

$$v_b' = v_a' + \varepsilon(v_a - v_b)$$

Insert into momentum:

$$m_a v_a + m_b v_b = m_a v_a' + m_b(v_a' + \varepsilon(v_a - v_b))$$

Solving for v_a':

$$v_a' = \frac{m_a v_a + m_b v_b - m_b \varepsilon(v_a - v_b)}{m_a + m_b}$$

Then v_b' becomes:

$$v_b' = \frac{m_a v_a + m_b v_b + m_a \varepsilon(v_a - v_b)}{m_a + m_b}$$

Substitute v_a', v_b' into T:

$$T = \frac{1}{2}m_a(v_a')^2 + \frac{1}{2}m_b(v_b')^2 - \left(\frac{1}{2}m_a v_a^2 + \frac{1}{2}m_b v_b^2\right)$$
$$\Rightarrow \varepsilon = \sqrt{1 - \frac{2\Delta T}{\mu(v_a - v_b)^2}}$$

□

4 Discussion

This expression offers a direct connection between a scalar energy loss quantity and a vectorial description of collision outcome. It clarifies that:

- $\varepsilon = 1$ implies $\Delta T = 0$ (perfectly elastic)
- $\varepsilon = 0$ implies maximum kinetic energy loss (perfectly inelastic)

• μ is the reduced mass The formula can be inverted to compute ε from empirical energy measurements, making it suitable for data-based physics modeling.

4.1 Corollaries: Collision Types and Their Kinetic Signatures

We explore the specific outcomes of collisions by evaluating the restitution coefficient and the change in kinetic energy ΔT for key cases. Each case provides a distinct form of the underlying energy equation.

4.2 Perfectly Elastic Collision

Corollary 1 (**Elastic Case** ($\varepsilon = 1$))**.** *The kinetic energy is fully conserved. Thus,*

$$\Delta T = 0$$

Velocities after collision:

$$v'_a = \frac{(m_a - m_b)v_a + 2m_b v_b}{m_a + m_b}$$
$$v'_b = \frac{(m_b - m_a)v_b + 2m_a v_a}{m_a + m_b}$$

These expressions are derived under conservation of both momentum and kinetic energy.

4.3 Perfectly Inelastic Collision

Corollary 2 (**Inelastic Case** ($\varepsilon = 0$))**.** *The objects stick together post-collision:*

$$v_f = \frac{m_a v_a + m_b v_b}{m_a + m_b}$$

The kinetic energy loss is maximized:

$$\Delta T = -\frac{1}{2}\frac{m_a m_b}{m_a + m_b}(v_a - v_b)^2$$

4.4 Partially Elastic Collision

Corollary 3 (**General Case** ($0 < \varepsilon < 1$))**.** *A fraction of the kinetic energy is lost:*

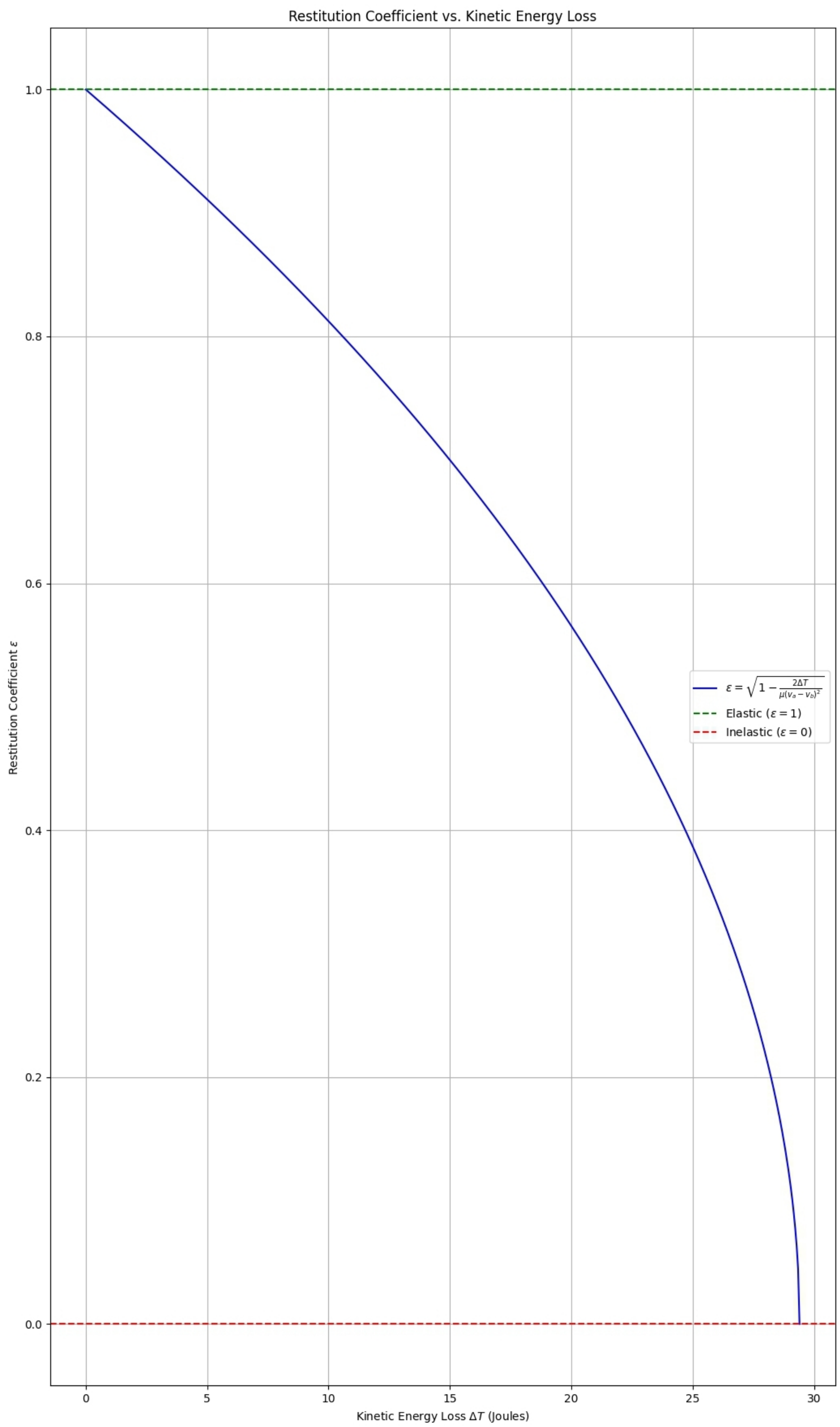
$$\Delta T = -\frac{1}{2}\frac{m_a m_b}{m_a + m_b}(1 - \varepsilon^2)(v_a - v_b)^2$$

Final velocities are:

$$v'_a = \frac{m_a v_a + m_b v_b - m_b \varepsilon (v_a - v_b)}{m_a + m_b}$$
$$v'_b = \frac{m_a v_a + m_b v_b + m_a \varepsilon (v_a - v_b)}{m_a + m_b}$$

These forms reduce to the elastic or inelastic cases under $\varepsilon = 1$ or $\varepsilon = 0$, respectively.

Graphical Summary



Restitution vs Energy Loss: Graphical relationship between kinetic energy loss and restitution coefficient. The curve illustrates nonlinear dissipation behavior under varying ΔT.

4.5 Limitations and Scope

While the derived expression for the restitution coefficient is both analytically elegant and physically consistent, it is essential to outline the constraints under which this formulation holds. The following limitations define the scope of its validity:

1. **One-Dimensional Collisions Only:** The derivation assumes purely linear motion along a single axis. Extension to two or three dimensions would require tensorial treatment of momentum and energy, accounting for angular degrees of freedom and vector projections [3, 2].
2. **Rigid Body Assumption:** The objects involved are considered point masses or rigid bodies. Internal deformations, flexing, and vibrational modes during impact are neglected, which are critical in real-world materials [4, **?**].
3. **Negligible External Forces:** The system is assumed isolated during collision, with no net external forces acting during the interaction time. This excludes gravitational gradients, electromagnetic effects, and friction from the environment [1].
4. **No Rotational Dynamics:** The derivation does not account for angular momentum or torque. For spinning or off-center impacts, the formulation becomes insufficient and would require moment of inertia and angular velocity analysis.
5. **Perfect Contact Model:** The collision is modeled as an instantaneous event with no surface compliance. Time-dependent contact forces or energy losses due to sound, heat, or microscopic friction are not included [4, **?**].
6. **Scalar Energy Representation:** The kinetic energy used is scalar and does not distinguish between directional motion. Thus, this model does not capture vectorial asymmetries or directional energy redistributions [3].
7. **Symmetric Frame Reference:** The derivation implicitly assumes an inertial frame where the total system momentum is defined consistently pre- and post-collision. Frame-dependent interpretations may distort results [1].

This model could be effective for idealized, symmetrical, and controlled collisions in elementary mechanics. For applications in materials science, robotics, or astrophysical dynamics, higher-order generalizations are necessary.

5 Conclusion

The restitution coefficient is often regarded as a purely kinematic descriptor. By deriving its dependence on kinetic energy loss, we establish a bridge between dynamics and energy methods in classical mechanics. This relationship adds interpretive power to both simulation and theoretical frameworks. This paper is open for generalizations to other concepts, such as in rotational mechanics, application to real world systems, and higher dimension generalizations.

References

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[4] W. J. Stronge, *Impact Mechanics*, Cambridge University Press, 2000.